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On partial sums of mock theta functions of order three

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Abstract. The object of this paper is to define and study the properties of partial mock theta functions of order three, on the lines Ramanujan had studied partial θ-functions. These new partial functions have been expressed in terms of basic hypergeometric function ${}_2\Phi_1$. Their continued fractions representations have also been given.

Keywords. Mock theta function; basic hypergeometric series; continued fractions.

1. Introduction

Ramanujan during the last days of his life discovered the mock theta functions which were communicated by him in his last letter to Hardy in February 1920. He classified these functions as of order 3, 5 and 7, without assigning any reason. In some of his unpublished works (now published as the 'Lost Note Book'), a number of identities and expansion formulae for, partial sums of the Jacobi's theta functions are mentioned without proof.

A study of these sums and expansions has been made by Andrews [3]. A little later, Agarwal [1, p. 121–128] showed that these relations, in fact, belong to a general class of basic hypergeometric identities and one could find a number of similar relations by making a study of three and four term relations between ${}_3\Phi_2$ series given by Sears [5].

Andrews in his work has shown that some of these partial theta function identities have interesting number theoretic interpretations.

The above considerations and the interest shown by a number of mathematician in the study of partial theta function identities motivated me to define partial mock theta functions and study of their associated identities.

The object of this paper is to define and study the relations between partial mock theta functions of order three. It has been shown that these partial mock theta functions are also interrelated with relations similar to the one for the mock theta function of order three.

It would be interesting to study the number theoretic interpretations, if any, of these partial mock theta functions.

2. Notation

Let

$$(a)_n = (1-a)(1-aq)(1-aq^2),\ldots,(1-aq^{n-1}), \quad n>0,$$

$$(a)_0 = 1.$$

Also

$$(a_1,a_2,\ldots,a_r)_n = (a_1)_n(a_2)_n,\ldots,(a_r)_n$$

and

$$(a)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

A basic hypergeometric series is then defined for $|q|<1$, as

$$ {}_{r+1}\Phi_r\left[\begin{matrix}\alpha_1,\ \alpha_2,\ldots,\ \alpha_{r+1};\\ \rho_1,\ \rho_2,\ldots,\ \rho_r\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1,\alpha_2,\ldots,\alpha_{r+1})_n Z^n}{(q,\rho_1,\rho_2,\ldots,\rho_r)_n}. \tag{2.1}$$

The series (2.1) converges for $|z|<1$.
Further, let

$$ {}_{r+1}\Phi_r\left[\begin{matrix}\alpha_1,\ \alpha_2,\ldots,\ \alpha_{r+1};\\ \rho_1,\ \rho_2,\ldots,\ \rho_r\end{matrix}\ z\right]_m $$

denote the partial sum

$$\sum_{n=0}^{m}\frac{(\alpha_1,\alpha_2,\ldots,\alpha_{r+1})_n z^n}{(q,\rho_1,\rho_2,\ldots,\rho_r)_n}.$$

of the generalized basic hypergeometric series.

3. Definitions of partial mock theta function of order three

Fine [4] represented all the third order mock theta functions as a limiting cases of a

$$ {}_2\Phi_1\left[\begin{matrix}\alpha,\ q;\\ \beta\end{matrix}\ x\right]. $$

By using Heine's transformation

$$ (x,e)_\infty\Phi\left[\begin{matrix}a,\ b;\\ e\end{matrix}\ x\right]=(b,ax)_\infty\Phi\left[\begin{matrix}\frac{e}{b},\ x;\\ ax\end{matrix}\ b\right], \tag{3.1}$$

we have given alternative forms for the third order mock theta functions as

$$ f(q)=2-\frac{(q)_\infty}{(-q)_\infty^2}\Phi\left[\begin{matrix}-1,\ -q;\\ 0\end{matrix}\ q\right], \tag{3.2}$$

$$ \phi(q)=(1+i)-\frac{i(q)_\infty}{(iq,-iq)_\infty}\Phi\left[\begin{matrix}-i,\ iq;\\ 0\end{matrix}\ q\right], \tag{3.3}$$

$$ \Psi(q)=\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\phi(q^2)\left[\begin{matrix}q,\ 0;\\ -q^3\end{matrix}\ q^2\right], \tag{3.4}$$

$$\chi(q)=(1+\omega)-\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\,\Phi\begin{bmatrix}-\omega, & -\omega^2 q; & \\ & & q\\ 0 & & \end{bmatrix}, \tag{3.5}$$

$$\omega(q)=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2}\,\Phi^{(q^2)}\begin{bmatrix}q, & q; & \\ & & q^2\\ 0 & & \end{bmatrix}, \tag{3.6}$$

$$\gamma(q)=\frac{(q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\,\Phi^{(q^2)}\begin{bmatrix}-q, & 0; & \\ & & q^2\\ -q^2 & & \end{bmatrix}, \tag{3.7}$$

and

$$\rho(q)=\frac{(q^2;q^2)_\infty}{(q\omega;q\omega^2;q^2)_\infty}\,\Phi^{(q^2)}\begin{bmatrix}\omega q, & \omega^2 q; & \\ & & q^2\\ 0 & & \end{bmatrix}. \tag{3.8}$$

Denoting the series

$$\sum_{n=0}^{m}\frac{(a,b)_n x^n}{(c,q)_n} \quad \text{as} \quad \Phi\begin{bmatrix}a, & b; & \\ & & x\\ c & & \end{bmatrix}_m$$

and by using the following transformation

$$\Phi\begin{bmatrix}D, & E; & \\ & & q\\ C & & \end{bmatrix}_N=\frac{(DEq^{N+1})_\infty(q^{N+1})_\infty}{(Eq^{N+1})_\infty(Dq^{N+1})_\infty}\Phi\begin{bmatrix}Cq^N, & D, & E; & \\ & & & q\\ C, & DEq^{N+1} & & \end{bmatrix}, \tag{3.9}$$

and (3.1) we have defined the partial mock theta functions of order three as

(i)
$$f_N(q)=2-\frac{(q)_\infty}{(-q)_\infty^2}\,\Phi\begin{bmatrix}-1, & -q; & \\ & & q\\ 0 & & \end{bmatrix}_N, \tag{3.10}$$

$$=2-\frac{(q)_\infty}{(-q)_\infty^2}\,\frac{(-q)_{N+1}(-q)_N}{(q)_N}\,\Phi\begin{bmatrix}-q^{1+N}, & q; & \\ & & -q\\ -q & & \end{bmatrix}, \tag{3.11}$$

(ii)
$$\phi_N(q)=(1+i)-\frac{i\,(q)_\infty}{(iq,-iq)_\infty}\,\Phi\begin{bmatrix}-i, & iq; & \\ & & q\\ 0 & & \end{bmatrix}_N, \tag{3.12}$$

$$=(1+i)-\frac{i(q)_\infty}{(iq,-iq)_\infty}\,\frac{i(q)_{N+1}(-iq)_N}{(q)_N}\,\Phi\begin{bmatrix}-iq^{1+N}, & q; & \\ & & iq\\ -iq & & \end{bmatrix}, \tag{3.13}$$

(iii) $$\Psi_N(q)=\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix} q, & 0; & \\ & & q^2 \\ -q^3 & & \end{bmatrix}_N, \tag{3.14}$$

$$=\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\frac{(q;q^2)_{N+1}}{(q^2;q^2)_N(-q^3;q^2)_{N+1}}\Phi\begin{bmatrix} -q^2, & q^2; & \\ & & q \\ -q^{2N+5} & & \end{bmatrix}, \tag{3.15}$$

(iv) $$\chi_N(q)=(1+\omega)-\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\Phi\begin{bmatrix} -\omega, & -\omega^2 q; & \\ & & q \\ 0 & & \end{bmatrix}_N, \tag{3.16}$$

$$=(1+\omega)-\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\frac{(-\omega q)_N(-\omega^2 q)_{N+1}}{(q)_N}$$

$$\times\Phi\begin{bmatrix} -\omega q^{N+1}, & q; & \\ & & -\omega^2 q \\ -\omega q & & \end{bmatrix}, \tag{3.17}$$

(v) $$\omega_N(q)=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2}\Phi^{(q^2)}\begin{bmatrix} q, & q; & \\ & & q^2 \\ 0 & & \end{bmatrix}_N, \tag{3.18}$$

$$=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2}\frac{(q;q^2)_{N+1}(q^3;q^2)_N}{(q^2;q^2)_N}\Phi^{(q^2)}\begin{bmatrix} q^{3+2N}, & q^2; & \\ & & q \\ q^3 & & \end{bmatrix}, \tag{3.19}$$

(vi) $$\gamma_N(q)=\frac{q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix} -q, & 0; & \\ & & q^2 \\ -q^2 & & \end{bmatrix}_N, \tag{3.20}$$

$$=\frac{(q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\frac{(-q;q^2)_{N+1}}{(q^2;q^2)_N(-q^2;q^2)_{N+1}}\Phi^{(q^2)}\begin{bmatrix} q^2, & q; & \\ & & -q \\ -q^{4+2N} & & \end{bmatrix}, \tag{3.21}$$

and

(vii) $$\rho_N(q)=\frac{(q^2;q^2)_\infty}{(q\omega,q\omega^2;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix} \omega q, & \omega^2 q; & \\ & & q^2 \\ 0 & & \end{bmatrix}_N, \tag{3.22}$$

$$=\frac{(q^2;q^2)_\infty}{(q\omega;q\omega^2;q^2)_\infty}\frac{(q\omega^2;q^2)_{N+1}(q^3\omega;q^2)_N}{(q^2;q^2)_N}\Phi\begin{bmatrix} \omega q^{3+2N}, & q^2; & \\ & & \omega^2 q \\ \omega q^3 & & \end{bmatrix}. \tag{3.23}$$

4. A summation formula and a lemma

A transformation formula of Sears [6, p.167] is

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, a, b, c; \\ d, e, f \end{matrix} q\right] = \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}\left(\frac{bc}{d}\right)^n {}_4\Phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c; \\ d, \frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f} \end{matrix}\right] \tag{4.1} $$

Putting $d = q^{-n}$ and $c = q$ in (4.1), we get for $ef = abq^2$ and using q-Saalschütz's theorem

$$ {}_3\Phi_2\left[\begin{matrix} a, b, q; \\ e, f \end{matrix} q\right]_n = \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}(bq^{1+n})^n $$

$$ {}_3\Phi_2\left[\begin{matrix} a, & q^{-n}/b, & q^{-n-1} \\ aq^{1-n}/e, & aq^{1-n}/f & \end{matrix}; q\right]_n $$

$$ = \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}(bq^{1+n})^n $$

$$ \left\{ {}_3\Phi_2\left[\begin{matrix} a, & q^{-n}/b, & q^{-n-1} \\ aq^{1-n}/e, & aq^{1-n}/f & \end{matrix} q\right]_{(n+1)} \right. $$

$$ \left. - \frac{(a)_{n+1}\left(\frac{q^{-n}}{b}\right)_{n+1}(q^{-n-1})_{n+1}q^{n+1}}{\left(\frac{aq^{1-n}}{e}\right)_{n+1}\left(\frac{aq^{1-n}}{f}\right)_{n+1}(q)_{n+1}} \right\} $$

$$ = \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}(bq^{1+n})^n \left\{ \frac{\left(\frac{q^{1-n}}{e}\right)_{n+1}\left(\frac{abq}{e}\right)_{n+1}}{\left(\frac{aq^{1-n}}{e}\right)_{n+1}\left(\frac{bq}{e}\right)_{n+1}} \right. $$

$$ \left. - \frac{(a)_{n+1}\left(\frac{q^{-n}}{b}\right)_{n+1}(q^{-n-1})_{n+1}q^{n+1}}{\left(\frac{aq^{1-n}}{e}\right)_{n+1}\left(\frac{aq^{1-n}}{f}\right)_{n+1}(q)_{n+1}} \right\} $$

$$ = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1 - \frac{(a,b)_{n+1}}{\left(\frac{e}{q}, \frac{abq}{e}\right)_{n+1}}\right]. \tag{4.2} $$

Equation (4.2) gives the required summation formula.

Next we prove the following Lemma:

Lemma. We know that

$$\sum_{m=0}^{p}\delta_m\sum_{r=0}^{m}\alpha_r=\sum_{r=0}^{p}\alpha_r\sum_{m=r}^{p}\delta_m=\sum_{r=0}^{p}\alpha_r\sum_{m=0}^{p}\delta_m-\sum_{r=0}^{p}\alpha_r\sum_{m=0}^{r-1}\delta_m. \tag{4.3}$$

COROLLARY

Taking $p\to\infty$ in the Lemma, subject to convergence, we get

$$\sum_{m=0}^{\infty}\delta_m\sum_{r=0}^{m}\alpha_r=\left(\sum_{r=0}^{\infty}\alpha_r\right)\left(\sum_{m=0}^{\infty}\delta_m\right)-\sum_{r=1}^{\infty}\alpha_r\sum_{m=0}^{r-1}\delta_m. \tag{4.4}$$

Choosing α_r such that $\sum_{r=0}^{m}\alpha_r$ is some partial mock theta function we choose δ_m such that $\sum_{m=0}^{r-1}\delta_m$ is summable to β_m (say) and $\sum_{r=1}^{\infty}\alpha_r\beta_r$ is some other mock theta function. Such a choice of δ_r and α_r gives us a relation between a given partial mock theta function, which is a complete form and possibly another mock theta function.

5. A general theorem

In this section we give two general theorems from which we shall derive a partial mock theta function identities.

Theorem. *Choosing*

$$\delta_m=\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\frac{(a,b)_m q^m}{\left(e,\dfrac{abq^2}{e}\right)_m}$$

in (4.4) *and evaluating β_r by* (4.2) *gives the following theorem:*

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\dfrac{abq^2}{e}\right)_m}\sum_{r=0}^{m}\alpha_r$$

$$=\left[1-\frac{(a,b)_\infty}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_\infty}\right]\sum_{m=0}^{\infty}\alpha_m-\sum_{m=1}^{\infty}\alpha_m\left[1-\frac{(a,b)_m}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_m}\right]$$

$$=-\frac{(a,b)_\infty}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_\infty}\sum_{m=0}^{\infty}\alpha_m+\sum_{m=0}^{\infty}\frac{(a,b)_m}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_m}\alpha_m. \tag{5.1}$$

6. Relations between partial mock theta function of order three

We use Theorem 1 (5·1) for particular values of α_r to get a number of partial mock theta function identities

(i) For $\alpha_r=(-1,-q)_r q^r/(q)_r$ in Theorem 1, we get

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\frac{abq^2}{e}\right)_m}\bar{f}_m(q)=-\frac{(a,b)_\infty}{\left(\frac{e}{q},\frac{abq}{e}\right)_\infty}\bar{f}(q)$$

$$+\frac{(q)_\infty}{(-q)_\infty^2}\sum_{m=0}^{\infty}\frac{(a,b,-1,-q)_m q^m}{\left(\frac{e}{q},\frac{abq}{e},q\right)_m}, \tag{6.1}$$

where $2-f_m(q)=\bar{f}_m(q), 2-f(q)=\bar{f}(q)$.

(ii) For $\alpha_r=(-i,iq)_r q^r/(q)_r$ in (5.1), we get

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\frac{abq^2}{e}\right)_m}\overline{\phi_m(q)}=-\frac{(a,b)_\infty}{\left(\frac{e}{q},\frac{abq}{e}\right)_\infty}\overline{\phi(q)}$$

$$+\frac{i(q)_\infty}{(iq,-iq)_\infty}\sum_{m=0}^{\infty}\frac{(a,b,-i,iq)_m q^m}{\left(\frac{e}{q},\frac{abq}{e},q\right)_m}, \tag{6.2}$$

where $\overline{\phi_m(q)}=(1+i)-\phi_m(q)$ and $\overline{\phi(q)}=(1+i)-\phi(q)$.

(iii) Letting $q\to q^2$ and then taking $\alpha_r=(q;q^2)_r q^{2r}/(q^2,-q^3;q^2)_r$ in (5.1), we get

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{\varepsilon};q^2\right)_m}\Psi_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\Psi(q)$$

$$+\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(a,b;q;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},q^2,-q^3;q^2\right)_m}. \tag{6.3}$$

(iv) Letting $q\to q^2$ and then taking $\alpha_r=(-q;q^2)_r q^{2r}/(-q^2,q^2;q^2)_r$ in (5.1), we get

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{e};q^2\right)_m}\gamma_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\gamma(q)$$

$$+\frac{(q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(a,b;-q;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},-q^2,q^2;q^2\right)_m}. \tag{6.4}$$

(v) Letting $q\to q^2$ and then taking $\alpha_r=(q;q^2)_r^2 q^{2r}/(q^2;q^2)_r$ in (5.1), we have

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{e};q^2\right)_m}\omega_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\omega(q)$$

$$+\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2}\sum_{m=0}^{\infty}\frac{(a,b,q,q;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},q^2;q^2\right)_m}. \tag{6.5}$$

(vi) Letting $q\to q^2$ and then taking $\alpha_r=(q\omega,q\omega^2;q^2)_r q^{2r}/(q^2;q^2)_r$ in (5.1), we get

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{e};q^2\right)_m}\rho_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\rho(q)$$

$$+\frac{(q^2;q^2)_\infty}{(q\omega,q\omega^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(a,b,q\omega,q\omega^2;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},q^2;q^2\right)_m}. \tag{6.6}$$

(vii) Letting $\alpha_r=(-\omega,-\omega^2 q)_r\, q^r/(q)_r$ in (5.1), we get

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\frac{abq^2}{e}\right)_m}\bar{\chi}_m(q)=-\frac{(a,b)_\infty}{\left(\frac{e}{q},\frac{abq}{e}\right)_\infty}\bar{\chi}(q)$$

$$+\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\sum_{m=0}^{\infty}\frac{(a,b,-\omega,-\omega^2 q)_m q^m}{\left(\frac{e}{q},\frac{abq}{e},q\right)_m} \tag{6.7}$$

where $(1+\omega)-\chi_m(q)=\bar{\chi}_m(q)$, $(1+\omega)-\chi(q)=\bar{\chi}(q)$.

7. Special cases

In this section we mention some special cases of our results in § 6.

(i) Putting $e=-q$, $a=b=0$ in (6.1), we get

$$(-q)_\infty\sum_{m=0}^{\infty}\frac{q^m\overline{f_m(q)}}{(-q)_m}+\overline{f(q)}=\frac{2}{1+q}.$$

(ii) Putting $e=-q$, $a=q/b$ and then $b=-i$ in (6.1), we get

$$\frac{(1+i)(1+qi)}{2i(1+q)}\sum_{m=0}^{\infty}\frac{(-i,qi)_m q^m}{(-q,-q^2)_m}\overline{f_m(q)}+\frac{(-i,qi)_\infty}{(-1,-q)_\infty}\overline{f(q)}=\frac{(iq,-iq)_\infty}{i(-q)_\infty^2}\overline{\phi(q)}.$$

(iii) Putting $e=-iq$, $a=b=0$ in (6.2), we get

$$\frac{i}{1+i}\sum_{m=0}^{\infty}\frac{q^m}{(-iq)_m}\overline{\phi_m(q)}+\frac{1}{(-i)_\infty}\overline{\phi(q)}=\frac{i}{(1-iq)(-iq)_\infty}.$$

(iv) Putting $e=-iq$, $a=q/b$, and then $b=-1$ in (6.2), we get

$$\frac{(q-i)(1-i)}{(1+i)(q+i)}\sum_{m=0}^{\infty}\frac{(-1,-q)_m q^m}{(-iq,iq^2)_m}\overline{\phi_m(q)}+\frac{(-1,-q)_\infty}{(-i,iq)_\infty}\overline{\phi(q)}=\frac{i(-q)_\infty^2}{(iq,-iq)_\infty}\overline{f(q)}.$$

(v) Putting $a=0$, $b=-q^3$ and then $e=0$ in (6.3), we get

$$-q^3\sum_{m=0}^{\infty}(-q^3;q^2)_m q^{2m}\Psi_m(q)+(-q^3;q^2)_\infty\,\Psi(q)=\frac{q(-q^3;q^2)_\infty}{(1-q)}.$$

(vi) Putting $a=-q^3$, $b=0$ and then $e=-q^4$ in (6.3), we get

$$\frac{q(1-q)}{1+q^2}\sum_{m=0}^{\infty}\frac{(-q^3;q^2)_m q^{2m}}{(-q^4;q^2)_m}\Psi_m(q)+\frac{(-q^3;q^2)_\infty}{(-q^2;q^2)_\infty}\Psi(q)=\frac{q(-q^3;q^2)_\infty}{(-q^2;q^2)_\infty}\gamma(-q).$$

(vii) Putting $a=-q^2$, $b=0$ and then $e=0$ in (6.4), we get

$$-q^2\sum_{m=0}^{\infty}(-q^2;q^2)_m q^{2m}\gamma_m(q)+(-q^2;q^2)_\infty\ \gamma(q)=\frac{(-q^2;q^2)_\infty}{(1+q)}.$$

(viii) Putting $e=q^5$, $a=-q^2$, $b=0$ in (6.4), we get

$$-\frac{(1+q)q^2}{(1-q^3)}\sum_{m=0}^{\infty}\frac{(-q^2;q^2)_m q^{2m}}{(q^5;q^2)_m}\gamma_m(q)+\frac{(-q^2;q^2)_\infty}{(q^3;q^2)_\infty}\gamma(q)=-\frac{(-q^2;q^2)_\infty}{q(q^3;q^2)_\infty}\Psi(-q).$$

(ix) Putting $e=q^3$, $a=b=0$ in (6.5), we get

$$-\frac{q}{(1-q)}\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^3;q^2)_m}\omega_m(q)+\frac{1}{(q;q^2)_\infty}\omega(q)=\frac{1}{(1-q)(q;q^2)_\infty}.$$

(x) Putting $e=q^3$, $a=q^2/b$ and then $b=-1$ in (6.5), we get

$$-\frac{(1+q)^2}{(1-q)^2}\sum_{m=0}^{\infty}\frac{(-1,-q^2;q^2)_m q^{2m}}{(q^3;q^2)_m^2}\omega_m(q)+\frac{(-1,-q^2;q^2)_\infty}{(q;q^2)_\infty^2}\omega(q)$$
$$=\frac{(-q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\overline{f(q^2)}.$$

(xi) Putting $a=b=0$, $e=q^3\omega^2$ in (6.6), we get

$$-\frac{q\omega^2}{(1-q\omega^2)}\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^3\omega^2;q^2)_m}\rho_m(q)+\frac{1}{(q\omega^2;q^2)_\infty}\rho(q)=\frac{1}{(1-q\omega)(q\omega^2;q^2)_\infty}.$$

(xii) Putting $e=q^3\omega$, $a=q^2/b$ and then $b=-1$, in (6.6), we get

$$\frac{(q+\omega)(1+q\omega)}{(1-q\omega)(q-\omega)}\sum_{m=0}^{\infty}\frac{(-1,-q^2;q^2)_m q^{2m}}{(q^3\omega,\,q^3\omega^2;q^2)_m}\rho_m(q)$$
$$+\frac{(-1,-q^2;q^2)_\infty}{(q\omega,q\omega^2;q^2)_\infty}\rho(q)=\frac{(-q^2;q^2)_\infty^2}{(q\omega,\,q\omega^2;q^2)_\infty}\overline{f(q^2)}.$$

(xiii) Putting $a=b=0$, $e=-\omega q$ in (6.7), we get

$$\frac{\omega}{(1+\omega)}\sum_{m=0}^{\infty}\frac{q^m}{(-\omega q)_m}\overline{\chi_m}(q)+\frac{1}{(-\omega)_\infty}\overline{\chi(q)}=\frac{\omega}{(1+\omega^2 q)(-\omega q)_\infty}.$$

(xiv) Change q to q^2, then $a=\omega q$, $b=\omega^2 q$, $e=-\omega q^2$ in (6.7) and we get

$$\frac{\omega^2(1+q)(1+\omega q)}{(1+\omega)(\omega+q^2)}\sum_{m=0}^{\infty}\frac{(\omega q,\omega^2 q;q^2)_m q^{2m}}{(-\omega q^2,-q^4\omega^2;q^2)_m}\bar{\chi}_m(q^2)$$
$$+\frac{(\omega q,\omega^2 q;q^2)_\infty}{(-\omega,-\omega^2 q^2;q^2)_\infty}\bar{\chi}(q^2)+\frac{\omega(q\omega,q\omega^2;q^2)_\infty}{(-\omega q^2,-q^2\omega^2)_\infty}\rho(q).$$

8. The continued fractions, Padé approximants and their convergents

Agarwal [2] represented the continued fractions, Padé approximants and their convergents for ${}_2\Phi_1\left[\begin{matrix}\alpha,\ q;\\ \gamma\end{matrix}\ x\right]$ as follows:

$$ {}_2\Phi_1\left[\begin{matrix}\alpha,\ q;\\ \\ \gamma\end{matrix}\ x\right]=\frac{1}{1+q\gamma_1}+\frac{1}{q\gamma_2}+\frac{1}{q\gamma_3}+\frac{1}{\text{---}}, \tag{8.1} $$

where

$$ \gamma_{2n+1}=\frac{(1-\alpha q^n)(1-\gamma q^{n-1})q^n}{(1-\gamma q^{2n-1})(1-\gamma q^{2n})} $$

and

$$ \gamma_{2n+2}=\frac{(\alpha-\gamma q^n)(1-q^{n+1})q^n}{(1-\gamma q^{2n})(1-\gamma q^{2n+1})}. $$

The Padé approximants (n,n) and $(n,n+1)$ are given by

$$ R_{n,n}=\frac{P_{n,n}}{Q_{n,n}}, \tag{8.2} $$

where

$$ P_{n,n}=\sum_{k=0}^{n}\frac{(\alpha)_k x^k}{(\gamma)_k}\sum_{s=0}^{n-k}\frac{(q^{-n})_s(q^{-n}/\alpha)_s}{(q)_s(q^{1-2n}/\gamma)_s}\left(\frac{x\alpha q}{\gamma}\right)^s $$

and

$$ Q_{n,n}={}_2\Phi_1\left[q^{-n},q^{-n}/\alpha;\ q^{1-2n}/\gamma;\frac{x\alpha q}{\gamma}\right], $$

and

$$ R_{n,n+1}=\frac{P_{n,n+1}}{Q_{n,n+1}}, \tag{8.3} $$

where

$$ P_{n,n+1}=\frac{(-x)^h q^{3n(1-n)/2}(q^2;q)_n(q\alpha;q)_n(q\gamma;q)_n}{(q;q)_n(q\gamma;q)_{2n}} $$

$$ \times\sum_{k=0}^{n}\frac{(q^{-n})_k(\alpha)_k(\gamma q^{n+1})_k q^k}{(\gamma)_k(\alpha q)_k(q^2)_k}\,{}_3\Phi_2\left[\begin{matrix}q^{-n+k},\ \gamma q^{n+k+1},\ q;\\ \\ q^{2+k},\quad \alpha q^{1+k}\end{matrix}\ \frac{q}{x}\right] $$

and

$$ Q_{n,\,n+1}={}_2\Phi_1\left[\begin{matrix}q^{-n-1},\ q^{-n}/\alpha;\\ \\ q^{-2n}/\gamma\end{matrix}\ \frac{x\alpha q}{\gamma}\right]. $$

From (7.1), (7.2) and (7.3) the continued fraction, Padé approximants and their covergents for the third order partial mock theta function $\Phi\left[\begin{matrix}-q^{1+m},\ q;\\ \\ -q\end{matrix}\ -q\right]$ are as follows:

$$ \frac{(-q)_\infty^2(q)_m}{(q)_\infty(-q)_{m+1}(-q)_m}\{2-f_m(q)\}=\frac{1}{1+q\gamma_1}+\frac{1}{q\gamma_2}+\frac{1}{q\gamma_3}+\frac{1}{\text{---}}, $$

where

$$\gamma_{2n+1}=\frac{(1+q^{m+n+1})(1+q^n)q^n}{(1+q^{2n})(1+q^{2n+1})}$$

and

$$\gamma_{2n+2}=\frac{(-q^{1+m}+q^{1+n})(1-q^{n+1})q^n}{(1+q^{2n+1})(1+q^{2n+2})},$$

and if

$$R_{n,n}=\frac{P_{n,n}}{Q_{n,n}}$$

then

$$P_{n,n}=\sum_{k=0}^{n}\frac{(-q^{1+m})_k q^k}{(-q)_k}\sum_{s=0}^{n-k}\frac{(q^{-n})_s(-q^{-n-m-1})_s(-q^{2+m})^s}{(q)_s(-q^{-2n})_s}$$

and

$$Q_{n,n}={}_2\Phi_1\left[\begin{matrix}q^{-n}, & -q^{-n-m-1}; \\ -q^{-2n} & \end{matrix}\;\middle|\; -q^{2+m}\right],$$

and

$$R_{n,n+1}=\frac{P_{n,n+1}}{Q_{n,n+1}},$$

where

$$P_{n,n+1}=\frac{q^n q^{3n(1-n)/2}(q^2)_n(-q^{2+m})_n(-q^2)_n}{(q)_n(-q^2)_{2n}}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n})_k(-q^{1+m})_k(-q^{2+n})_k q^k}{(-q)_k(-q^{2+m})_k(q^2)_k}\;{}_3\Phi_2\left[\begin{matrix}q^{-n+k}, & -q^{n+k+2}, & q; \\ q^{2+k}, & -q^{m+k+2} & \end{matrix}\;\middle|\; -1\right]$$

and

$$Q_{n,n+1}={}_2\Phi_1\left[\begin{matrix}q^{-n-1}, & -q^{-n-m-1}; \\ -q^{-2n-1} & \end{matrix}\;\middle|\; -q^{m+2}\right].$$

Similarly, one can get continued fractions, Padé approximants and their convergents for the other six partial third order mock theta functions $\phi_N(q)$, $\Psi_N(q)$, $\chi_N(q)$, $\omega_N(q)$, $\gamma_N(q)$ and $\rho_N(q)$. The Padé approximants are useful in finding the bounds for the partial mock theta functions by taking simple values of n.

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On degrees of maps between Grassmannians

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Abstract. Let $\tilde{G}_{n,k}$ denote the oriented grassmann manifold of oriented k-planes in $\mathbb{R}^n$. It is shown that for any continuous map $f: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$, $\dim \tilde{G}_{n,k} = \dim \tilde{G}_{m,l} = l(m-l)$, the Brouwer's degree is zero, provided $l > 1$, $n \neq m$. Similar results for continuous maps $g: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $h: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$, $1 \leqslant l < k \leqslant n/2$, $k(n-k) = l(m-l)$ are also obtained.

Keywords. Grassmann manifolds; Brouwer degree; characteristic classes.

1. Introduction

Let $\tilde{G}_{n,k}$ denote the oriented grassmann manifold of *oriented* k-dimensional vector subspaces of $\mathbb{R}^n$. For $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, let $\mathbb{F}G_{n,k}$ denote the $\mathbb{F}$-grassmannian of k-dimensional (left) $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$. In their work on self maps of homogeneous varieties, Paranjape and Srinivas [7] prove, among other things, that if $f: \mathbb{C}G_{n,k} \to \mathbb{C}G_{m,l}$, $l \geqslant 2$, $\dim \mathbb{C}G_{m,l} = l(m-l) = k(n-k) = \dim \mathbb{C}G_{n,k}$ is a finite morphism of projective varieties, then $(n, k) = (m, l)$ and f is an *isomorphism*. It is obvious that if $f: \mathbb{C}G_{n,k} \to \mathbb{C}G_{m,l}$ is any morphism of projective varieties, then f is complex analytic when the varieties involved are regarded as complex analytic manifolds. It follows that f is orientation preserving and that its Brouwer degree can be calculated as $\#f^{-1}(x)$ for most points $x \in \mathbb{C}G_{m,l}$. In particular its (Brouwer) degree must be positive. It is a well-known fact that if $f: N \to M$ is any continuous map of non-zero degree between compact, connected, oriented manifolds, then the induced map in the rational cohomology $f^*: H^*(M; \mathbb{Q}) \to H^*(N; \mathbb{Q})$ is a monomorphism. Using this observation we are able to show the following.

Theorem 1. *Let $h: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$, be any map between oriented grassmann manifolds of the same dimension, where $(n, k) \neq (m, l)$, $2 \leqslant l \leqslant m/2$, $1 \leqslant k \leqslant n/2$. Then* $\deg(h) = 0$.

Theorem 2. *Let $f: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $g: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$, $1 \leqslant l < k \leqslant [n/2]$ be any map between the complex (respectively quaternionic) grassmannians of the same dimension. Then* $\deg(f) = 0 = \deg(g)$.

In the case, when $h: \tilde{G}_{n,k} \to \tilde{G}_{m,1} = S^d$, $k(n-k) = d = m-1$, is a continuous map, one knows by Hopf–Whitney theorem (Theorem 5, ch.1, [6]), that there are maps of every degree $m \in \mathbb{Z}$, from $\tilde{G}_{n,k}$ to S^d and that any two maps of the same degree are homotopic.

The restriction $1 \leqslant l < k \leqslant [n/2]$ is needed in our proof in the case of complex and quaternionic grassmannians. However it seems plausible that the theorem will continue to be true, even for $2 \leqslant k < l \leqslant [m/2]$. We were able to verify this only for small values of l, k, m and n.

Note that Theorem 2 implies part of the result of Paranjape and Srinivas quoted earlier, namely, if $f: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $2 \leqslant l \leqslant k \leqslant n/2$, is a finite surjective morphism, then we deduce from Theorem 2 that $(n, k) = (m, l)$. On the other hand, our theorem applies to any *continuous* map f. It is not true in general that any given continuous map can be homotoped to a complex analytic map. Hence, Theorem 2 does not follow from the work of Paranjape and Srinivas [7].

Our proofs are in fact quite elementary, and for the most part follow from a purely algebraic lemma (see Lemma 4 (v)). The cases when $f: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$, $g: \mathbb{H}G_{n,k} \to \mathbb{H}P^d$, $k(n-k) = d$, are any continuous maps between the complex (respectively quaternionic) grassmannians will be discussed at the end of §2. We give applications to K-theory in some cases in §3.

For the sake of completeness, we give a proof of the following well-known lemma, quoted earlier.

Lemma 3. Let $f: N \to M$ be any continuous map between two compact connected oriented manifolds of the same dimension d such that $\deg(f) \neq 0$. *Then $f^*: H^*(M; \mathbb{Q}) \to H^*(N; \mathbb{Q})$ is a monomorphism.*

Proof. Let $[M]$ denote the orientation class in $H_d(M; \mathbb{Z}) \subsetneq H_d(M; \mathbb{Q}) \cong \mathbb{Q}$. One has a non-degenerate pairing

$$\begin{aligned} H^p(M; \mathbb{Q}) \times H^{d-p}(M; \mathbb{Q}) &\to \mathbb{Q} \\ (\alpha, \beta) &\mapsto \langle \alpha \cup \beta, [M] \rangle \\ &= \langle \alpha, \beta \cap [M] \rangle. \end{aligned}$$

Let $\deg(f) = \lambda \neq 0$, $\lambda \in \mathbb{Z}$. Thus $f_*([N]) = \lambda[M]$. Now if $0 \neq \alpha \in H^p(M; \mathbb{Q})$, choose $\beta \in H^{d-p}(M; \mathbb{Q})$ such that $\langle \alpha, \beta \rangle = 1$. Then $\langle f^*(\alpha) \cup f^*(\beta), [N] \rangle = \langle \alpha, \cup \beta, f_*[N] \rangle = \langle \alpha \cup \beta, \lambda[M] \rangle = \lambda \neq 0$. Therefore f^* is a monomorphism. □

2. Proofs of main results

Let $1 \leqslant k \leqslant n/2$, $n, k \in \mathbb{Z}$. Let $H_{n,k}$ denote the graded $\mathbb{Q}$-algebra with generators $x_1, x_2, \ldots, x_k$, $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-k}$; $\deg(x_i) = i = \deg(\bar{x}_i)$ with relations given by the inhomogeneous relation

$$(1 + x_1 + x_2 + \cdots + x_k)(1 + \bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_{n-k}) = 1 \tag{1}$$

(i.e. the relations are all generated by the basic relations $\Sigma_{0 \leqslant i \leqslant r} x_i \bar{x}_{r-i} = 0$, $1 \leqslant r \leqslant n$). Note that one can regard the basic relations $\Sigma_{i+j=r} x_i \bar{x}_j = 0$ as defining the $\bar{x}_r$ as a polynomial in x_i, inductively for $1 \leqslant r \leqslant n-k$. Therefore $H_{n,k} = \mathbb{Q}[x_1, x_2, \ldots, x_k]/\sim$. $H_{n,k}$ is readily recognized as being isomorphic to the cohomology algebra $H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$ by an isomorphism that doubles the gradation, sending x_i to c_i, the ith Chern class of the canonical k-plane bundle over $\mathbb{C}G_{n,k}$. We denote the vector space of homogeneous elements of $H_{n,k}$ of degree r by $H^r_{n,k}$.

Lemma 4. With the above notation

(i) $H^r_{n,k} = 0$ *if* $r > k(n-k)$,

(ii) $x_1^{k(n-k)}$ *is a generator of* $H^{k(n-k)}_{n,k}$, *which is a 1-dimensional $\mathbb{Q}$-vector space,*

(iii) $\dim_{\mathbb{Q}} H_{n,k} = \binom{n}{k}$,

(iv) *there are no algebraic relations among* $x_1, x_2, \ldots, x_k$ *up to degree* $n-k$.

(v) *If* $1 \leqslant l < k \leqslant n/2$, *and* $m \in \mathbb{Z}$ *is such that* $k(n-k) \leqslant l(m-l)$, *then for any homomorphism* $\phi : H_{n,k} \to H_{m,l}$ *of graded* $\mathbb{Q}$*-algebras,* $\ker(\phi) \neq 0$.

Proof. Parts (i), (ii), (iii) and (iv) follow from the isomorphism of graded algebras $H_{n,k} \cong H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$, and well-known facts about the cohomology of $\mathbb{C}G_{n,k}$.

Proof of (v): Let $z_1, z_2, \ldots, z_l$ denote the defining algebra generator of $H_{m,l}$, $\deg(z_j) = j$, $1 \leqslant j \leqslant l$. Let, if possible, $\phi : H_{n,k} \to H_{m,l}$ be a monomorphism of $\mathbb{Q}$-algebras. Since $k > l$, there exists an $i \leqslant l+1$ such that $\phi(x_i)$ is in the subalgebra generated by $z_1, z_2, \ldots, z_{i-1}$. We may assume that $1 \leqslant i$ is the smallest such integer. If $i = 1$, then $\phi(x_1) = 0$. Therefore $i \geqslant 2$ and

$$\phi(x_1) = \lambda_1 z_1, \quad \lambda_1 \neq 0, \tag{2}$$

$$\phi(x_j) = \lambda_j z_j + P_j(z_1, z_2, \ldots, z_{j-1}), \quad \lambda_j \neq 0 \tag{3}$$

and

$$\phi(x_i) = P_i(z_1, z_2, \ldots, z_{i-1}), \tag{4}$$

for suitable polynomials P_j, $1 \leqslant j \leqslant i$. In view of (2) and (3), one can express the z_j as a polynomial in $\phi(x_1), \phi(x_2), \ldots, \phi(x_j)$ for $1 \leqslant j < i$. Thus

$$\mathbb{Q}[z_1, z_2, \ldots, z_{i-1}] = \mathbb{Q}[\phi(x_1), \phi(x_2), \ldots, \phi(x_{i-1})]. \tag{5}$$

In particular, for a suitable polynomial Q, one has

$$P_i(z_1, z_2, \ldots, z_{i-1}) = Q(\phi(x_1), \phi(x_2), \ldots, \phi(x_{i-1})) \tag{6}$$

and hence

$$\phi(x_i) = Q(\phi(x_1), \phi(x_2), \ldots, \phi(x_{i-1})) = \phi(Q(x_1, x_2, \ldots, x_{i-1})). \tag{7}$$

Therefore, $x_i - Q(x_1, x_2, \ldots, x_{i-1}) \in \ker(\phi) = 0$. But this contradicts (iv). Therefore, we must have $\ker(\phi) \neq 0$. □

Let $2 \leqslant k \leqslant n/2$. We recall the structure of the cohomology algebra $H^*(\tilde{G}_{n,k}; \mathbb{Q})$. Let $\gamma_{n,k}$ (resectively $\beta_{n,k}$) denote the canonical real k-plane (respectively $(n-k)$-plane) bundle over $\tilde{G}_{n,k}$. Denote by p_i the Pontrjagin class $p_i(\gamma_{n,k}) \in H^{4i}(\tilde{G}_{n,k}; \mathbb{Q})$, $1 \leqslant i \leqslant [k/2]$ and denote by $\bar{p}_j$ the class $p_j(\beta_{n,k})$, $1 \leqslant j \leqslant [(n-k)/2]$. Note that since $\gamma_{n,k} \oplus \beta_{n,k} \approx n\varepsilon$, the trivial bundle of rank n, one has

$$p(\gamma_{n,k}) \cdot p(\beta_{n,k}) = 1, \tag{8}$$

where p denotes the total Pontrjagin class. The subalgebra generated by $p_1, p_2, p_3, \ldots, p_s, \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_t$ is isomorphic to the graded algebra $H_{s+t,s}$ where $k = 2s + \varepsilon$, $n - k = 2t + \eta, \varepsilon, \eta \in \{0, 1\}$ under the isomorphism $p_i \mapsto x_i$; $1 \leqslant i \leqslant s$. Note that since $\gamma_{n,k}$ and $\beta_{n,k}$ are oriented in a natural way, one has the rational Euler classes $e_k := e(\gamma_{n,k}) \in H^k(\tilde{G}_{n,k}; \mathbb{Q})$, $e_{n-k} := e(\beta_{n,k}) \in H^{n-k}(\tilde{G}_{n,k}; \mathbb{Q})$. (Our notation is ambiguous when $2k = n$, but as it is unlikely to cause confusion, we retain this notation throughout.) Since the (integral) Euler class of an oriented bundle of odd rank is of order two, e_k (respectively e_{n-k}) is zero if k (respectively $n-k$) is odd. Again using the relation $\gamma_{n,k} \oplus \beta_{n,k} \approx n\varepsilon$, we get $e(\gamma_{n,k}) . e(\beta_{n,k}) = 0$. That is, $e_k . e_{n-k} = 0$. Furthermore, $e_k^2 = p_s$, (respectively $e_{n-k}^2 = \bar{p}_t$), when $k = 2s$ (respectively $n - k = 2t$).

In case n is even and k is odd there is a cohomology class $\sigma = \sigma_{n,k} \in H^{n-1}(\tilde{G}_{n,k}; \mathbb{Q})$, which transgresses to $p_{[n/2]}(\gamma_{\infty,n}) \in H^{2n}(BSO(n); \mathbb{Q})$ in the fibration

$$\tilde{G}_{n,k} \subsetneq BSO(k) \times BSO(n-k) \to BSO(n)$$

as can be seen using a spectral sequence argument and known facts about the rational cohomology of the classifying spaces $BSO(n)$ [5]. Here $\gamma_{\infty,n}$ denotes the universal oriented n-plane bundle over $BSO(n)$. We are ready to describe the cohomology algebra $H^*(\tilde{G}_{n,k}; \mathbb{Q})$:

PROPOSITION 5

With the above notation

(i) $H^*(\tilde{G}_{2s+2t,2s}; \mathbb{Q}) \cong H_{s+t,s}[e_{2s}, e_{2t}]/\sim$ where $e_{2s}^2 = p_s$, $e_{2t}^2 = \bar{p}_t$, $e_{2s} \cup e_{2t} = 0$.

(ii) $H^*(\tilde{G}_{2s+2t+1,2s}; \mathbb{Q}) \cong H_{s+t,s}[e_{2s}]/\sim$ where $e_{2s}^2 = p_s$.

(iii) $H^*(\tilde{G}_{2s+2t+2,2s+1}; \mathbb{Q}) \cong H_{s+t,s}[\sigma]$, $\sigma^2 = 0$, $\sigma \in H^{2s+2t+1}(\tilde{G}_{2s+2t+2,2s+1}; \mathbb{Q})$.

Here $H_{s+t,s} \cong \mathbb{Q}[p_1, p_2, \ldots, p_s]/\sim$ is as defined earlier.

We omit the proof of this proposition. For (i) and (ii) see Théorème 26.1 [2]. □

Note. In $H^*(\tilde{G}_{2s+2t,2s}; \mathbb{Q})$ one has $e_{2s}\bar{p}_t = e_k \cup e_{n-k} \cup e_{n-k} = 0$, and similarly, $e_{2t}p_s = 0$. However, there are no linear relations over $H_{s+t,s}$ satisfied by e_{2s} in dimensions less than $2s + 4t = 2n - k$. □

Proof of Theorem 1. Let $1 < l < k \leqslant n/2 < m/2$, $k(n-k) = l(m-l) = d$. Let $f: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$, $g: \tilde{G}_{m,l} \to \tilde{G}_{n,k}$ be any continuous map. We must show that $\deg(f) = 0 = \deg(g)$.

In view of Lemma 3, it suffices to show that neither f^* nor g^* is a monomorphism in the rational cohomology.

Write $k = 2s + \varepsilon$, $n - k = 2t + \eta$ with $\varepsilon, \eta \in \{0, 1\}$, $l = 2a + \varepsilon'$, $m - l = 2b + \eta'$, $\varepsilon', \eta' \in \{0, 1\}$. Let q_i, $1 \leqslant i \leqslant a$ denote the ith Pontrjagin class of $\gamma_{m,l}$ and let p_j, $1 \leqslant j \leqslant s$ denote the jth Pontrjagin class of $\gamma_{n,k}$.

First consider the case when $d = \dim \tilde{G}_{n,k}$ is odd. In this case n, m are both even and k, l are both odd. Thus $\varepsilon = \varepsilon' = \eta = \eta' = 1$, $n = 2s + 2t + 2$ and $m = 2a + 2b + 2$. Clearly, $g^*(p_1) = \lambda q_1$ for some $\lambda \in \mathbb{Z}$ and $f^*(q_1) = \mu p_1$ for some $\mu \in \mathbb{Z}$. Recall, for a nilpotent element x in any algebra, the height of x is defined to be the smallest integer n such that $x^n \neq 0$ and $x^{n+1} = 0$. Using Lemma 4(ii) and the above Proposition, note that the height of p_1 is st, whereas the height of q_1 is ab. Since $st = (d - (n-1))/4 > (d - (m-1))/4 = ab$ it follows that $g^*(p_1^{ab+1}) = \lambda^{ab+1} q_1^{ab+1} = 0$ and so g^* is not a monomorphism. On the other hand $f^*(q_1) = 0$ because f^* is a ring homomorphism and $st > ab$.

Now assume $d = \dim \tilde{G}_{n,k}$ is even. Again, one can see easily that the height of p_1 equals st and the height of q_1 equals ab. But st equals ab only when $\varepsilon = \eta = \varepsilon' = \eta' = 0$. Therefore, only the case $n = 2s + 2t$, $k = 2s$, $m = 2a + 2b$, $l = 2a$ remains to be considered. Consider any continuous maps $f: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$ and $g: \tilde{G}_{m,l} \to \tilde{G}_{n,k}$.

First we show that $g^*(e_k) = 0$. Observe that using the relations in $H^*(\tilde{G}_{m,l}; \mathbb{Q})$ one can write

$$g^*(e_k) = e_l P + Q \tag{9}$$

for suitable homogeneous polynomials $P = P(q_1, q_2, \ldots, q_a)$ and $Q = Q(q_1, q_2, \ldots, q_a)$ of degrees $k - l$ and l respectively. Similarly,

$$g^*(e_{n-k}) = e_l P' + Q', \tag{10}$$

where $P'$and Q' are homogeneous polynomials in $q_1, q_2, \ldots, q_a$ of degrees $n-k-l$ and $n-k$ respectively. Since $e_k \cup e_{n-k} = 0$, one has

$$0 = g^*(e_k \cup e_{n-k}) = (e_l P + Q)(e_l P' + Q') \tag{11}$$

$$= q_a PP' + QQ' + e_l(PQ' + P'Q) \tag{12}$$

in $H^n(\tilde{G}_{m,l}; \mathbb{Q})$. Note that as there are no *linear* relations satisfied by the Euler class e_l over $H_{a+b,a}$ up to dimension $2m-l-1$, and since $n < 2m-l$, we obtain

$$q_a PP' + QQ' = 0, \tag{13}$$

$$PQ' + P'Q = 0. \tag{14}$$

Since by Lemma 4 (iv), there are no algebraic relations satisfied by q_j in degrees up to $4b$, the above relations actually hold in the *polynomial* algebra $A = \mathbb{Q}[q_1, q_2, \ldots, q_a]$. Therefore, multiplying (13) by Q and substituting for $P'Q$ from (14), we obtain

$$Q'Q^2 = q_a P^2 Q' \tag{15}$$

in A. This implies $Q^2 = q_a P^2$ in A. Since A is a UFD this is clearly a contradiction, unless $P = Q = 0$. Hence $g^*(e_k) = 0$. Now by Lemma 4 (v), g^* cannot be a monomorphism and so $\deg(g) = 0$.

To show $\deg(f) = 0$ note that since $a < s, f^*(e_{2a})$ can be expressed as a polynomial $P(p_1, p_2, \ldots, p_{[a/2]})$ in $p_1, p_2, \ldots, p_{[a/2]}$. To obtain a contradiction assume that f^* is a monomorphism. Then proceeding as in the proof of Lemma 4 (v) one can express p_i, $1 \leqslant i \leqslant [a/2]$ as a polynomial in $f^*(q_1), f^*(q_2), \ldots, f^*(q_i)$. Hence

$$f^*(e_{2a}) = P(p_1, \ldots, p_{[a/2]}) = P'(f^*q_1), f^*(q_2), \ldots, f^*(q_{[a/2]})) \tag{16}$$

for some suitable polynomial P'. In particular $e_{2a} - P'(q_1, q_2, \ldots, q_{[a/2]})$ is in $\ker(f^*) = 0$. That is, $e_{2a} = P'(q_1, \ldots, q_{[a/2]})$. But this contradicts Proposition 5. Hence $\deg(f)$ has to be zero. This completes the proof of the theorem. □

Proof of Theorem 2. Let $1 \leqslant l < k \leqslant [n/2]$, $k(n-k) = l(m-l)$. Let $f: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$ be any continuous map. Then f induces an algebra homomorphism $f^*: H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \to H^*(\mathbb{C}G_{m,l}; \mathbb{Q})$. As $H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \cong H_{n,k}$, it is immediate from Lemma 4(v) that $\ker(f^*) \neq 0$. Hence $\deg(f) = 0$. Proof for $g: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$ is similar. □

Remark 6. Let $f: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$, $d = k(n-k)$ be any continuous map. Then $f^*(c_1(\gamma_{d+1,1})) = \lambda_f c_1(\gamma_{n,k})$ for some integer λ_f. Using the fact that $\mathbb{C}P^d$ is the $2d+1$-skeleton of $\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$, one sees readily that if $g: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$ is another continous map, then f is homotopic to g if and only if $\lambda_f = \lambda_g$. Moreover, there exists a map $f: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$ with λ_f as any pre-assigned integer. Note that with respect to the orientation obtained from the complex structure on $\mathbb{C}P^d$, the positive generator of $H^{2d}(\mathbb{C}P^d, \mathbb{Z})$ is $(-c_1(\gamma_{d+1,1}))^d$. The degree of f can be determined to be $(\lambda_f)^d \cdot \langle (-c_1(\gamma_{n,k}))^d, [\mathbb{C}G_{n,k}] \rangle$ which equals $(\lambda_f)^d [(1!2!\ldots(k-1)!d!)/((n-k)!\ldots(n-1)!)]$, using (Eg. 14.7.11, [3]).

By an entirely analogous argument one shows that the set $[\mathbb{H}G_{n,k}; \mathbb{H}P^d] \cong [\mathbb{H}G_{n,k}; \mathbb{H}P^\infty] \cong [\mathbb{H}G_{n,k}; BSp(1)]$. But $Sp(1) \cong SU(2) \cong S^3$. So, $[\mathbb{H}G_{n,k}; \mathbb{H}P^d]$ is in bijective correspondence with the set of isomorphism cases of $SU(2)$-bundles over

$\mathbb{H}G_{n,k}$. If $f: \mathbb{H}G_{n,k} \to \mathbb{H}P^d$, then the degree of f is given by the same formula as in the case of complex grassmannians, where λ_f is defined by

$$c_2(f^*(\gamma_{d+1,1})) = \lambda_f c_2(\gamma_{n,k}).$$

We do not know if there exists a continuous map f with λ_f as any pre-assigned integer. □

3. Application to K-theory

Let $f: X \to Y$ be any continuous map between two finite CW complexes. One has a commutative diagram [1]

$$\begin{array}{ccc} \tilde{K}(Y) & \xrightarrow{f^*} & \tilde{K}(X) \\ \downarrow \mathrm{ch}(Y) & & \downarrow \mathrm{ch}(X) \\ \tilde{H}^{ev}(Y;\mathbb{Q}) & \xrightarrow{f^*} & \tilde{H}^{ev}(X;\mathbb{Q}) \end{array}$$

where $\mathrm{ch}(-)$ denotes the Chern character. In case X has cells only in even dimensions then Chern character is well-known to be a monomorphism. In any case $\mathrm{ch}_{\mathbb{Q}}(X) := \mathrm{ch}(X) \otimes \mathbb{Q} : \tilde{K}(X) \otimes \mathbb{Q} \to \tilde{H}^{ev}(X;\mathbb{Q})$ is a monomorphism ([4], p. 238).

Lemma 7. Suppose that $f^*: \tilde{H}^{ev}(Y;\mathbb{Q}) \to \tilde{H}^{ev}(X;\mathbb{Q})$ is zero and that $K(X)$ and $K(Y)$ are free abelian groups, then $f^*: \tilde{K}(Y) \to \tilde{K}(X)$ is zero.

Proof. It suffices to show that $f^*: \tilde{K}(Y) \otimes \mathbb{Q} \to \tilde{K}(X) \otimes \mathbb{Q}$ is zero. This follows from the fact that $\mathrm{ch}_{\mathbb{Q}}(X)$ is a monomorphism and the hypothesis that $f^*: \tilde{H}^{ev}(Y) \to \tilde{H}^{ev}(X)$ is zero. □

Lemma 8. Let $1 \leqslant l < k \leqslant n/2 < m/2$, $k(n-k) \leqslant l(m-l)$. Assume that $n \geqslant k^2/(k-l)$. Then any graded $\mathbb{Q}$-algebra homomorphism $\phi: H_{n,k} \to H_{m,l}$ has image in $\mathbb{Q}$, the elements of degree zero in $H_{m,l}$.

Proof. Let us write the canonical generators of $H_{n,k}$ (respectively $H_{m,l}$) as $x_1, x_2, \ldots, x_k$ (respectively $y_1, y_2, \ldots, y_l$). Write $u_i = \phi(x_i) \in H_{m,l}$, $1 \leqslant i \leqslant k$. We must show that $u_i = 0$ for each i. To obtain a contradiction, assume that $u_i \neq 0$ for some $i \geqslant 1$. Then $\bar{u}_j := \phi(\bar{x}_j) \neq 0$ for some j, $1 \leqslant j \leqslant n-k$. Let p and q be the largest integers so that $u_p \neq 0$, $\bar{u}_q \neq 0$. Applying ϕ to both sides of the relation

$$(1 + x_1 + x_2 + \ldots + x_k)(1 + \bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_{n-k}) = 1 \tag{17}$$

and collecting the terms of degree $p+q$ we get $u_p \cdot \bar{u}_q = 0$ in $H_{m,l}$. Since $p \leqslant k$, $q \leqslant n-k$, we get $p+q \leqslant n$. But, $n \geqslant k^2/(k-l) \Rightarrow n(k-l) \geqslant k^2 \Rightarrow k(n-k) \geqslant nl \Rightarrow l(m-l) \geqslant nl \Rightarrow (m-l) \geqslant n$. Hence $(p+q) \leqslant m-l$. This contradicts Lemma 4 (iv).

□

As in the proof of Theorem 1, we write $k = 2s + \varepsilon$, $n-k = 2t+\eta$, $l = 2a+\varepsilon'$, $m-l = 2b+\varepsilon'$, with $\varepsilon, \eta, \varepsilon', \eta' \in \{0,1\}$.

Theorem 9. (i) *Let $1 \leqslant l < k \leqslant n/2 < m/2$, and let $k(n-k) \leqslant l(m-l)$. Assume that $n \geqslant k^2/(k-l)$. Then for any continuous maps, $g: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $h: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$, one has $g^*: \tilde{K}(\mathbb{C}G_{n,k}) \to \tilde{K}(\mathbb{C}G_{m,l})$ and $h^*: \tilde{K}(\mathbb{H}G_{n,k}) \to K(\mathbb{H}G_{m,l})$ are zero.* (ii) *Let $1 \leqslant l \leqslant m/2$, $2 \leqslant k \leqslant n/2$, and let $1 \leqslant a < s \leqslant t$, $st \leqslant ab$. Assume that $(s+t) \geqslant s^2/(s-a)$. If $l \not\equiv 0 \bmod 4$ and $(m-l) \not\equiv 0 \bmod 4$, then for any continuous map $f: \tilde{G}_{m,l} \to \tilde{G}_{n,k}$, one has $f^*: \tilde{K}(\mathbb{C}G_{n,k}) \to \tilde{K}(\mathbb{C}G_{m,l})$ is zero.*

Proof. By Theorem 3.6 [1] one knows that $K(X)$ is a free abelian when $X = G/K$, where G is any compact connected Lie group and K is a connected subgroup of maximal rank. In particular, this shows that $K(X)$ is a free abelian for $X = \mathbb{C}G_{n,k}$, $\mathbb{H}G_{n,k}$, $\tilde{G}_{p,q}$ when p is odd or q is even. $K^*(\tilde{G}_{p,q})$ has been calculated in [8] for any p and q, and in particular, it follows from Theorem 3.6, [8] that $K(\tilde{G}_{p,q})$ is a free abelian for p is even and q is odd. (i) The above result shows $g^* = 0 = h^*$ by applying Lemmas 7 and 8. (ii) Now let $l \not\equiv 0 \bmod 4$, $(m-l) \not\equiv 0 \bmod 4$. Then by a straightforward dimension argument, $f^* : H^{ev}(\tilde{G}_{n,k}; \mathbb{Q}) \to H^{ev}(\tilde{G}_{m,l}; \mathbb{Q})$ must map the subalgebra $H_{s+t,s} \subset H^{ev}(\tilde{G}_{n,k}; \mathbb{Q})$ into the subalgebra $H_{a+b,a} \subset H^{ev}(\tilde{G}_{m,l}; \mathbb{Q})$. By our hypotheses on s, t, a, b it follows from Lemma 8 that $f^* | H_{s+t,s}$ is zero.

In case $\eta = 0$, so that $n-k = 2t$, one has $f^*(e_{n-k}) \in H_{a+b,a}$ if $n-k \equiv 0 \bmod 4$, and $f^*(e_{n-k}) \in e_l H_{a+b,a} \subset H^{ev}(\tilde{G}_{m,l}; \mathbb{Q})$ if $n-k \equiv 2 \bmod 4$. (Here $e_l = 0$ if l is odd). This is because l is *not* divisible by 4. Suppose $f^*(e_{n-k}) = P \in H_{a+b,a}$. Then

$$0 = f^*(\bar{p}_t) = f^*(e_{n-k}^2) = P^2 . \tag{18}$$

Since $st \leqslant ab$ and $a < s$, one has $t < b$. This implies $2(n-k) < 4b$. Hence, $P^2 = 0$ implies $P = 0$ by Lemma 4(iv) and hence $f^*(e_{2t}) = 0$. Similarly, when $n-k \equiv 2 \bmod 4$, we show that $f^*(e_{n-k}) = 0$. By an analogous argument, when $\varepsilon = 0$, so that $k = 2s$, we show that $f^*(e_{2s}) = 0$. Hence, from Lemma 7, $f^* : \tilde{K}(\tilde{G}_{n,k}) \to \tilde{K}(\tilde{G}_{m,l})$ is zero.

□

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Elementary counterexamples to Kodaira vanishing in prime characteristic

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Abstract. Using methods from the modular representation theory of algebraic groups one can construct [1] a projective homogeneous space for SL_4, in prime characteristic, which violates Kodaira vanishing. In this note we show how elementary algebraic geometry can be used to simplify and generalize this example.

Keywords. Cohomology; line bundles; Kodaira vanishing.

1. Introduction

Let X be a smooth projective variety of dimension m over an algebraically closed field of characteristic zero and let $\mathscr{L}$ be an ample line bundle on X. The Kodaira vanishing theorem [3] states that $H^i(X, \mathscr{L}^{-1}) = 0$ for $i \neq m$. It is well known that the result is false in characteristic $p > 0$; Raynaud constructed [8] a smooth projective surface in positive characteristic with an ample line bundle for which Kodaira vanishing failed (Mumford [7] had earlier constructed a normal non-smooth projective surface counter-example). Raynaud posed two questions:

(1) Are there counter-examples where the vanishing fails for a very ample line bundle?
(2) Are there pairs $(X, \mathscr{L})$ (X smooth projective, $\mathscr{L}$ ample) for which

$$\chi_i(\mathscr{L} \otimes \omega_X) := h^i(X, \mathscr{L} \otimes \omega_X) - h^{i+1}(X, \mathscr{L} \otimes \omega_X) + \cdots$$

is < 0 for some i?

The first author, studying proper homogeneous spaces in characteristic p [4, 5], answered both questions affirmatively using methods from modular representation theory of algebraic groups. The object of this note is to generalize, via elementary algebraic geometry, the simplest counterexample ([1], Example 4) answering question (1) without using Jantzen's sum formula from modular representation theory. The simplest of our examples is as follows. Let Y be the incidence correspondence of points lying on planes in projective three space $\mathbb{P}(V)$. There is a natural bundle $\mathscr{G}$ of rank 2 on Y such that the projectivization $\mathbb{P}(\mathscr{G})$ of $\mathscr{G}$ is the variety of flags in V. Now, let $X = \mathbb{P}(F^*\mathscr{G})$ be the projectivization of $F^*\mathscr{G}$ (the Frobenius pull-back of $\mathscr{G}$). Then X can be embedded in $\mathbb{P}(V) \times \mathbb{P}(\wedge^2 V^\vee) \times \mathbb{P}(V^\vee)$, and the line bundle $\mathscr{L} = \mathscr{O}(1, 3, 1)$ is very ample on X and $H^5(X, \mathscr{L}^{-1}) \neq 0$.

It would be interesting to see if such elementary calculations also yield examples to Raynaud's second question. The example known so far [5] uses computer intensive calculations. Also the examples of this note are of dimension six or more (and with Picard group of rank three), whereas the example of Raynaud (and Mumford's normal variety, earlier) is a surface. While the examples here are more elementary than Raynaud's example, it would be nice to find other examples of smaller dimension (or with Picard group of rank two). Since it is only the penultimate $H^i(X, \mathscr{L}^{-1})$ which is non-zero, this failure of Kodaira vanishing is not necessarily inherited by hyperplane sections.

2. Preliminaries

2.1. Let V be an $n+1$-dimensional vector space over a field k. We will consider $\mathbb{P}(V)$, $\mathbb{P}(\wedge^2 V^\vee)$ and $\mathbb{P}(V^\vee)$. The tautological line (quotient) bundle of each will be denoted as $\mathscr{O}(1,0,0)$, $\mathscr{O}(0,1,0)$ and $\mathscr{O}(0,0,1)$ respectively. So on $\mathbb{P}(V)$, there is an exact sequence of vector bundles

$$0 \to \mathscr{A} \to V \otimes_k \mathscr{O}_{\mathbb{P}(V)} \to \mathscr{O}(1,0,0) \to 0.$$

This identifies $H^0(\mathbb{P}(V), \mathscr{O}(1,0,0))$ with V. If we fix a basis $X_0, X_1, \ldots, X_n$ for V and the dual basis $Y_0, Y_1, \ldots, Y_n$ for $V^\vee$, the dual of the above sequence gives the homomorphism

$$\mathscr{O}_{\mathbb{P}(V)} \to \mathscr{O}(1,0,0) \otimes_k V^\vee$$

which determines the global section $\Sigma X_i Y_i$.

2.2. Let Y be a scheme, and let $\mathscr{E}$ be a vector bundle of rank r on Y. Let $X = \mathbb{P}(\mathscr{E})$ be the projectivized bundle. It comes with a morphism $\pi: X \to Y$, such that X is smooth over Y and X has a tautological line bundle $\mathscr{O}_\pi(1)$ which appears in a sequence

$$0 \to \mathscr{F} \to \pi^* \mathscr{E} \to \mathscr{O}_\pi(1) \to 0.$$

$\omega_{X/Y}$ can be identified with $\wedge^r(\pi^*\mathscr{E}) \otimes \mathscr{O}_\pi(-r)$ ([2], III, Ex. 8.4).

2.3. Let Y be a scheme over a field k of characteristic $p \neq 0$. The absolute Frobenius morphism $F: Y \to Y$ is defined on the level of affine rings by mapping the function a to a^p, and has the property that if $\mathscr{L}$ is a line bundle on Y, then $F^*\mathscr{L} \cong \mathscr{L}^p$, and if there is a homomorphism $\mathscr{O}_X \to \mathscr{L}$ defining a section s, it pulls back to a homomorphism $\mathscr{O}_X \to \mathscr{L}^p$ defining the section s^p [6].

3. The example

Let V be a vector space of dimension $n+1$ over a field k of characteristic $p \neq 0$ where $n \geqslant 3$ and $p \geqslant n-1$. On $\mathbb{P}(V)$ there is the sequence of bundles

$$0 \to \mathscr{A} \to V \otimes_k \mathscr{O}_{\mathbb{P}(V)} \to \mathscr{O}(1,0,0) \to 0.$$

Let $Y = \mathbb{P}(\mathscr{A}^\vee)$, with the morphism $\alpha: Y \to \mathbb{P}(V)$. We have

$$0 \to \mathscr{G} \to \alpha^* \mathscr{A}^\vee \to \mathscr{O}_\alpha(1) \to 0,$$

with $\mathscr{G}$ a vector bundle of rank $n-1$. The surjection $V^\vee \otimes_k \mathscr{O}_{\mathbb{P}(V)} \to \mathscr{A}^\vee$ induces an inclusion of $\mathbb{P}(V)$-schemes $Y \subsetneq \mathbb{P}(V) \times \mathbb{P}(V^\vee)$.

Let π_1, π_2 be the two projections defined on $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$. There is the natural map on the product

$$\pi_1^*(V^\vee \otimes_k \mathscr{O}_{\mathbb{P}(V)}) \to \pi_2^*(\mathscr{O}(0,0,1)) \to 0$$

and the composite of this map with the inclusion $\pi_1^*(\mathscr{O}(-1,0,0)) \subsetneq \pi_1^*(V^\vee \otimes_k \mathscr{O}_{\mathbb{P}(V)})$ defines a homomorphism $\pi_1^*(\mathscr{O}(-1,0,0)) \to \pi_2^*(\mathscr{O}(0,0,1))$. Y is the zero scheme on the product $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ of the global section of $\mathscr{O}(1,0,1)$ induced by this homomorphism and this section can be seen to be just $\Sigma X_i Y_i$. So, Y has bi-homogeneous coordinate ring

$$k[X_0, \ldots, X_n; Y_0, \ldots, Y_n] \Big/ \left(\sum X_i Y_i\right)$$

and has canonical line bundle $\omega_Y = \mathscr{O}(-n, 0, -n)$.

If $\beta : Y \to \mathbb{P}(V^\vee)$ is induced by projection π_2 on the second factor, we see that $\mathscr{O}_\alpha(1)$ is the pull-back of $\mathscr{O}(0,0,1)$. On Y we have the commuting diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathscr{G} & \longrightarrow & \alpha^*\mathscr{A}^\vee & \longrightarrow & \mathscr{O}(0,0,1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{B} & \longrightarrow & V^\vee \otimes \mathscr{O}_Y & \longrightarrow & \mathscr{O}(0,0,1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathscr{O}(-1,0,0) & = & \mathscr{O}(-1,0,0) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The middle horizontal sequence is part of the Koszul complex, hence there is a surjection $\wedge^2 V^\vee \otimes \mathscr{O}(0,0,-1) \to \mathscr{B} \to 0$.

Now consider $F^*\mathscr{G}'$ where $\mathscr{G}' = \mathscr{G} \otimes \mathscr{O}(0,0,1)$. Let $X = \mathbb{P}(F^*\mathscr{G}')$ and let π: $X \to Y$ be the projection. X is smooth over k and since there is a surjection $\wedge^2 V^\vee \otimes \mathscr{O}_Y \to F^*\mathscr{G}' \to 0$, there is an inclusion of Y-schemes

$$X \subsetneq Y \times \mathbb{P}(\wedge^2 V^\vee).$$

It is evident that via projection onto the second factor, $\mathscr{O}_\pi(1)$ can be identified with the pull-back of $\mathscr{O}(0,1,0)$. Hence the line bundle $\mathscr{O}(1,1,1)$ on X is very ample. Since $\mathscr{O}(0, n-1, 0)$ is globally generated on X, the line bundle $\mathscr{L} = \mathscr{O}(1, n, 1)$ is also very ample on X ([2] II, Ex. 7.5.).

Claim. The very ample line bundle $\mathscr{L}$ *on the smooth variety* X (smooth over k of dimension $3n-3$) violates Kodaira vanishing.

Proof. We will compute $H^i(X, \mathscr{L}^{-1})$, where $\mathscr{L} = \mathscr{O}(1,0,1) \otimes \mathscr{O}_\pi(n)$. Since $\omega_X = \omega_{X/k} = \omega_{X/Y} \otimes \pi^*(\omega_Y) = \wedge^{n-1} F^*\mathscr{G}' \otimes \mathscr{O}_\pi(-n+1) \otimes \mathscr{O}(-n, 0, -n) = \mathscr{O}(p, 0, p(n-2)) \otimes \mathscr{O}_\pi(-n+1) \otimes \mathscr{O}(-n, 0, -n) = \mathscr{O}(p-n, 0, p(n-2)-n) \otimes \mathscr{O}_\pi(-n+1)$

we get

$$\begin{aligned} H^i(X, \mathscr{L}^{-1}) &= H^{3n-3-i}(X, \mathscr{L} \otimes \omega_X)^\vee \\ &= H^{3n-3-i}(X, \mathscr{O}(p-n+1, 0, p(n-2)-n+1) \otimes \mathscr{O}_\pi(1))^\vee \\ &= H^{3n-3-i}(Y, \mathscr{O}(p-n+1, 0, p(n-2)-n+1) \otimes F^*\mathscr{G})^\vee \\ &= H^{3n-3-i}(Y, \mathscr{O}(p-n+1, 0, p(n-2)-n+1+p) \otimes F^*\mathscr{G})^\vee \\ &= H^{3n-3-i}(Y, \mathscr{O}(p-n+1, 0, (p-1)(n-1)) \otimes F^*\mathscr{G})^\vee \end{aligned}$$

hence clearly 0 when $3n-3-i > 2n-1$, the dimension of Y, ie. when $i < n-2$.

Let $\mathscr{M} = \mathscr{O}(p-n+1, 0, (p-1)(n-1))$ on Y. The Frobenius pull-back of the commuting diagram above, tensored with $\mathscr{M}$ gives

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ & & & & \mathscr{M} \otimes F^*\mathscr{G} & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & \mathscr{M} \otimes F^*\mathscr{B} & \longrightarrow & \mathscr{V}^\vee \otimes \mathscr{M} & \longrightarrow & \mathscr{M} \otimes \mathscr{O}(0,0,p) & \longrightarrow & 0. \\ & & \uparrow & & & & & & \\ & & \mathscr{M} \otimes \mathscr{O}(-p,0,0) & & & & & & \\ & & \uparrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

By running along the exact sequence on $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$

$$0 \to \mathscr{O}(-1,0,-1) \to \mathscr{O} \to \mathscr{O}_Y \to 0,$$

we see that $H^j(Y, \mathscr{M} \otimes \mathscr{O}(-p,0,0)) = H^j(Y, \mathscr{O}(1-n, 0, (p-1)(n-1)))$ is always zero, hence

$$H^i(X, \mathscr{L}^{-1}) = H^{3n-3-i}(Y, \mathscr{M} \otimes F^*\mathscr{B})^\vee.$$

$V^\vee \otimes \mathscr{M}$ and $\mathscr{M} \otimes \mathscr{O}(0,0,p)$ have only H^0 as nonzero cohomology (H^0 is non-zero since $p \geq n-1$) hence $F^*\mathscr{B} \otimes \mathscr{O}(p+1-n, 0, (p-1)(n-1))$ has zero cohomology except possibly H^0 and H^1 which are to be studied by considering the map

$$\begin{aligned} A: V^\vee \otimes H^0(Y, \mathscr{O}(p+1-n, 0, (p-1)(n-1))) \\ \to H^0(Y, \mathscr{O}(p+1-n, 0, (p-1)(n-1)+p)). \end{aligned}$$

The homomorphism A is given by the matrix $[Y_0^p, Y_1^p, \ldots, Y_n^p]$ and it is easy to see that it is not onto. Consider the element

$$t = X_n^{p+1-n} \cdot Y_0 Y_1^{p-1} Y_2^{p-1} \ldots Y_n^{p-1}$$

which is well defined in $H^0(Y, \mathscr{O}(p+1-n, 0, (p-1)(n-1)+p))$ modulo multiples of $X_0Y_0 + X_1Y_1 + \cdots + X_nY_n$. No element in its equivalence class can be expressed as a sum of multiples of $Y_0^p, Y_1^p, \ldots, Y_n^p$, i.e. t is not in the image of A.

It follows that with $3n-3-i=1$, $H^i(X, \mathscr{L}^{-1}) \neq 0$. Hence $H^{3n-4}(X, \mathscr{L}^{-1}) \neq 0$, where X has dimension $3n-3$ and $H^i(X, \mathscr{L}^{-1}) = 0$ for all $i < 3n-4$. □

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Double convolution integral equations involving a general class of multivariable polynomials and the multivariable H-functions

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Abstract. In this paper we have solved a double convolution integral equation whose kernel involves the product of the H-functions of several variables and a general class of multivariable polynomials. Due to general nature of the kernel, we can obtain from it, solutions of a large number of double and single convolution integral equations involving products of several classical orthogonal polynomials and simpler functions. We have also obtained here solutions of two double convolution integral equations as special cases of our main result. Exact reference of three known results, which are obtainable as particular cases of one of these special cases, have also been included.

Keywords. Double Laplace transform; convolution theorem for Laplace transform; the double convolution integral equation; a general class of multivariable polynomials; multivariable H-function.

1. Introduction

A large number of convolution integral equations (double or single) involving various functions as their kernel have been solved by several research workers. An extensive and systematic presentation has been given in the books by Srivastava and Buschman [6, 7]. In the present paper we consider the generalizations of the results obtained earlier by Gupta *et al* [2], Jain [3], and Raina and Koul [4].

We shall require the following definitions and results later on. (i) A general class of multivariable polynomials [9, p. 686, eq. (1.4)] defined as follows

$$S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]$$
$$= \sum_{k_{s+1},\ldots,k_n=0}^{M_{s+1}k_{s+1}+\cdots+M_nk_n\le N} (-N)_{M_{s+1}k_{s+1}+\cdots+M_nk_n} A_1(N; k_{s+1},\ldots,k_n)$$
$$\times \frac{(-z_{s+1})^{k_{s+1}}}{k_{s+1}!}\cdots\frac{(-z_n)^{k_n}}{k_n!}, \tag{1.1}$$

where $M_{s+1},\ldots,M_n$ are arbitrary positive integers and the coefficients $A_1(N; k_{s+1},\ldots,k_n)$, $(N; k_i \ge 0,\ i = s+1,\ldots,n)$ are arbitrary constants, real or complex.

Another class of polynomials $S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]$ occurring in (2.1) is defined in a similar manner with changed parameters. (ii) A particular case of the multivariable H-function defined by Srivastava and Panda (cf. [10, p. 251, eq. (C.1)]) possessing the following integral representation

$$H_1\begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix} = H^{0,0\ :1,n_1;\ldots;1,n_s}_{p^{(1)},q^{(1)};p_1,q_1+1;\ldots;p_s,q_s+1}\begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix}.$$

$$\left.\begin{matrix}(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(s)})_{1,p^{(i)}};(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots;(c_j^{(s)},\gamma_j^{(s)})_{1,p_s}\\ (b_j:\beta_j^{(1)},\ldots,\beta_j^{(s)})_{1,q^{(i)}};(0,1),(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots;(0,1),(d_j^{(s)},\delta_j^{(s)})_{1,q_s}\end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\phi_1(\xi_1)\cdots\phi_s(\xi_s)\psi(\xi_1,\ldots,\xi_s)$$

$$\times\Gamma(-\xi_1)\cdots\Gamma(-\xi_s)z_1^{\xi_1}\cdots z_s^{\xi_s}\,d\xi_1\cdots d\xi_s,\quad w=\sqrt{-1} \tag{1.2}$$

or, equivalently, in the series form as follows [11, p. 64, eq. (1.3)]

$$H_1\left[\begin{matrix}z_1\\ \vdots\\ z_s\end{matrix}\right]=\sum_{k_1,\ldots,k_s=0}^{\infty}\phi_1(k_1)\cdots\phi_s(k_s)\psi(k_1,\ldots,k_s)\frac{(-z_1)^{k_1}}{k_1!}\cdots\frac{(-z_s)^{k_s}}{k_s!}, \tag{1.3}$$

where

$$\phi_i(k_i)=\frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)}{\prod_{j=1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}k_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}k_i)},\qquad (i=1,\ldots,s) \tag{1.4}$$

and

$$\psi(k_1,\ldots,k_s)=\left\{\prod_{j=1}^{p^{(i)}}\Gamma\left(a_j-\sum_{i=1}^{s}\alpha_j^{(i)}k_i\right)\prod_{j=1}^{q^{(i)}}\Gamma\left(1-b_j^{(i)}+\sum_{i=1}^{s}\beta_j^{(i)}k_i\right)\right\}^{-1}. \tag{1.5}$$

For the convergence, and existence conditions and other details of the multivariable H-function, the conditions referred to in [10, pp. 251–253, eqs ((C.2)–(C.8))] are assumed to be satisfied throughout the present work.

During our study we shall also need a second multivariable H-function with different variables and parameters. We shall denote it by H_2.

(iii) The results recorded in the book by Ditkin and Prudnikov [1, p. 4, eqs ((1.2), (1.3)); p. 93, eq. (25); p. 99, eq. (68)] as referred to in [3, p. 99, eqs ((1.2)–(1.8))].

2. Main result

Theorem. *The double convolution integral equation*

$$\int_0^x\int_0^y(x-u)^{\rho_1-1}(y-v)^{\rho_2-1}H_1\left[\begin{matrix}z_1(x-u)\\ \vdots\\ z_s(x-u)\end{matrix}\right]H_2\left[\begin{matrix}w_1(y-v)\\ \vdots\\ w_r(y-v)\end{matrix}\right]$$

$$\times S_N^{M_{s+1},\ldots,M_n}[-z_{s+1}(x-u),\ldots,-z_n(x-u)]$$

$$\times S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1}(y-v),\ldots,-w_m(y-v)]f(u,v)\,du\,dv=g(x,y) \tag{2.1}$$

has the solution given by

$$f(x,y)=\int_0^x\int_0^y(x-u)^{a-\rho_1-\mu_1-1}(y-v)^{b-\rho_2-\mu_2-1}W[x-u,y-v]$$

$$\times g_{u^a,v^b}^{(a+b)}(u,v)\,du\,dv, \tag{2.2}$$

where $\min \operatorname{Re}(\rho_1, \rho_2) > 0, a$ and b are positive integers such that $a > \operatorname{Re}(\rho_1 + \mu_1) > 0$, $b > \operatorname{Re}(\rho_2 + \mu_2) > 0$,

$$\bar{g}_{1,y^k}(p,0) = \bar{g}_{2,x^c}(0,q) = g^{(c+k)}_{x^c,y^k}(0,0) = 0 \quad \text{for} \quad 0 \leqslant c < a \quad \text{and} \quad 0 \leqslant k < b \tag{2.3}$$

and

$$W[x,y] = \sum_{\nu_1,\nu_2=0}^{\infty} \frac{H^{(1)}_{\nu_1} H^{(2)}_{\nu_2} x^{\nu_1} y^{\nu_2}}{\Gamma(a+\nu_1-\rho_1-\mu_1)\,\Gamma(b+\nu_2-\rho_2-\mu_2)}, \tag{2.4}$$

where $H^{(1)}_{\nu_1}$ and $H^{(2)}_{\nu_2}$ are given by the recurrence relations

$$\lambda^{(i)}_{\mu_i} H^{(i)}_0 = 1 \quad \text{and for} \quad \mathscr{G}_i > 0, \sum_{\nu_i=0}^{\mathscr{G}_i} H^{(i)}_{\nu_i} \lambda^{(i)}_{\mathscr{G}_i+\mu_i-\nu_i} = 0, \quad \mathscr{G}_i = 1,2,3,\ldots, i = 1,2. \tag{2.5}$$

μ_i is the least B_i for which $\lambda^{(i)}_{\mu_i} \neq 0$.

$$\lambda^{(1)}_{B_1} = (-1)^{B_1} \sum_{k_1+\cdots+k_n=B_1} \Delta_1(k_1,\ldots,k_n) \frac{(z_1)^{k_1}}{k_1!} \cdots \frac{(z_n)^{k_n}}{k_n!}, \tag{2.6}$$

where

$$\Delta_1(k_1,\ldots,k_n) = \phi_1(k_1)\cdots\phi_s(k_s)\,\psi(k_1,\ldots,k_s)\,\phi(k_{s+1},\ldots,k_n) \times \Gamma\left(\rho_1 + \sum_{i=1}^{n} k_i\right). \tag{2.7}$$

$\phi_i(k_i)$ for $i = 1,\ldots,s$ and $\psi(k_1,\ldots,k_s)$ are defined by (1.4) and (1.5) respectively, and

$$\phi(k_{s+1},\ldots,k_n) = \begin{cases} (-N)_{\sum_{i=s+1}^{n}(M_i k_i)} A_1(N;k_{s+1},\ldots,k_n), & 0 \leqslant \sum_{i=s+1}^{n} M_i k_i \leqslant N \\ 0, & \sum_{i=s+1}^{n} M_i k_i > N. \end{cases} \tag{2.8}$$

Similarly,

$$\lambda^{(2)}_{B_2} = (-1)^{B_2} \sum_{K_1+\cdots+K_m=B_2} \Delta_2(K_1,\ldots,K_m) \frac{w_1^{K_1}}{K_1!} \cdots \frac{w_m^{K_m}}{K_m!}, \tag{2.9}$$

where

$$\Delta_2(K_1,\ldots,K_m) = \Phi_1(K_1)\cdots\Phi_r(K_r)\Psi(K_1,\ldots,K_r)\Phi(K_{r+1},\ldots,K_m) \times \Gamma\left(\rho_2 + \sum_{i=1}^{m} K_i\right), \tag{2.10}$$

$$\Phi_i(K_i) = \frac{\prod_{j=1}^{N_i}\Gamma(1-u_j^{(i)} + U_j^{(i)}K_i)}{\prod_{j=1}^{Q_i}\Gamma(1-v_j^{(i)}) + V_j^{(i)}K_i) \prod_{j=N_i+1}^{P_i}\Gamma(u_j + U_j^{(i)}K_i)} \quad (i = 1,\ldots,r), \tag{2.11}$$

$$\Psi(K_1,\ldots,K_r)=\left\{\prod_{j=1}^{P}\Gamma\left(g_j-\sum_{i=1}^{r}G_j^{(i)}K_i\right)\prod_{j=1}^{Q}\Gamma\left(1-h_j+\sum_{i=1}^{r}H_j^{(i)}K_i\right)\right\}^{-1} \tag{2.12}$$

and

$$\Phi(K_{r+1},\ldots,K_m)=\begin{cases}(-N')_{\sum_{i=r+1}^{m}(M_iK_i)}A_2(N';K_{r+1},\ldots,K_m), & 0\leqslant\sum_{i=r+1}^{m}M_iK_i\leqslant N'\\ 0, & \sum_{i=r+1}^{m}M_iK_i>N'.\end{cases} \tag{2.13}$$

Also

$$\bar{g}_{1,y^k}(p,0)=\int_0^\infty e^{-p\xi}\frac{\partial^k}{\partial y^k}g(\xi,y)\Big|_{y=0}d\xi\,, \tag{2.14}$$

$$\bar{g}_{2,x^c}(0,q)=\int_0^\infty e^{-q\eta}\frac{\partial^c}{\partial x^c}g(x,\eta)\Big|_{x=0}d\eta \tag{2.15}$$

and

$$g^{(c+k)}_{x^c,y^k}(x,y)=\frac{\partial^{c+k}}{\partial x^c\,\partial y^k}g(x,y). \tag{2.16}$$

Proof. To solve the double convolution integral equation (2.1), we first take the double Laplace transform of both its sides, use the convolution theorem for the two dimensional Laplace transform and arrive at the following:

$$\left\{\int_0^\infty\int_0^\infty e^{-px-qy}x^{\rho_1-1}y^{\rho_2-1}H_1\begin{bmatrix}z_1x\\ \vdots\\ z_sx\end{bmatrix}H_2\begin{bmatrix}w_1y\\ \vdots\\ w_ry\end{bmatrix}\right.$$
$$\times S_N^{M_{s+1},\ldots,M_n}[-z_{s+1}x,\ldots,-z_nx]S_{N'}^{M_{r+1},\ldots,M_m}$$
$$\left.\times[-w_{r+1}y,\ldots,-w_my]\,dx\,dy\right\}\cdot\bar{f}(p,q)=\bar{g}(p,q), \tag{2.17}$$

where $\bar{f}(p,q)$ and $\bar{g}(p,q)$ denote the double Laplace transform of functions $f(x,y)$ and $g(x,y)$, respectively.

Now we express the H-function of several variables and the general class of polynomials occurring in the left-hand side of (2.17) in series form using (1.3) and (1.1), interchange the order of series and integrations and evaluate the x and y integrals. By making use of a known formula [11, p. 67, eq. (2.3)], expressing the equation in terms of $\bar{f}(p,q)$, reciprocating the two series occurring on the right-hand side, proceeding in a manner as given in a recent paper [2] and taking the inverse of double Laplace transform of both sides of the resulting equation and applying convolution theorem in its right-hand side, we arrive at the desired result. The details of the proof can be seen in [2].

3. Special cases

(i) If we take

$$A_1(N;k_{s+1},\ldots,k_n)=\frac{\prod_{j=1}^{E_1}(e_j)_{k_{s+1}\theta_j^{(s+1)}+\cdots+k_n\theta_j^{(n)}}}{\prod_{j=1}^{G_1}(g_j)_{k_{s+1}\psi_j^{(s+1)}+\cdots+k_n\psi_j^{(n)}}}$$
$$\times\frac{\prod_{j=1}^{U_1^{(s+1)}}(u_j^{(s+1)})_{k_{s+1}\phi_j^{s+1}}\cdots\prod_{j=1}^{U_1^{(n)}}(u_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{V_1^{(s+1)}}(v_j^{(s+1)})_{k_{s+1}\zeta_j^{s+1}}\cdots\prod_{j=1}^{V_1^{(n)}}(v_j^{(n)})_{k_n\zeta_j^{(n)}}} \tag{3.1}$$

in (2.1), $S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]$ reduces to the generalized Lauricella function of Srivastava and Daoust [8, p. 454] as follows:

$$S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]=F_{G_1}^{1+E_1:\,U_1^{(s+1)},\ldots,U_1^{(n)}}$$
$$\left[\begin{matrix}(-N:M_{s+1},\ldots,M_n),(e:\theta_1^{(s+1)},\ldots,\theta_1^{(n)}):((u^{(s+1)}):\phi^{(s+1)};\ldots;((u^{(n)}):\phi^{(n)});\\ (g:\psi_1^{(s+1)},\ldots,\psi_1^{(n)}):((v^{(s+1)}):\zeta^{(s+1)};\ldots;((v^{(n)}):\zeta^{(n)});\end{matrix}\;-z_{s+1},\ldots,-z_n\right]. \tag{3.2}$$

Similarly choosing the coefficients $A_2(N';K_{r+1},\ldots,K_m)$ appropriately $S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]$ reduces to another generalized Lauricella function given by

$$S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]=F_{G_2}^{1+E_2:\,U_2^{(r+1)},\ldots,U_2^{(m)}}$$
$$\times\left[\begin{matrix}(-N':M_{r+1},\ldots,M_m),(e':\theta_2^{(r+1)},\ldots,\theta_2^{(m)}):((f^{(r+1)}):\eta^{(r+1)});\ldots;((f^{(m)}):\eta^{(m)});\\ (g':\psi_2^{(r+1)},\ldots,\psi_2^{(m)}):((t^{(r+1)}):\nu^{(r+1)});\ldots;((t^{(m)}):\nu^{(m)});\end{matrix}\;-w_{r+1},\ldots,-w_m\right] \tag{3.3}$$

and our theorem yields the following result. (For the sake of brevity, here we shall write the functions occurring on the right-hand side of (3.2) and (3.3) as $F(-z_{s+1},\ldots,-z_n)$ and $F^*(-w_{r+1},\ldots,-w_m)$, respectively.

1. COROLLARY

The double convolution integral equation

$$\int_0^x\int_0^y(x-u)^{\rho_1-1}(y-v)^{\rho_2-1}H_1\begin{bmatrix}z_1(x-u)\\ \vdots\\ z_s(x-u)\end{bmatrix}H_2\begin{bmatrix}w_1(y-v)\\ \vdots\\ w_r(y-v)\end{bmatrix}$$
$$\times F[-z_{s+1}(x-u),\ldots,-z_n(x-u)]F^*[-w_{r+1}(y-v),\ldots,-w_m(y-v)]$$
$$\times f(u,v)\,du\,dv=g(x,y) \tag{3.4}$$

possesses the solution

$$f(x,y)=\int_0^x\int_0^y(x-u)^{a-\rho_1-\mu_1-1}(y-v)^{b-\rho_2-\mu_2-1}W_1[x-u,y-v]$$
$$\times g_{u^a,v^b}^{(a+b)}(u,v)\,du\,dv, \tag{3.5}$$

where

$$W_1[x,y]=\sum_{\nu_1,\nu_2=0}^{\infty}\frac{G_{\nu_1}^{(1)}G_{\nu_2}^{(2)}x^{\nu_1}y^{\nu_2}}{\Gamma(a+\nu_1-\rho_1-\mu_1)\Gamma(b+\nu_2-\rho_2-\mu_2)}, \tag{3.6}$$

the coefficients $G_{v_1}^{(1)}$ *and* $G_{v_2}^{(2)}$ *can easily be obtained from* $H_{v_1}^{(1)}$ and $H_{v_2}^{(2)}$, *respectively. The result holds under the conditions easily derived from the conditions mentioned with the main result.*

(ii) Again if we take $M_i = 0$, $z_i \to 0$ $(i = s+2,\ldots,n)$ in (2.1) and replace $A_1(N; k_{s+1}, 0,\ldots,0)$ by $A_{N,k_{s+1}}$. The general class of multi-variable polynomials $S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]$ reduces to a general class of polynomials $S_N^{M_{s+1}}[-z_{s+1}]$ [5, p. 1, eq. (1)] which includes almost all the well known orthogonal polynomials as its special cases.

Similarly the multivariable polynomial $S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]$ can be reduced to $S_{N'}^{M_{r+1}}[-w_{r+1}]$ and we arrive at the following result:

2. COROLLARY

The double convolution integral equation

$$\int_0^x \int_0^y (x-u)^{\rho_1-1}(y-v)^{\rho_2-1} S_N^{M_{s+1}}[-z_{s+1}(x-u)]\, S_{N'}^{M_{r+1}}[-w_{r+1}(y-v)] \times H_1\begin{bmatrix} z_1(x-u) \\ \vdots \\ z_s(x-u) \end{bmatrix} H_2\begin{bmatrix} w_1(y-v) \\ \vdots \\ w_r(y-v) \end{bmatrix} f(u,v)\,du\,dv = g(x,y) \tag{3.7}$$

has the solution given by

$$f(x,y) = \int_0^x \int_0^y (x-u)^{a-\rho_1-\mu_1-1}(y-v)^{b-\rho_2-\mu_2-1} W_2[x-u, y-v] \times g_{u^a,v^b}^{(a+b)}(u,v)\,du\,dv, \tag{3.8}$$

where

$$W_2[x,y] = \sum_{v_1,v_2=0}^{\infty} \frac{J_{v_1}^{(1)} J_{v_2}^{(2)} x^{v_1} y^{v_2}}{\Gamma(a+v_1-\rho_1-\mu_1)\Gamma(b+v_2-\rho_2-\mu_2)}, \tag{3.9}$$

the coefficients $J_{v_1}^{(1)}$ and $J_{v_2}^{(2)}$ *can easily be derived from* $H_{v_1}^{(1)}$ and $H_{v_2}^{(2)}$, *respectively. The result holds under the conditions easily obtained from the conditions mentioned with the main result.*

Corollary 2 itself is quite general in nature and includes the following results as it special cases.

(iii) If we take $s = r = 1$, $p^{(1)} = q^{(1)} = 0 = P = Q$ in (ii), then each of the multivariable H_1 and H_2-functions reduce to Fox's H-function. Also, if we take $N = N' = 0$, then the polynomials $S_0^{M_2}[-z_2]$ and $S_0^{M_2}[-w_2]$ will reduce to $A_1(0,0)$ and $A_2(0,0)$, which can be taken to be unity without loss of generality and we arrive at a result which in essence is same as that given by Raina and Koul [4, p. 366, Cor. (1)].

(iv) If we take $s = r = 1$, $z_2 = w_2 = -1$, $p^{(1)} = q^{(1)} = 0 = P = Q$ in (ii) and further reduce the Fox's H-functions thus obtained to $\exp(-z_1)$ and $\exp(-w_1)$ [10, p. 18, eq. (2.6.2)] and let $z_1 \to 0$ and $w_1 \to 0$, these functions reduce to unity and we obtain a result given by Jain [3, p. 100, eq. (2.1)].

(v) The one variable-analogue of Corollary 2 corresponds to the result obtained recently by Gupta *et al* [2, p. 188, eq. (2.1)].

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On a new geometric property for Banach spaces

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Abstract. In this paper we study a geometric property for Banach spaces called condition (*), introduced by de Reyna *et al* in [3]. A Banach space has this property if for any weakly null sequence $\{x_n\}$ of unit vectors in X, if $\{x_n^*\}$ is any sequence of unit vectors in X^* that attain their norm at x_n's, then $x_n^* \xrightarrow{w^*} 0$. We show that a Banach space satisfies condition (*) for all equivalent norms iff the space has the Schur property. We also study two related geometric conditions, one of which is useful in calculating the essential norm of an operator.

Keywords. Banach spaces; Schur property.

1. Introduction

In a recent work [3], de Reyna *et al* considered a new geometric property for a Banach space called "condition (*)". They have introduced it as a necessary condition for a Banach space X so that $C(K,X)$, the space of X-valued continuous functions is an M-ideal in $WC(K,X)$ the space of X-valued functions on K that are continuous when X has the weak topology (i.e. there is a projection P: $WC(K,X)^* \to WC(K,X)^*$ whose range is $C(K,X)^\perp$ and $\|P\wedge\| + \|\wedge - P\wedge\| = \|\wedge\| \forall \wedge \in WC(K,X)^*$ for the ordinal space $K = [0,\omega]$.

Motivated by some results in M-structure theory (see [10], [11]), this author has proved in [12] that condition (*) is also a necessary condition for $C(\beta\mathbb{N},X)$ to be an M-ideal in $WC(\beta\mathbb{N},X)$. It is known that this condition is not a sufficient condition for continuous functions to be a M-ideal (see [3]).

Among other non-trivial examples of Banach spaces with condition (*) are the ℓ^p-spaces for $1<p<\infty$ and spaces with property (m_∞), (see [7]) the Bloch space as well as c_0 direct sum of spaces with the Schur property (see [3]). Any Banach space with the Schur approximation property or its modified version (see [3], [12]) satisfies condition (*). Clearly condition (*) is hereditary and is a separably determined property.

It is clear that any Banach space that satisfies the Schur property (i.e. weak and norm sequential convergence coincide) has (*). We show that if a Banach space is such that it satisfies condition (*) in all equivalent renormings, then X has the Schur property. This comes as a consequence of our result on the stability of this condition under ℓ^1-direct sums. We next show that this condition is invariant under c_0-direct sums. If X has the Dunford and Pettis property then again condition (*) for X^* implies the Schur property on X^*.

We next consider a weaker form of condition (*) which we shall call as condition (*) with extreme points. For a Banach space that has this property, zero is a ω^*-accumulation point of $\partial_e X_1^*$, the set of extreme points of the dual unit ball. Motivated by the example $c_0(\Gamma)$ (Γ is a discrete set) we show that a Banach space X

whose dual is isometric to a $L^1(\mu)$ has this weaker condition iff it is isometric to $c_0(\Gamma)$ for a discrete set Γ. This is also a hereditary property but we do not know if it is a separably determined one. However we show that if X is a Asplund space, this weaker form is also a seperably determined property.

For a Banach space X, let $\mathscr{L}(X)$, $\mathscr{K}(X)$ denote the space of bounded and compact linear operators respectively. In [2], Axler *et al* consider Banach spaces for which the essential norm of an operator $\|T\|_e = d(T, \mathscr{K}(X))$ equals that of the adjoint. Motivated by a formula for the essential norm obtained by Werner, in the case when compact operators form an M-ideal in the space of bounded operators [14], we introduce a stronger form of condition (*) to show that if X is a reflexive Banach space whose dual has this property and Y is a separable space such that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$, then $\|T\|_e = \|T^*\|_e$ $\forall T \in \mathscr{L}(X, Y)$.

Since any space with the Schur property satisfies all these properties, unless otherwise specified we consider only space that do not have the Schur property.

All the Banach spaces considered are over the real scalar field. For basic Banach space theory we shall refer to [4, 5] and for L^1-predual theory [8] and for results about M-structure of Banach spaces the fundamental paper [1] and the latest monograph [6].

2. Main results

Our first result shows that ℓ^p-spaces $(1 < p < \infty)$ and any Banach space that has property (m_∞) satisfies condition (*).

Recall from [7] that a Banach space has "property (m_p)" if $\overline{\lim}\|x + x_n\| = \|(\|x\|, \overline{\lim}\|x_n\|)\|_p$ whenever $x_n \to 0$ weakly.

PROPOSITION 1

Any Banach space that has property (m_∞) *and* ℓ^p for $1 < p < \infty$, *satisfies condition* (*).

Proof. Let $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and $1 = x_n^*(x_n) = \|x_n^*\|$. Let $\|x\| = 1$. We shall show that $x_n^*(x) \to 0$. Passing through a subsequence if necessary, assume that $x_n^*(x) \to \alpha$.

$$|1 \pm \alpha| = \lim_n |x_n^*(x_n) \pm x_n^*(x)|$$

$$\leqslant \overline{\lim}\|x_n \pm x\| = 1.$$

Hence $\alpha = 0$. That ℓ^p satisfies condition (*) can be seen easily using the facts that the x_n^*'s are uniquely determined in terms of the x_n's and the weak convergence is pointwise convergence on the unit ball.

For the case $p = 1$ the following proposition shows that if μ is not purely atomic, $L^1(\mu)$ fails (*).

PROPOSITION 2

Any Banach space X *satisfying condition* (*) *has trivial L-structure i.e. there is no nontrivial projection* P *in* X *such that* $\|Px\| + \|x - Px\| = \|x\|$ $\forall x \in X$.

Proof. As part of our general assumption we have that X is an infinite dimensional space failing the Schur property. Suppose $X = M \oplus N$ for nontrivial M and N such that $\|m+n\| = \|m\| + \|n\|$ $\forall m \in M$, $n \in N$.

Since X fails the Schur property, we assume w.l.o.g. that there exists a sequence $\{m_k\}$ in M with $\|m_k\| = 1$, $m_k \xrightarrow{w} 0$. Choose $\|m_k^*\| = 1 = m_k^*(m_k)$, $m_k^* \in M^*$. Since N is nontrivial we can choose $\{n^*\}$ a unit vector in N^* and n in N such that $n^*(n) = 1$.

Now consider the sequence $\{m_k^* + n^*\}$ in $X^*(= M^* \oplus N^*$ equipped with the maximum norm).

For any k, $\|(1-(1/k))m_k + (1/k)n\| = 1$ and $(m_k^* + n^*)\ ((1-(1/k))m_k + (1/k)n) = 1-(1/k)+(1/k) = 1$. Also $(1-(1/k))m_k + (1/k)n \xrightarrow{\omega} 0$. Since X has property $(*)$, we get, $m_k^* + n^* \xrightarrow{\omega^*} 0$, a contradiction.

Therefore there is no nontrivial L-projection in X.

COROLLARY

An infinite dimensional Banach space has the Schur property iff it satisfies condition $(*)$ *for each equivalent norm.*

Proof. Suppose X satisfies condition $(*)$ for each equivalent norm.

Let $X = M \oplus N$ where M and N are two proper closed subspaces.

Now $\|x\|' = \|m\| + \|n\|$ where $x = m+n$, $m \in M$, $n \in N$ is an equivalent norm and since $(X, \|\ \|')$ has a non-trivial L-projection, we conclude that X has the Schur property.

We next consider the stability of condition $(*)$ under c_0-direct sums.

PROPOSITION 3

If $\{X_i\}$ *is a family of Banach spaces satisfying condition* $(*)$ *then so does* $Y = \oplus_{c_0} X_i$.

Proof. Let us note that if each X_i has the Schur property then Y satisfies condition $(*)$ (see [3]).

Let $y_n \in Y$, $\|y_n\| = 1$ and $y_n \xrightarrow{w} 0$. Clearly $y_n(i) \xrightarrow{w} 0 \quad \forall i$.

Let $y_n^* \in Y^* = \oplus_1 X_i^*$, $\|y_n^*\| = 1 = y_n^* \cdot (y_n)$. It is enough to show that $y_n^*(i) \xrightarrow{w^*} 0$ $\forall i$.

Now since $\sum_i \|y_n^*(i)\| = 1 = \sum y_n^*(i)\,(y_n(i))$,

we have

$$\sum_i \|y_n^*(i)\| \frac{y_n^*(i)}{\|y_n^*(i)\|}(y_n(i)) = 1 \quad \forall n.$$

Since

$\|y_n\| = 1$ $\forall n$, we get

$$\frac{y_n^*(i)}{\|y_n^*(i)\|}(y_n(i)) = 1$$

whenever $y_n^*(i) \neq 0$ $\forall i$ and this implies $\|y_n(i)\| = 1$. However since $y_n(i) \xrightarrow{\omega} 0$ we get $y_n^*(i) \xrightarrow{\omega^*} 0$ as $n \to \infty$.

Therefore Y satisfies condition $(*)$.

PROPOSITION 4

Suppose Y has the Dunford and Pettis property, then Y^* has condition (*) iff it has the Schur property.

Proof. Suppose Y^* fails the Schur property. Using the Bishop–Phelps theorem [4], we may assume that $\exists y_n \in Y$, $\|y_n\| = 1$ and $\|y_n^*\| = 1 = y_n^*(y_n)$, $y_n^* \in Y^*$, $y_n^* \xrightarrow{w} 0$. Since Y^* has condition (*), $y_n \xrightarrow{w} 0$. Also since Y has the Dunford and Pettis property, $y_n^*(y_n) \longrightarrow 0$, a contradiction. Hence, Y^* has the Schur property.

Remark. This should be compared with Corollary 2 of [12], where we have proved that if X has the MSAP then any subspace of X that is isomorphic to a dual space has the Schur property.

We now introduce a weaker form of condition (*) which we shall call as condition (*) with extreme points.

(A) DEFINITION

A Banach space X satisfies condtion (*) with extreme points if for any weakly null sequence $\{x_n\}$ of unit vectors in X such that $x_n^*(x_n) = 1$, $x_n^* \in \partial_e X_1^*$ implies $x_n^* \xrightarrow{w^*} 0$.

Since by our general assumption on X, there is at least one sequence $\{x_n\}$ of unit vectors with $x_n \xrightarrow{w} 0$. By choosing $x_n^* \in \partial_e X_1^*$, $1 = x_n^*(x_n)$ we conclude that 0 is in the w^*-closure of $\partial_e X_1^*$. Thus c, the space of convergent sequences fails this property.

Since for any closed subspace $M \subset X$, any $f \in \partial_e M_1^*$ can be extended to a $g \in \partial_e X_1^*$, this is a hereditary property.

Theorem 1. *Suppose X is a Asplund space. If all separable subspace of X have condition (*) with extreme points then X has the same property.*

Proof. Let $x_n \in X$, $x_n^* \in \partial_e X_n^*$, $\|x_n\| = 1 = x_n^*(x_n)$, $x_n \xrightarrow{w} 0$.

Fix a $x_0 \in X$. Let $Y = \overline{\text{Span}}\{x_n\}_{x \geq 0}$ and $Y' = \overline{\text{Span}}\{x_n^*\}_{n \geq 1}$.

It follows from the results in [13] that there exists a separable subspace Z of X containing Y and a linear operator T: $Z^* \to X^*$ such that $T(Z^*) \supseteq Y'$ and $T(z^*)$ is a norm preserving extension of z^* for each $z^* \in Z^*$.

Fix n and let $F_n = \{z^* \in Z_1^*\colon T(z^*) = x_n^*\}$. Since T is linear and x_n^* is an extreme point, F_n is an extreme subset of Z_1^*. Since X is a Asplund space it follows that (see [4], Chapter 6) there is a $z_n^* \in \partial_e Z_1^* \cap F_n$.

Since Z is separable it follows from the hypothesis that $z_n^* \xrightarrow{w^*} 0$ in Z^*. As $x_0 \in Z$, we get $x_n^*(x_0) \to 0$. Since x_0 is arbitrary we conclude that $x_n^* \to 0$. Hence X has condition (*) with extreme points.

Theorem 2. *Let X be an L^1-predual space. Suppose X satisfies condition (*) with extreme points then X is isometric to $c_0(\Gamma)$ for some discrete set Γ.*

Proof. We shall first prove the theorem when X is a separable Banach space. Since c, the space of convergent sequences, fails (*) with extreme points and since this is a

hereditary property, clearly X has no isometric copy of c. Hence X^* is isometric to ℓ^1 (see [8], p. 226). Let $\partial_e X_1^*$ denote the set of extreme points of the dual unit ball X_1^* of X. We can choose a sequence $\{x_n^*\}$ of linearly independent vectors such that

$$\partial_e X_1^* = \pm \{x_n^*\}_{n \geq 1}.$$

If we can show that $x_n^* \xrightarrow{w^*} 0$, then it would follow that X is a c_0-space.

Since any linear functional defined on the span of finitely many x_n^*'s has a norm preserving extension to an element of X (see [1], Corollary 5.5), inductively define a sequence $\{x_n\} \subset X$ such that $x_n^*(x_n) = 1 = \|x_n\|$, $x_n^*(x_m) = 0$ for $m > n$. Clearly then for each $x_n^* \in \partial_e X_1^*$,

$$x_n^*(x_m) \to 0 \quad \text{as} \quad m \to \infty.$$

Hence, by Rainwater's theorem [5], $x_n \to 0$ weakly. Therefore by condition (*) we have that $x_n^* \xrightarrow{w^*} 0$. Since any infinite w^*-convergent sequence in $\partial_e X_1^*$ is going to give an isometric copy of c in X (see [9]), it is clear that $\partial_e X_1^*$ is discrete. Therefore X is isometric to c_0.

To see the general case, let $Y \subset X$ be any separable subspace. Then there exist a separable L^1-predual space Z such that $Y \subset Z \subset X$ (see [8], Chapter 7). From what we saw above, Z is isometric to c_0. At this stage instead of using purely L^1-predual theory techniques, we shall use the following argument using results about Banach spaces that are M-ideals in their biduals to conclude that X is isometric to $c_0(\Gamma)$.

Since c_0 is a Banach space that is an M-ideal in its bidual and since this property is hereditary, we can conclude from the arguments above that X is such that every separable subspace of it is an M-ideal in its bidual. Therefore it follows from Proposition 2.8 of [10] that X is an M-ideal in its bidual and being an L^1-predual it therefore is isometric to $c_0(\Gamma)$ (see [6] Chapter 3).

Remark. The motivation for the above theorem comes from a result of [10] where the authors prove that if X is a Banach space such that the space of compact operators $\mathscr{K}(\ell^1, X)$ forms an M-ideal in $\mathscr{L}(\ell^1, X)$, the space of bounded operators, then X is an M-ideal in its bidual and in particular the only L^1-predual space in the above class is $c_0(\Gamma)$. This author has observed in [12] that for such an X, $C(\beta\mathbb{N}, X)$ is an M-ideal in $WC(\beta\mathbb{N}, X)$ and further this latter class is a strictly smaller class. Hence it is natural to ask if $c_0(\Gamma)$ is the only L^1-predual in the latter class as well.

c_0 is an M-ideal in its bidual ℓ^∞ and ℓ^∞ fails condition (*) with extreme points. Also $0 \notin \overline{\partial_e(\ell^1)_1^*}^w$. The ideas of Theorem 1 can be used to prove the following result.

PROPOSITION 5

Suppose X is an infinite dimensional Banach space which is an M-ideal in its bidual. If $0 \notin \overline{\partial_e X_1^*}^w$ (closure in the weak topology) then X^{**} fails condition (*) with extreme points.

Proof. This is clear when X is a reflexive space. So assume that $\exists x_n^{**} \in X^{**}, \|x_n^{**}\| = 1$, $x_n^{**} \xrightarrow{w} 0$ and X is non-reflexive. Note that since X has an isomorphic copy of c_0, X and hence X^{**} cannot have the Schur property. It is also known (see [6], Chapter 3) that such an X is a Asplund space. So, using this fact and the Bishop–Phelps theorem and imitating the arguments given before we can choose,

$y_n^* \in \partial_e X_1^*$ such that $y_n^{**}(y_n^*) = \|y_n^{**}\| = 1$, $y_n^{**} \in X^{**}$ and $y_n^{**} \xrightarrow{w} 0$. Now if X^{**} has condition (*), we get that $y_n^* \xrightarrow{w} 0$, a contradiction.

Delving into a different aspect of the geometry of $c_0(\Gamma)$ we get the next result.

Theorem 3. *Suppose X is such that $\partial_e X_1^* \cup \{0\}$ is ω^*-closed. Suppose X has either* (a) *Dunford and Pettis property and ω^* and weak topologies agree on $S(X^*) = \{x^* : \|x^*\| = 1\}$ or* (b) *ω^* and norm topologies agree on $S(X^*)$, then X has condition (*) with extreme points.*

Proof. It is easy to see that under either of the conditions, every separable subspace has a separable dual. Hence by Theorem 6 on page 230 of [5], X_1^* is ω^*-sequentially compact.

If X has (a), let $x_n^* \in \partial_e X_1^*$, $x_n^*(x_n) = \|x_n\| = 1$ and $x_n \xrightarrow{w} 0$. By hypothesis if $x_n^* \xrightarrow{w^*} x^* \neq 0$, then $x^* \in \partial_e X_1^*$. Hence $x_n^* \xrightarrow{w} x^*$. But this contradicts the fact that X has the Dunford and Pettis property. Therefore $x_n^* \xrightarrow{w^*} 0$. If X has (b) then again if $x_n^* \xrightarrow{\|\|} x^*$, $x^* \in \partial_e X_1^*$ we get a contradiction to the fact that $x_n^*(x_n) = 1$ so that $x_n^* \xrightarrow{w^*} 0$.

We conclude with an application to operator theory of a stronger form of (*) which is satisfied by the (m_∞) spaces.

For any Banach space X, Y let $\mathscr{K}(X, Y)$ and $\mathscr{L}(X, Y)$ denote the space of compact and bounded operators respectively. For any $T \in \mathscr{L}(X, Y)$, the essential norm of T, $\|T\|_e$ is defined by $\|T\|_e = d(T, \mathscr{K}(X, Y))$.

It can be deduced from the observations made by Axler *et al* [2] that

1. $\|T\|_e \geq \|T^*\|_e$,
2. $\|T\|_e = \|T^*\|_e$ if Y is a dual space.

The following formula for the essential norm when X is a Banach space such that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$ was given by Werner in [14].

$$\|T\|_e = \max\{\omega(T), w^*(T)\},$$

where

$$w(T) = \sup\{\overline{\lim}_\alpha \|T(x_\alpha)\| : \{x_\alpha\} \text{ is a net in } X, \|x_\alpha\| = 1 \ \forall\alpha \text{ and } x_\alpha \xrightarrow{w} 0\},$$

$$w^*(T) = \sup\{\overline{\lim} \|T^*(y_\alpha^*)\| : \{y_\alpha^*\} \text{ is a net in } Y^*, \|y_\alpha^*\| = 1 \ \forall\alpha \text{ and } y_\alpha^* \xrightarrow{w} 0.\}$$

and at least one of the involved suprema is actually attained.

First we prove a general proposition regarding these weights $w(T), w^*(T)$.

PROPOSITION 6

Suppose $\|T\|_e = \overline{\lim}_\alpha \|T(x_\alpha)\|$ for a net $\{x_\alpha\}$ of unit vectors in X such that $x_\alpha \xrightarrow{w} 0$, then $\|T\|_e = \|T^*\|_e$.

Proof. For any $S \in \mathscr{K}(Y^*, X^*)$, since S^* is a compact operator, $S^*(x_\alpha) \longrightarrow 0$ in the norm.

Thus $\|T^* - S\| = \|T^{**} - S^*\| \geq \overline{\lim} \|T(x_\alpha)\| = \|T\|_e$. Hence $\|T\|_e \leq \|T^*\|_e$ and therefore $\|T\|_e = \|T^*\|_e$

Now we are ready to formulate a stronger form of condition (*). As before X is a Banach space failing the Schur property.

(B) DEFINITION

If $\{x_n\}$ is a sequence of vectors in the unit ball of X such that $x_n \xrightarrow{\omega} 0$ and $\overline{\lim}_n x_n^*(x_n) = 1$ for a sequence of unit vectors $\{x_n^*\}$ in X^*, then $x_n^* \xrightarrow{\omega^*} 0$.

Theorem 4. *Let X be a reflexive Banach space such that X^* satisfies the above property and Y be a separable Banach space such that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$. Then $\|T\|_e = \|T^*\|_e \; \forall T \in \mathscr{L}(X, Y)$.*

Proof. Since $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$ we know that

$$\|T\|_e = \overline{\lim}_\alpha \|T(x_\alpha)\| \quad \text{for a net of unit vectors} \quad x_\alpha \xrightarrow{w} 0$$

or

$$\|T\|_e = \overline{\lim}_\alpha \|T^*(y_\alpha^*)\| \quad \text{for a net of unit vectors} \quad y_\alpha^* \xrightarrow{w^*} 0.$$

In view of the above Proposition, our theorem is proved if the first case happens. In the second case, again using the hypothesis that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$, $\|T\|_e = \|T - S\|$ for some $S \in \mathscr{K}(X, Y)$. So we may assume w.l.o.g that $\|T\| = 1 = \|T\|_e$.

Since X is reflexive and Y is separable we get

$$1 = \overline{\lim}_n T^*(y_n^*)(x_n) = \overline{\lim}_n y_n^*(T(x_n))$$

for a sequence of unit vectors $\{y_n^*\}$ in Y with $y_n^* \xrightarrow{w^*} 0$ (see the proof of lemma 7 in [14]) and sequence $\{x_n\}$ of unit vectors in X.

Since $\|T\| = 1$, $T^*(y_n^*)$ is a sequence in the unit ball of X. Also $T^*(y_n^*) \xrightarrow{w} 0$. By our hypothesis on X we get that $x_n \xrightarrow{w} 0$.

Also

$$\overline{\lim} \|T(x_n)\| = 1.$$

Hence by applying the above Proposition again we get $\|T\|_e = \|T^*\|_e$.

Remark. It is easy to see that a suitable version of (B) can be a imposed on Y to derive the same conclusion when $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$.

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Intrinsic geometry of curves and the Bonnor's equation

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Abstract. The exact form of the paths of charged particles moving according to Bonnor's equation, for the case of constant electromagnetic field, is found. The method employed is fully relativistic and is based on systematic use of the Frenet–Serret equations which determine the world line geometry in Minkowskian space.

Keywords. Frenet–Serret formulae; charged particle path; constant electromagnetic field; Bonnor equation.

1. Introduction

An equation to describe the motion of charged macroscopic particles has been proposed by Bonnor [1], which intends to replace the Lorentz–Dirac equation [2, 3, 7] in order to eliminate the self-accelerated solutions and also the causality violations of this last equation [4, 9]. It is assumed that the rest mass of the particle decreases as a consequence of the lost energy by radiation in the Bonnor equation.

Here two main purposes of the Bonnor's equation are raised simultaneously: first, to determine the path for a charged particle under the action of a constant electromagnetic field, and secondly, to set the conditions to be satisfied by such field in order to be in agreement with the found curve. To do that, one makes use of the Frenet–Serret formulae [8, 10] for the world line of the charge in Minkowski space-time system and of the antisymmetric tensor properties [5, 6, 11] in this system, together with the particle motion equation describing its interaction with the external field. It is shown in general that half of the components for the electromagnetic field tensor may be determined, and that the Bonnor equation can be integrated at all when such tensor is a constant.

2. Frenet–Serret formulae for a time-like world line

In special relativity, the 4-space is flat and hence the complete coordinate system $(x^r) = (ct, x, y, z), r = 0, \ldots, 3$, is such that the squared distance between the events x^r and $x^r + \mathrm{d}x^r$ is given by the invariant (c = light velocity in the vacuum)

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b, \tag{1}$$

where we assume the dummy indices convention and η_{ab} is the Minkowski metric tensor

$$\eta_{rc} = \begin{cases} 1 \text{ if } r = c = 0 \\ -1 \text{ if } r = c = 1, 2, 3 \\ 0 \text{ if } r \neq c. \end{cases} \tag{2}$$

η_{rc} allows us to distinguish a four-vector A^r when its inner product is performed [10]

$$\eta_{rc}A^rA^c \text{ is } \begin{cases} >0: A^r \text{ is time-like} \\ <0: A^r \text{ is space-like} \\ 0: A^r \text{ is null.} \end{cases} \tag{3}$$

The four-vectors which are time-like and null, are the possible tangents to the world lines of the particles with non-zero mass and massless photons, respectively. Further, the four-vectors A^r and B^r are orthogonal to each other by definition if $\eta_{rj}A^rB^j = 0$. In this sense, we recall [11] that every 4-vector being orthogonal to a time-like one, must be space-like.

The systematic treatment of curved paths is conducted in a better way through the Frenet–Serret formulae [8, 10]. Either in a flat or a curved space-time, these equations make use of a main orthonormal tetrad formed by the tangent to the world line plus three normal lines to this tangent with three associated curvatures. Then the world line is determined by means of the Frenet–Serret equations in the case when the main tetrad is initially specified and the corresponding curvatures are given as functions of the interval s on the path.

A "particle" in special relativity means a curve $x^r(s)$ with a time-like unitary 4-vector $T^i(s)$ defined by

$$T^r = \frac{dx^r}{ds} \quad \text{with} \quad T_rT^r = \eta_{ab}T^aT^b = 1. \tag{4}$$

Therefore one has three space-like unitary normals $\underset{r}{N^j}(s)$, that is (there is no sum over r)

$$T_j\underset{r}{N^j} = 0, \quad \underset{r}{N_a}\underset{r}{N^a} = -1, \quad r = 1,2,3 \tag{5}$$

which together with the tangent line give rise to the main tetrad $(T^r, \underset{1}{N^r}, \underset{2}{N^r}, \underset{3}{N^r})$.

The development (translation + rotation) of the tetrad on the path $x^r(s)$ can be described through the Frenet–Serret formulae [8, 10]

$$\frac{dT^r}{ds} = k_1\underset{1}{N^r}, \tag{6}$$

$$\frac{d\underset{1}{N^r}}{ds} = k_1T^r + k_2\underset{2}{N^r}, \tag{7}$$

$$\frac{d\underset{2}{N^r}}{ds} = -k_2\underset{1}{N^r} + k_3\underset{3}{N^r}, \tag{8}$$

$$\frac{d\underset{3}{N^r}}{ds} = -k_3\underset{2}{N^r}, \tag{9}$$

where $k_i(s) > 0$, $i = 1,2,3$ are the curvatures of the world line.

Now we emphasize the following remarks:

1. If k_1, k_2 and k_3 have zeros on the path, then we let these scalars to have negative values in order to be sure of the continuity of the 4-vectors $\underset{r}{N^j}$, $j = 1,2,3$.

2. We must pay attention to the degeneration, by giving up the arbitrary elements not defined on the path, see [10]. For instance:

If $dT^r/ds = 0$, then $k_1 = 0$ by (6) and strictly speaking we are free to choose $\overset{1}{N}{}^r$ arbitrarily (exception made of $\overset{1}{N}_c \overset{1}{N}{}^c = -1$ and $\overset{1}{N}_c T^c = 0$); the remaining Frenet–Serret equations are used in turn. This freedom is achieved by taking $d\overset{1}{N}{}^r/ds = 0$, which is always true in a flat space. From here, (7) implies $k_2 = 0$ and $\overset{1}{N}{}^r$ becomes an arbitrary element not defined on the path. It is chosen such that $d\overset{2}{N}{}^r/ds = 0$ and by (8) we get $k_3 = 0$, which combined with (9) to give $d\overset{3}{N}{}^r/ds = 0$. So the main tetrad undergoes parallel transportation on $x^r(s)$.

In conclusion

$$\text{if } k_1 = 0 \text{ then } k_2 = k_3 = 0, \tag{10}$$

and similarly

$$k_2 = 0 \text{ implies } k_3 = 0. \tag{11}$$

Finally, we would stress the equivalence of determining $k_i(s)$ to know the path in space-time. With the functions $k_i(s)$ and the Frenet–Serret equations, we find the required curve. In this way, when the determination and analysis of the world lines for the charged particles are over, we are ready to translate our problem in terms of curvatures.

3. Faraday tensor and Bonnor equation

We consider the path of a non-zero mass particle which is moving in the Minkowski 4-space, and assume the coordinate system x^r where the metric has the form (1)

$$ds = \frac{c\,dt}{\gamma}, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{12}$$

with

$$v^2 = v_x^2 + v_y^2 + v_z^2 \tag{13}$$

and the components of the ordinary 3-velocity defined as

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}. \tag{14}$$

Furthermore, by defining the element $d\tau$ of the proper time for the particle as

$$d\tau = \frac{dt}{\gamma}, \tag{15}$$

one has

$$ds = c\,d\tau, \tag{16}$$

and the elements of the space interval and proper time interval are proportionals through c.

The particle 4-velocity u^r is defined by $u^r = cT^r$, with T^i being the unitary tangent vector to the world line, that is

$$u^r = cT^r = c\frac{dx^r}{ds} = \frac{dx^r}{d\tau} \tag{17}$$

or, by eqs (13–15) we have

$$(u^r) = (\gamma c, \gamma \mathbf{v}). \tag{18}$$

The quantities introduced till now allow us to describe the kinematics of the motion.

On the other hand, the electromagnetic field is denoted by the Faraday antisymmetric tensor F_{ab}, which is written in this coordinate system x^r as [9,11] ($a, b = 0, \ldots, 3$ stand for rows and columns, respectively)

$$[F_{ab}] = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}, \tag{19}$$

$$[F^{ab}] = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}, \tag{20}$$

$$[F^a_b] = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}, \tag{21}$$

where $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ are the electric and magnetic field 3-vectors, respectively.

The Maxwell equations in Gaussian units are

$$\nabla \cdot \mathbf{B} = 0, \tag{22}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{23}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{24}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{25}$$

where ρ and $\mathbf{J} = (J^1, J^2, J^3)$ are the charge and current densities, respectively.

Equations (22), (23) can be cast in terms of F_{ab} as [9,11]

$$F_{ab,c} + F_{bc,a} + F_{ca,b} = 0, \quad a, b, c = 0, \ldots, 3, \tag{26}$$

where $_{,r} = \partial/\partial x^r$. Similarly, (24), (25) are

$$F^{ab}_{\ \ ,a} = \frac{4\pi}{c} J^b \tag{27}$$

with the four-vector $(J^r) = (c\rho, J^1, J^2, J^3)$.

Now, with the help of the Levi–Civita totally antisymmetric tensor

$$\varepsilon^{abcd} = -\varepsilon_{abcd} = \begin{cases} 1 \text{ if } abcd \text{ is an even permutation of } 0123 \\ -1 \text{ if } abcd \text{ is an odd permutation of } 0123 \\ 0 \text{ in other case,} \end{cases} \tag{28}$$

we can define the simple dual of F^{ij} as [5, 6]

$$^*F_{ab} = -{}^*F_{ba} = \tfrac{1}{2}\varepsilon_{abcd}F^{cd} \tag{29}$$

which by (20), (28) is represented as

$$[^*F_{ab}] = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}. \tag{30}$$

This expression in turn with (19), (20) allows us to build up the only independent invariants for the electromagnetic field

$$I_1 \equiv F_{ab}F^{ab} = 2(B^2 - E^2) = 2H\cos\phi, \tag{31}$$

$$I_2 \equiv {}^*F_{ab}F^{ab} = 4\mathbf{E}\cdot\mathbf{B} = 2H\sin\phi, \tag{32}$$

$$H \geqslant 0, \quad 0 \leqslant \phi < 2\pi,$$

where E and B are the electric and magnetic field magnitudes, respectively.

These invariants are the starting points for an algebraic classification of the electromagnetic field [6].

Null case: $H = 0$, then $I_1 = I_2 = 0$, and we call it type C-field.
Non-null case: $H \neq 0$. Then

a) $\cos(\phi/2) = 0$, that is $\phi = \pi$, therefore $I_1 < 0$, $I_2 = 0$ type B-field,
b) $\sin(\phi/2) = 0$, so $\phi = 0$ and $I_1 > 0$, $I_2 = 0$ type D-field,
c) $\sin(\phi/2) \neq 0$, $\cos(\phi/2) \neq 0$, so $I_2 \neq 0$ type A-field.

Hence we conclude that

$$\begin{aligned} &\text{type A-field: } \mathbf{E}\cdot\mathbf{B} \neq 0; \\ &\text{type B-field: } \mathbf{E}\cdot\mathbf{B} = 0, \quad B^2 - E^2 < 0; \\ &\text{type C-field: } \mathbf{E}\cdot\mathbf{B} = 0, \quad B^2 - E^2 = 0; \\ &\text{type D-field: } \mathbf{E}\cdot\mathbf{B} = 0, \quad B^2 - E^2 > 0. \end{aligned} \tag{33}$$

Now we suppose that the interaction between a particle with charge e, and the electromagnetic field, is given by the Bonnor equation [1]

$$m_0\frac{du^r}{d\tau} = \frac{e}{c}F^{ri}u_i + \frac{2e^2}{3c^5}\frac{du^i}{d\tau}\frac{du_i}{d\tau}u^r - \frac{dm_0}{d\tau}u^r \tag{34}$$

or

$$m_0\frac{dT^r}{ds} = aF^{ri}T_i + b\frac{dT^i}{ds}\frac{dT_i}{ds}T^r - \frac{dm_0}{ds}T^r, \tag{35}$$

where m_0 is the rest mass of the macroscopic particle, $a = e/c^2$, $b = (2/3)ea$, and we have used eqs (16), (17).

From Frenet–Serret formula (6), (35) simplifies to

$$\left(\frac{dm_0}{ds}+bk_1^2\right)T^r+k_1m_0N^r=aF^{ri}T_i \tag{36}$$

which together with (5) give us an expression relating k_1 and m_0 fulfilled by every electromagnetic field, that is

$$\left(\frac{dm_0}{ds}+bk_1^2\right)^2-k_1^2m_0^2=\frac{a^2}{c^2}F^i_rF^r_du_iu^d \tag{37}$$

$$=-\frac{\gamma^2}{c^6}[c^2\mathbf{f}^2-(\mathbf{f}\cdot\mathbf{v})^2] \quad \text{by (18),(19),(21)} \tag{38}$$

where the Lorentz force under the particle with 3-velocity $\mathbf{v}$ and charge e appears naturally i.e.

$$\mathbf{f}=e\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right). \tag{39}$$

Furthermore, if we make use of the Schwarz inequality in 3-space, we have

$$(\mathbf{f}\cdot\mathbf{v})^2\leqslant v^2\mathbf{f}^2<c^2\mathbf{f}^2 \quad \text{since} \quad v<c.$$

Therefore

$$c^2\mathbf{f}^2-(\mathbf{f}\cdot\mathbf{v})^2>0 \quad \text{for} \quad \mathbf{f}\neq\mathbf{0}. \tag{40}$$

In this way, because of (38), (40), we arrive at

$$\left(\frac{dm_0}{ds}+bk_1^2\right)^2-k_1^2m_0^2=0 \quad \text{if and only if } \mathbf{f}=\mathbf{0}. \tag{41}$$

in other words, if we have an event x^r with $\mathbf{f}=\mathbf{0}$, that is

$$\mathbf{E}=-\frac{1}{c}\mathbf{v}\times\mathbf{B} \tag{42}$$

it means that in such an event certainly

$$\frac{dm_0}{ds}+bk_1^2=\pm k_1m_0. \tag{43}$$

Notice that we have not required F_{ab} to be a constant.

From (42) we also see that $\mathbf{E}\cdot\mathbf{B}=0$ and $E<B$ since $v<c$. Hence

only for a D-field type, we have (43) valid. (44)

It makes sense to ask what kind of fields permit the motion of a particle with $m_0=\text{constant}=m_0(0)$.

In this case (43) gives $k_1(bk_1\mp m_0)=0$, and from (10)

$$k_1=k_2=k_3=0 \tag{45}$$

or

$$k_1 = \pm \frac{m_0}{b} = \text{constant.} \tag{46}$$

The first possibility leads to a straight line in space-time, or better, to a motion with the 3-velocity **v** constant. We have such a situation if

$$\mathbf{E} = \mathbf{B} = \mathbf{0} \text{ and (42) is verified,} \tag{47}$$

$$\mathbf{E} = \mathbf{0} \text{ and } \mathbf{v} \parallel \mathbf{B}, \text{ and (42) is again fulfilled.} \tag{48}$$

Later on, we will show that (46) actually does not appear. The expressions (40–43) cannot avoid this possibility, since (37–43) have less content than (36).

The essential aim is to study the Bonnor equation (36), and to do that we should analyze the behavior of F_{ab} in the main tetrad. This is done in the next section.

4. The Frenet–Serret tetrad and the Faraday tensor

The main tetrad $(T^r, \underset{1}{N}{}^r, \underset{2}{N}{}^r, \underset{3}{N}{}^r)$ has positive direction

$$\varepsilon_{abcd} T^a \underset{1}{N}{}^b \underset{2}{N}{}^c \underset{3}{N}{}^d = 1. \tag{49}$$

It helps us to construct the following antisymmetric basis from which the electromagnetic field tensor is expanded

$$\underset{r}{W}{}^{ab} = T^a \underset{r}{N}{}^b - T^b \underset{r}{N}{}^a, \quad r = 1, 2, 3,$$

$$\underset{4}{W}{}^{ab} = \underset{1}{N}{}^a \underset{2}{N}{}^b - \underset{1}{N}{}^b \underset{2}{N}{}^a, \quad \underset{5}{W}{}^{ab} = \underset{1}{N}{}^a \underset{3}{N}{}^b - \underset{1}{N}{}^b \underset{3}{N}{}^a,$$

$$\underset{6}{W}{}^{ab} = \underset{2}{N}{}^a \underset{3}{N}{}^b - \underset{2}{N}{}^b \underset{3}{N}{}^a. \tag{50}$$

This tensor obeys the algebraic properties (without sum over r)

$$\underset{r}{W}_{ij} \underset{r}{W}{}^{ij} = 2(-1)^n, \quad r = 1, 2, 3, \quad n = 1; \quad r = 4, 5, 6, \quad n = 2$$

$$\underset{r}{W}_{ij} \underset{b}{W}{}^{ij} = 0, \quad \text{if } b \neq r \tag{51}$$

which arise from (4), (5), (50).

We define the dual to $\underset{r}{W}{}^{ij}$ as

$$*\underset{r}{W}_{ij} = \tfrac{1}{2} \varepsilon_{ijab} \underset{r}{W}{}^{ab}, \quad r = 1, 2, \ldots, 6 \tag{52}$$

such that

$$*\underset{r}{W}_{ab} \underset{r}{W}{}^{ab} = 0, \quad r = 1, 2, \ldots, 6,$$

$$*\underset{r}{W}_{ab} \underset{i}{W}{}^{ab} = \underset{r}{W}_{ab} *\underset{i}{W}{}^{ab} = 0 \text{ except}$$

$$*\underset{1}{W}_{ab} \underset{6}{W}{}^{ab} = - *\underset{2}{W}_{ab} \underset{5}{W}{}^{ab} = *\underset{3}{W}_{ab} \underset{4}{W}{}^{ab} = 2. \tag{53}$$

By (6)–(9), (50) we obtain the differential properties of the tensor $\underset{i}{W^{jk}}$ as

$$\frac{d}{ds}\underset{1}{W^{br}} = k_2 \underset{2}{W^{br}},$$

$$\frac{d}{ds}\underset{2}{W^{br}} = k_1 \underset{4}{W^{br}} - k_2 \underset{1}{W^{br}} + k_3 \underset{3}{W^{br}},$$

$$\frac{d}{ds}\underset{3}{W^{br}} = k_1 \underset{5}{W^{br}} - k_3 \underset{2}{W^{br}},$$

$$\frac{d}{ds}\underset{4}{W^{br}} = k_1 \underset{2}{W^{br}} + k_3 \underset{5}{W^{br}},$$

$$\frac{d}{ds}\underset{5}{W^{br}} = k_1 \underset{3}{W^{br}} + k_2 \underset{6}{W^{br}} - k_3 \underset{4}{W^{br}},$$

$$\frac{d}{ds}\underset{6}{W^{br}} = -k_2 \underset{5}{W^{br}}. \tag{54}$$

It is clear that relations (54) are also valid when we replace $\underset{r}{W^{ab}}$ by $*\underset{r}{W^{ab}}$ since the Levi–Civita tensor is a constant. We also stress that (54) are true for any trajectory.

The basis $\underset{r}{W^{ij}}$ contains a number of 6 elements, that is, the same number of independent components of F_{ab}. Let p_j, $j = 1, \ldots, 6$ be the real components of F_{ir} in this basis. Then

$$F^{ab} = \sum_{r=1}^{6} p_r \underset{r}{W^{ab}}. \tag{55}$$

If the path is given, to determine F^{ir} is equivalent to know p_j. In the same way

$$*F^{ab} = \sum_{r=1}^{6} p_r *\underset{r}{W^{ab}}. \tag{56}$$

Therefore, the invariants of the electromagnetic field become, from (31), (32)

$$I_1 = 2(p_4^2 + p_5^2 + p_6^2 - p_1^2 - p_2^2 - p_3^2) = 2(B^2 - E^2),$$

$$I_2 = 4(p_1 p_6 + p_3 p_4 - p_2 p_5) = 4\mathbf{E}\cdot\mathbf{B}, \tag{57}$$

and by the use of (54), (55), one sees how F^{ij} is changing over the world line

$$\begin{aligned}\frac{d}{ds}F^{ab} = &\left(\frac{dp_1}{ds} - k_2 p_2\right)\underset{1}{W^{ab}} + \left(\frac{dp_2}{ds} + k_1 p_4 + k_2 p_1 - k_3 p_3\right)\underset{2}{W^{ab}} \\ &+ \left(\frac{dp_3}{ds} + k_1 p_5 + k_3 p_2\right)\underset{3}{W^{ab}} + \left(\frac{dp_4}{ds} + k_1 p_2 - k_3 p_5\right)\underset{4}{W^{ab}} \\ &+ \left(\frac{dp_5}{ds} + k_1 p_3 - k_2 p_6 + k_3 p_4\right)\underset{5}{W^{ab}} + \left(\frac{dp_6}{ds} + k_2 p_5\right)\underset{6}{W^{ab}}.\end{aligned} \tag{58}$$

If F^{ab} is constant, then in (58) each term in the parentheses is zero.

When (50), (55) and (56) are substituted in the Bonnor equation (36), we find

$$\frac{dm_0}{ds} + bk_1^2 = 0, \tag{59}$$

$$k_1 m_0 + ap_1 = 0, \tag{60}$$

$$p_2 = p_3 = 0, \tag{61}$$

which are fulfilled for every curve under an arbitrary electromagnetic field. So, (36) involves explicitly to p_r, $r = 1,2,3$, that is, the motion equation does not tell us any information about p_4, p_5 and p_6, such that these coefficients remain arbitrary. So

$$F^{ab} = -\frac{k_1 m_0}{a} \overset{1}{W}{}^{ab} + \sum_{r=4}^{6} p_r \overset{r}{W}{}^{ab} \tag{62}$$

and on the other hand the invariants (57) are

$$B^2 - E^2 = p_4^2 + p_5^2 + p_6^2 - \frac{k_1 m_0^2}{a^2}, \tag{63}$$

$$\mathbf{E}\cdot\mathbf{B} = -\frac{k_1 p_6}{a} m_0. \tag{64}$$

Hence we realize that when $k_1 = 0$ in an event, then

$$B^2 - E^2 > 0 \quad \text{and} \quad \mathbf{E}\cdot\mathbf{B} = 0 \quad \text{type D} \tag{65}$$

which coincides with (44), (45). Notice that if the field is absent, $p_1 = 0$ and (60) gives $k_1 = 0$, and from here $k_2 = k_3 = 0$ by (10). The trajectory then has constant 3-velocity. This means that there never appears the possibility of self-acceleration like in Lorentz–Dirac equation [2, 7, 9].

5. Constant electromagnetic field

Now we study the case when F_{ij} is constant on the world line. Then (58) implies by (59)–(61)

$$\frac{dm_0}{ds} + bk_1^2 = 0, \qquad k_1 m_0 = k_1(0) m_0(0),$$

$$k_1\left(p_4 - \frac{k_2 m_0}{a}\right) = 0, \qquad k_1 p_5 = 0,$$

$$\frac{d}{ds} p_4 - k_3 p_5 = 0, \qquad \frac{d}{ds} p_5 - k_2 p_6 + k_3 p_4 = 0,$$

$$\frac{d}{ds} p_6 + k_2 p_5 = 0, \tag{66}$$

which are a set of seven equations for the seven quantities $m_0, k_1, k_2, k_3, p_4, p_5$ and p_6.

Let us consider two cases:

Case 1. $k_1 = 0$. Then $k_2 = k_3 = 0$ and (60), (66) yield

$$m_0 = m_0(0), \quad p_1 = 0, \quad p_r = p_r(0), \quad r = 4,5,6. \tag{67}$$

This is the same to say that the rest mass of the particle does not change and that the components p_4, p_5 and p_6 remain constants, but on the other hand they are conditioned by the field invariants, see (63), (64)

$$\mathbf{E}\cdot\mathbf{B}=0, \quad B^2-E^2=p_4^2+p_5^2+p_6^2\geqslant 0. \tag{68}$$

Further

$$F^{ab}=\sum_{r=4}^{6} p_r \underset{r}{W}{}^{ab} \tag{69}$$

with $\underset{r}{W}{}^{ab}=\underset{r}{W}{}^{ab}(0)$ because the main tetrad is constant and the path is a straight line in Minkowski space

$$x^r(s)=x^r(0)+T^r(0)s. \tag{70}$$

There exist two possibilities:

a) If $p_4=p_5=p_6=0$, then $F^{ab}=0$ and the particle is free.
b) If $p_4^2+p_5^2+p_6^2\neq 0$ we get the type D-field.

If we wish to consider this possibility in a deeper way, one has to choose properly the coordinate axes, such that

$$\mathbf{E}=(E_x,0,0), \quad \mathbf{B}=(0,B_y,0) \quad \text{with } B_y>E_x \tag{71}$$

according to (68). Therefore (20) simplifies to

$$F^{ab}=\begin{bmatrix} 0 & -E_x & 0 & 0 \\ E_x & 0 & 0 & B_y \\ 0 & 0 & 0 & 0 \\ 0 & -B_y & 0 & 0 \end{bmatrix}. \tag{72}$$

By substitution of (72) in (69) one finds p_4, p_5 and p_6, as functions of E_x, B_y and $(T^r(0), \underset{1}{N}{}^r(0), \underset{2}{N}{}^r(0), \underset{3}{N}{}^r(0))$, which in turn are already known in a given problem. That is, we must always give the initial motion conditions of the particle, or better, its tetrad at $s=0$, although we have to say that not necessarily any tetrad choice will lead us to a straight line. In other words, how do we "shoot" the particle in the field (72) to generate the path (70)? The answer to this question comes from (36) which in this case yields

$$F^{ri}T_i=0 \tag{73}$$

which together with (72) lead to the restrictions

$$B_y T^1=0, \quad E_x T^1=0, \quad B_y T^3-E_x T^0=0. \tag{74}$$

But $B_y\neq 0$ from (71). Therefore

$$\mathbf{T}=(T^0(0),0,T^2(0),\frac{E_x}{B_y}T^0(0))=\left(\gamma_0,0,T^2(0),\frac{E_x}{B_y}\gamma_0\right), \tag{75}$$

where $\gamma_0=(1-(v_0^2/c^2))^{-1/2}$ by (18), and since T^r is unitary, we have

$$T^2(0)=\pm\gamma_0\left(\frac{v_0^2}{c^2}-\frac{E_x^2}{B_y^2}\right)^{1/2}. \tag{76}$$

So (75), (76) allow us to obtain the components of the ordinary velocity

$$v_{x_0} = 0,\ v_{y_0} = \pm\left(v_0^2 - \frac{E_x^2}{B_y^2}c^2\right)^{1/2},\quad v_{z_0} = \frac{E_x}{B_y}c, \tag{77}$$

such that the initial velocity must fulfill

$$c > v_0 \geqslant \frac{E_x}{B_y}c \tag{78}$$

and has to be directed as dictated by (77) in order that the path be a straight line in the field (72). This result is not found in the paper of Bonnor [1]. For the particular case $E_x = 0$, (77) implies $v_{x_0} = v_{z_0} = 0$ and $v_{y_0} = v_0$ is arbitrary but less than c, that is, the particle must be going off parallel to the electromagnetic field $\mathbf{B} = (0, B_y, 0)$ as we hoped.

The expressions (75), (76) do restrict to $\mathbf{T}(0)$, however we can choose $\underset{r}{\mathbf{N}}(0)(r = 1, 2, 3)$ arbitrarily, but not so the orthonormality conditions (5).

Case 2. $k_1 \neq 0$. Since the first curvature is different from zero at every point of the world line, (66) give

$$\frac{dm_0}{ds} + bk_1^2 = 0, \tag{79}$$

$$k_1 m_0 = k_1(0)m_0(0), \tag{80}$$

$$p_4 = \frac{k_2 m_0}{a}, \tag{81}$$

$$p_5 = 0, \tag{82}$$

$$p_4 = p_4(0) \text{ and } p_6 = p_6(0), \tag{83}$$

$$k_3 p_4 - k_2 p_6 = 0, \tag{84}$$

and the invariants (63), (64) become

$$\frac{1}{2}I_1 = p_4^2 + p_6^2 - \frac{k_1^2(0)m_0^2(0)}{a^2} \quad \text{and} \quad \frac{1}{4}I_2 = -\frac{k_1(0)m_0(0)}{a}p_6, \tag{85}$$

from which

$$p_6 = -\frac{aI_2}{4k_1(0)m_0(0)}, \tag{86}$$

$$p_4 = \left[\frac{I_1}{2} + \frac{k_1^2(0)m_0^2(0)}{a^2} - \frac{a^2 I_2^2}{16k_1^2(0)m_0^2(0)}\right]^{1/2}, \tag{87}$$

and we also recall that, by (60), (80)

$$p_1 = -\frac{k_1(0)m_0(0)}{a}. \tag{88}$$

In this way we have found all the components of the electromagnetic field tensor

$$F^{ab} = p_1 \underset{1}{W}^{ab} + p_4 \underset{4}{W}^{ab} + p_6 \underset{6}{W}^{ab} \tag{89}$$

which will be completely specified once $k_r(s)$, $r = 1, 2, 3$ and the Frenet–Serret tetrad are determined. But first of all an important remark has to be made. In a given problem, the known quantities are $\mathbf{E}$, $\mathbf{B}$, $x^r(0)$, $\mathbf{v}_0$ and $m_0(0)$. Nevertheless, $k_1(0)$ is not given by us directly, but rather it has to be calculated from the previous quantities by means of (38), (79)

$$k_1^2(0)m_0^2(0) = \frac{\gamma_0^2}{c^6}[c^2\mathbf{f}_0^2 - (\mathbf{f}_0\cdot\mathbf{v}_0)^2], \tag{90}$$

where $\mathbf{f}_0 = e(\mathbf{E} + \frac{1}{c}\mathbf{v}_0 \times \mathbf{B})$ is the initial Lorentz force.

On the other hand, p_4 is real such that the radicand of (87) must be positive or zero, thus imposing a restriction on $k_1(0)$, that is

$$k_1^2(0)m_0^2(0) \geqslant \frac{a^2}{4}(-I_1 + \sqrt{I_1^2 + I_2^2}) = a^2 H \sin\frac{\phi}{2} \quad \text{by (31), (32).} \tag{91}$$

It is clear that (91) fulfills for both the null case ($H = 0$) and type D ($\sin\phi/2 = 0$) fields. But it is natural to ask oneself if (91) is always true in every situation when $k_1(0)$ is given by (90). The answer is yes, but we shall not give the proof here.

Now we find $k_1(s)$ and $m_0(s)$. From (80) we have

$$k_1 = \frac{k_1(0)m_0(0)}{m_0} \tag{92}$$

which in (79) gives

$$m_0(s) = m_0(0)\left[1 - \frac{2e^2k_1^2(0)}{c^2m_0(0)}s\right]^{1/3}. \tag{93}$$

This equation gives us the exact law for the decrease of the particle rest mass. We note that $m_0(s) \to 0$ as $s \to c^2m_0(0)/2e^2k_1^2(0)$, that is, the rest mass goes to zero for long proper times; that is why we do not "feel" the decrease of m_0 with its motion, when macroscopic particles (described by the Bonnor equation) are considered. From (80), (81) we see that k_1 and k_2 go to infinity when $m_0 \to 0$, and the particle decays as a helix.

From (81) we obtain the second curvature

$$k_2 = \frac{ap_4}{m_0} \tag{94}$$

with p_4 and m_0 given by (87) and (93), respectively.

By using (84), k_3 can be got according to the following two cases:

Case 1: $p_4 = 0$. Hence $k_2 = 0$ by (94) and $k_3 = 0$ due to (1), so that only k_1 remains different from zero because of (92), (93).

Case 2: $p_4 \neq 0$. Therefore, (84) allows to get the third curvature

$$k_3 = \frac{p_6}{p_4}k_2 = a\frac{p_6}{m_0(s)} = -\frac{a^2I_2}{4k_1(0)m_0(0)m_0(s)} \tag{95}$$

with m_0 given by (93). Then we see that for type A field ($I_2 \neq 0$) one has three curvatures different from zero as given by (92), (94), (95). However, for the types B, C and D ($I_2 = 0$) fields, it always happen that $k_3 = 0$, $p_6 = 0$ and the value of p_4 as given by (87) will depend on I_1. The tetrad can be obtained through quadratures from (6)–(9).

With this we culminate the analysis of the solutions to the Bonnor equation when the electromagnetic field is a constant. We also hope to have shown explicitly how efficient is the Frenet–Serret method.

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Radial displacements of an infinite liquid saturated porous medium with spherical cavity

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Abstract. The components of radial displacements in solid and liquid parts of a liquid-saturated porous medium with spherical cavity subjected to an arbitrary time dependent force have been obtained. Laplace transform technique has been used to solve the problem. Numerical calculation has been performed for two specific models. Variations of displacement in solid and liquid parts of the medium have been shown graphically.

Keywords. Radial displacement; liquid-saturated porous; spherical cavity; Laplace transform.

1. Introduction

Porous material in view of their importance in numerous applications in the geotechnical engineering, exploration geo-physics and in the prediction of behaviour of sound-absorbing materials have received considerable attention in the literature devoted to mechanics. Propagation of elastic waves in liquid-saturated porous media has been a subject of continued interest due to its importance in seismology and foundation engineering. The constitutive and dynamical equations for fluid-saturated porous solids were formulated by Biot [1, 2]. He demonstrated the existence of two kinds of compressional waves along with one shear wave. The boundary conditions appropriate for the boundaries of liquid-saturated porous solid have been discussed by Deresiewicz and Skalak [5]. In recent years a number of problems regarding wave propagation in poroelastic solid have been studied, e.g., Yew and Wang [15], Sharma *et al* [10, 11].

The problem of propagation of elastic waves from a spherical cavity in an unbounded elastic medium has been discussed by several investigators. Notable among them were Eringen [6], Eason [7], Chakraborty and Roy [4], Blake [3], Vodicka [12], Olsson [9] etc. Wengler [13] studied the propagation of waves from a spherical cavity in an unbounded linear viscoelastic solid.

In spite of these studies no attempt has been made for the problem of radial vibration of an inifinite liquid-saturated porous solid with spherical cavity. In the present investigation, the problem of radial displacement of an unbounded liquid-saturated porous medium due to a spherical cavity, the boundary surface of which is acted upon by time dependent surface force has been discussed.

2. Basic equations

The field equations for the liquid-saturated porous solid in the absence of dissipation, are given by Biot [1, 2] as

$$N\nabla^2\mathbf{u} + \text{grad}\{(D+N)e + Q\varepsilon\} = \frac{\partial^2}{\partial t^2}(\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}), \tag{1}$$

$$\text{grad}\{Qe + R\varepsilon\} = \frac{\partial^2}{\partial t^2}(\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}), \tag{2}$$

where D, N, Q, R are elastic constants for the solid–liquid aggregate; ρ_{11}, ρ_{12}, ρ_{22} ar the dynamical coefficients. $\mathbf{u}$ and $\mathbf{U}$ are the displacements in the solid and liquid part respectively, and

$$e = \text{div}\,\mathbf{u},$$

$$\varepsilon = \text{div}\,\mathbf{U}$$

are the corresponding dilatations.

The stresses in the solid σ_{ij} and in the liquid σ are given by

$$\sigma_{ij} = (De + Q\varepsilon)\delta_{ij} + 2N\varepsilon_{ij}, \tag{3}$$

$$\sigma = Qe + R\varepsilon, \tag{4}$$

where δ_{ij} is the Kronecker delta and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

3. Formulation of the problem

We assume a spherical cavity of radius a in an isotropic, homogeneous, liquic saturated porous solid of an infinite extent. We take the spherical polar co-ordina (r, θ, ϕ) and the origin of the system is assumed at the centre of cavity. The surface spherical cavity is assumed to be acted upon by time dependent pressure $f(t)$. We sha assume that the initial displacements and their corresponding velocities are zer throughout the medium.

As the problem is radially symmetric, the quantities are independent of θ and ϕ an therefore the displacements in the solid and liquid can be written as

$$\mathbf{u} = u(r,t)\hat{e}_r, \tag{5}$$

$$\mathbf{U} = U(r,t)\hat{e}_r. \tag{6}$$

With the help of (5) and (6), eqs (1) and (2) reduces to

$$P\left\{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right\} + Q\left\{\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\left(\frac{\partial U}{\partial r} - \frac{U}{r}\right)\right\} = \frac{\partial^2}{\partial t^2}(\rho_{11}u + \rho_{12}U),$$

$$Q\left\{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right\} + R\left\{\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\left(\frac{\partial U}{\partial r} - \frac{U}{r}\right)\right\} = \frac{\partial^2}{\partial t^2}(\rho_{12}u + \rho_{22}U).$$

As per our assumption

$$\left.\begin{aligned} u(r,0) = u_t(r,0) = 0 \\ U(r,0) = U_t(r,0) = 0 \end{aligned}\right\} \quad r > a,$$

where subscript t denotes derivative with respect to time t; the radiation conditi imply that

$$\lim u(r,t) = \lim U(r,t) = 0, \quad r \to \infty, \quad t > 0; \tag{1}$$

and the boundary conditions for this problem are that the radial stress at the surface of the cavity is equal to the applied force and the stress in the liquid part vanishes. Mathematically, these can be written as

$$\sigma_{rr} = -f(t), \quad r = a, \quad t > 0, \tag{11}$$

$$\sigma = 0, \quad r = a, \quad t > 0, \tag{12}$$

where

$$\sigma_{rr} = P\frac{\partial u}{\partial r} + 2D\frac{u}{r} + Q\left(\frac{\partial U}{\partial r} + 2\frac{U}{r}\right), \tag{13}$$

$$\sigma = Q\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + R\left(\frac{\partial U}{\partial r} + \frac{2U}{r}\right). \tag{14}$$

Let

$$u = \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{1}{r^2}\phi, \tag{15}$$

$$U = \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\psi, \tag{16}$$

where $\phi(r, t)$ and $\psi(r, t)$ are the potentials.

Substituting (15) and (16) in eqs (7) and (8) yields

$$\left(P\frac{\partial^2}{\partial r^2} - \rho_{11}\frac{\partial^2}{\partial t^2}\right)\phi + \left(Q\frac{\partial^2}{\partial r^2} - \rho_{12}\frac{\partial^2}{\partial t^2}\right)\psi = 0, \tag{17}$$

$$\left(Q\frac{\partial^2}{\partial r^2} - \rho_{12}\frac{\partial^2}{\partial t^2}\right)\phi + \left(R\frac{\partial^2}{\partial r^2} - \rho_{22}\frac{\partial^2}{\partial t^2}\right)\psi = 0. \tag{18}$$

The above equations show the coupling between ϕ and ψ. Solving them, we get

$$\left\{A\frac{\partial^4}{\partial r^4} - B\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial r^2} + C\frac{\partial^4}{\partial t^4}\right\}(\phi, \psi) = 0, \tag{19}$$

where

$$A = PR - Q^2, \quad B = P\rho_{22} + R\rho_{11} - 2Q\rho_{12}$$

$$C = \rho_{11}\rho_{22} - \rho_{12}^2, \quad P = D + 2N. \tag{20}$$

4. Solution of the problem

To solve (19), we define the Laplace transform and its inverse as

$$\bar{g}(r, p) = \int_0^\infty e^{-pt} g(r, t)\,dt, \quad g(r, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\bar{g}(r, p)\,dp. \tag{21}$$

Applying the Laplace transform (21) in eq. (19), the solutions of the transformed equations satisfying the radiation condition (10) can be written as

$$\bar{\phi} = A_1 e^{-ps_1 r} + A_2 e^{-ps_2 r}, \tag{22}$$

$$\bar{\psi} = E_1 e^{-ps_1 r} + E_2 e^{-ps_2 r}, \tag{23}$$

where

$$s_1^2 = \frac{1}{\alpha_1^2} = \frac{B - (B^2 - 4AC)^{1/2}}{2A}, \quad s_2^2 = \frac{1}{\alpha_2^2} = \frac{B + (B^2 - 4AC)^{1/2}}{2A} \tag{24}$$

and α_1, α_2 are the velocities of fast P(or P_f) and slow $P(P_s)$ waves respectively; A_1, A_2, E_1 and E_2 are the arbitrary constants.

Now, using the Laplace transform (21) in eqs (17) and (18), and making use of eqs (22) and (23), we obtain

$$E_j = m_j A_j, \quad j = 1, 2, \tag{25}$$

where

$$m_j = \frac{P - \rho_{11}\alpha_j^2}{\rho_{12}\alpha_j^2 - Q} = \frac{Q - \rho_{12}\alpha_j^2}{\rho_{22}\alpha_j^2 - R}, \quad j = 1, 2. \tag{26}$$

Using (15), (16), (21), (22) and (23), we obtain

$$\bar{u} = -\frac{1}{r^2}[A_1(1 + ps_1 r)e^{-ps_1 r} + A_2(1 + ps_2 r)e^{-ps_2 r}], \tag{27}$$

$$\bar{U} = -\frac{1}{r^2}[m_1 A_1(1 + ps_1 r)e^{-ps_1 r} + m_2 A_2(1 + ps_2 r)e^{-ps_2 r}]. \tag{28}$$

Using (21) on boundary conditions (11)–(14) and (27) and (28), we get the following equations

$$\{(P + Qm_1)p^2 s_1^2 a^2 + 4Naps_1 + 4N\}e^{-ps_1 a} A_1 + \{(P + Qm_2)p^2 s_2^2 a^2 + 4Naps_2 + 4N\}e^{-ps_2 a} A_2 = -a^3 \bar{f}(p), \tag{29}$$

$$\{(Q + Rm_1)p^2 s_1^2\}e^{-ps_1 a} A_1 + \{(Q + m_2 R)p^2 s_2^2\}e^{-ps_2 a} A_2 = 0. \tag{30}$$

Solving eq. (29) and (30), we get

$$A_1 = -\frac{a^3 \bar{f}(p)(Q + m_2 R)s_2^2}{\Delta} e^{ps_1 a}, \tag{31}$$

$$A_2 = \frac{a^3 \bar{f}(p)(Q + m_1 R)s_1^2}{\Delta} e^{ps_2 a}, \tag{32}$$

where

$$\Delta = B_1 p^2 + B_2 p + B_3 \tag{33}$$

and

$$B_1 = a^2 C(m_2 - m_1),$$

$$B_2 = 4aN\sqrt{\frac{C}{A}}\{Q(s_2 - s_1) + R(m_2 s_2 - m_1 s_1)\},$$

$$B_3 = 4N\{Q(s_2^2 - s_1^2) + R(s_2^2 m_2 - s_1^2 m_1)\}. \tag{34}$$

Substituting the values of A_1, A_2 from (31) and (32) in eqs (27) and (28), we get

$$\bar{u} = \frac{a^3}{r^2}[\bar{f}(p) \cdot \bar{w}_1(r, p) - \bar{f}(p) \cdot \bar{w}_2(r, p)], \tag{35}$$

$$\bar{U} = \frac{a^3}{r^2}[m_1 \bar{f}(p) \cdot \bar{w}_1(r, p) - m_2 \bar{f}(p) \cdot \bar{w}_2(r, p)], \tag{36}$$

where

$$\bar{w}_1(r,p)=\frac{C_2(1+ps_1r)}{\Delta}e^{-ps_1(r-a)},\quad \bar{w}_2(r,p)=\frac{C_1(1+ps_2r)}{\Delta}e^{-ps_2(r-a)}, \tag{37}$$

and

$$C_j=(Q+m_jR)s_j^2,\quad j=1,2. \tag{38}$$

Making use of convolution theorem on (35) and (36), we obtain

$$u(r,t)=\frac{a^3}{r^2}[f(t)*w_1(r,t)-f(t)*w_2(r,t)], \tag{39}$$

$$U(r,t)=\frac{a^3}{r^2}[m_1f(t)*w_1(r,t)-m_2f(t)*w_2(r,t)], \tag{40}$$

where $*$ denotes convolution and $w_j(r,t), j=1,2$ are the functions whose transforms are $\bar{w}_j(r,p), j=1,2$ respectively.

The solutions $u(r,t)$ and $U(r,t)$ given by (39) and (40) shall be known if $w_1(r,t)$ and $w_2(r,t)$ are known. Thus, our problem reduces to the determination of $w_1(r,t)$ and $w_2(r,t)$.

Applying inverse transform (21) on eq. (37), we get

$$w_1(r,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{p\{t-(r-a)/\alpha_1\}}K_1(r,p)\,dp, \tag{41}$$

where

$$K_1(r,p)=\frac{C_2(1+ps_1r)}{B_1p^2+B_2p+B_3}. \tag{42}$$

It is evident that

$$w_1(r,t)=0,\quad t<\frac{r-a}{\alpha_1}, \tag{43}$$

and following the method of calculus of residues, $w_1(r,t)$ equals to the sum of residues of integrand in (41) at its poles for $t>(r-a)/\alpha_1$. Thus, we obtain

$$w_1(r,t)=\frac{C_2}{S}\left[\frac{rS}{B_1\alpha_1}\cos\theta_1+\frac{2B_1\alpha_1-rB_2}{B_1\alpha_1}\sin\theta_1\right]e^{-(B_2/2B_1)\{t-(r-a)/\alpha_1\}},\quad t>\frac{r-a}{\alpha_1}, \tag{44}$$

where

$$\theta_1=\frac{S}{2B_1}\left(t-\frac{r-a}{\alpha_1}\right). \tag{45}$$

Using (43) and (44), we obtain the expression of $w_1(r,t)$ for all values of t as

$$w_1(r,t)=\frac{C_2}{S}H\left(t-\frac{r-a}{\alpha_1}\right)V_1(r,t)e^{-(B_2/2B_1)\{t-(r-a)/\alpha_1\}}, \tag{46}$$

where

$$V_1(r,t)=\frac{rS}{B_1\alpha_1}\cos\theta_1+\frac{2B_1\alpha_1-rB_2}{B_1\alpha_1}\sin\theta_1, \tag{47}$$

and H is the Heaviside's unit function.

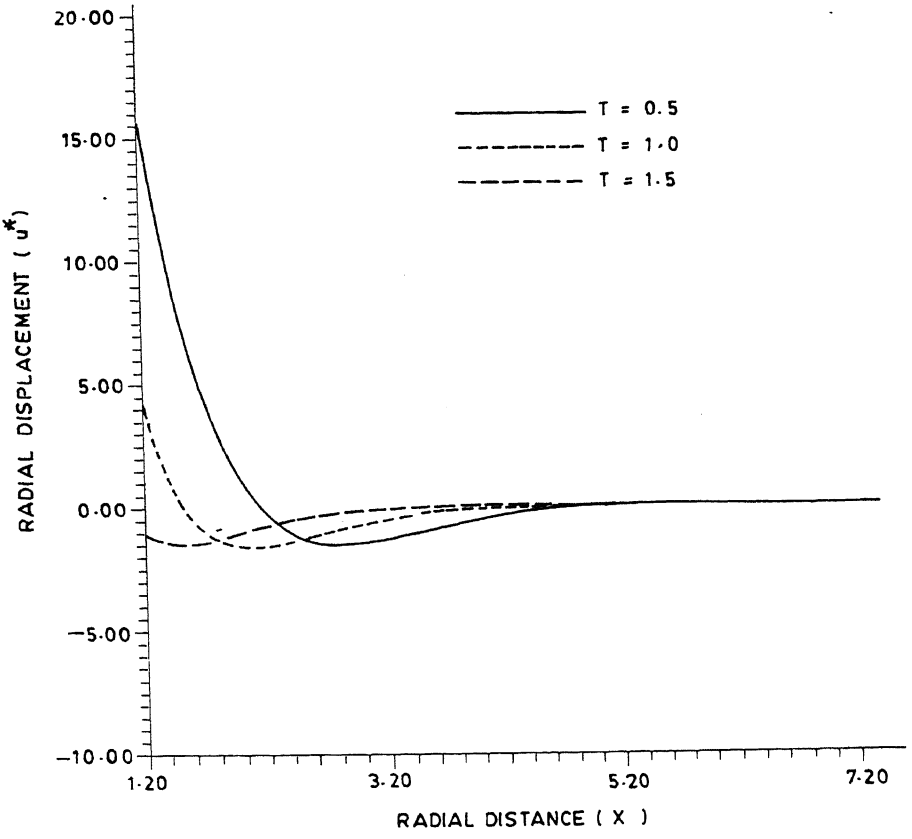

Figure 1. Variation of radial displacement ($u^* = (2Nu/Fa) \times 10^2$) versus radial distance ($X = r/a$).

Similarly, we obtain

$$w_2(r,t) = \frac{C_1}{S} H\left(t - \frac{r-a}{\alpha_2}\right) V_2(r,t) e^{-(B_2/2B_1)\{t-(r-a)/\alpha_2\}}, \tag{48}$$

where

$$V_2(r,t) = \frac{rS}{B_1\alpha_2}\cos\theta_2 + \frac{2B_1\alpha_2 - rB_2}{B_1\alpha_2}\sin\theta_2 \tag{49}$$

and

$$\theta_2 = \frac{S}{2B_1}\left(t - \frac{r-a}{\alpha_2}\right). \tag{50}$$

Substituting (46) and (48) in eqs (39) and (40), using convolution theorem, we get

$$u(r,t) = \frac{a^3}{r^2 S}\Bigg[C_2 e^{(B_2/2B_1)(r-a)/\alpha_1} H\left(t - \frac{r-a}{\alpha_1}\right) \int_{(r-a)/\alpha_1}^{t} e^{-(B_2/2B_1)\tau} f(t-\tau) \times V_1(r,\tau)\,d\tau - C_1 e^{(B_2/2B_1)(r-a)/\alpha_2} H\left(t - \frac{r-a}{\alpha_2}\right) \int_{(r-a)/\alpha_2}^{t} e^{-(B_2/2B_1)\tau} \times f(t-\tau)\, V_2(r,\tau)\,d\tau \Bigg], \tag{51}$$

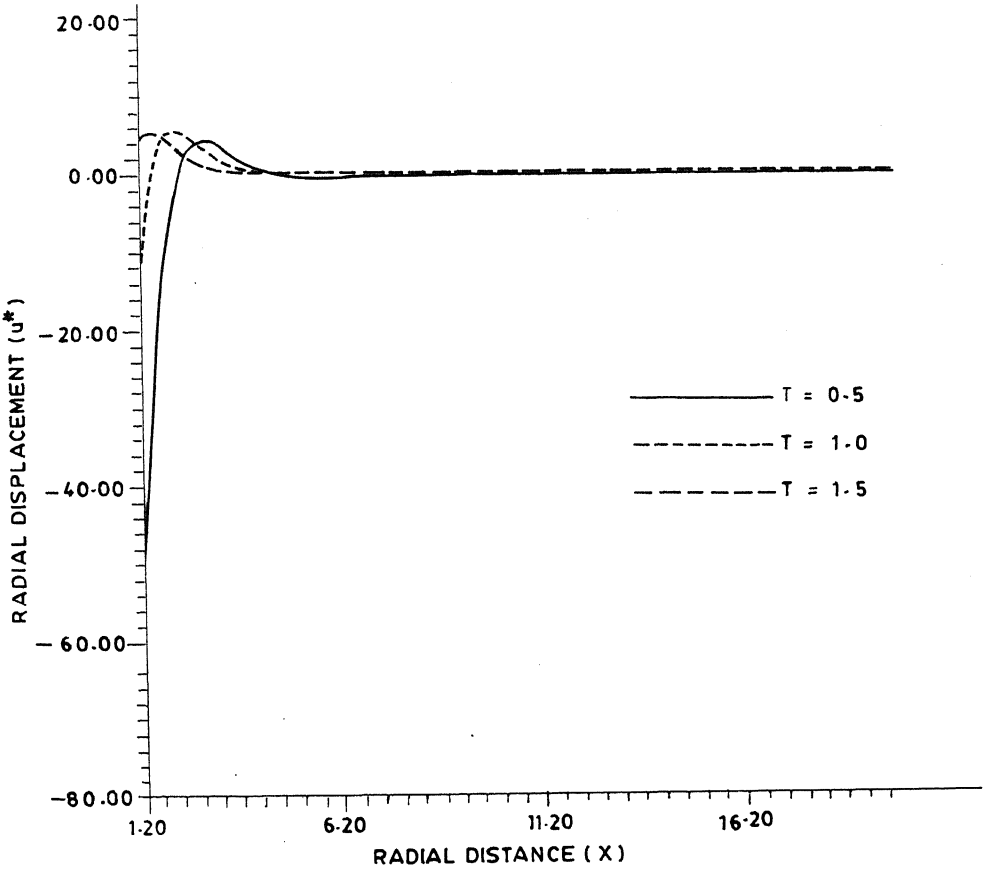

Figure 2. Variation of radial displacement ($U^* = (2Nu/Fa) \times 10^2$) versus radial distance ($X = r/a$).

$$U(r,t) = \frac{a^3}{r^2 S}\Bigg[m_1 C_2 e^{(B_2/2B_1)(r-a)/\alpha_1} H\left(t - \frac{r-a}{\alpha_1}\right) \int_{(r-a)/\alpha_1}^{t} e^{-(B_2/2B_1)\tau} f(t-\tau) \times V_1(r,\tau)\,d\tau - m_2 C_1 e^{(B_2/2B_1)(r-a)/\alpha_2} H\left(t - \frac{r-a}{\alpha_2}\right) \int_{(r-a)/\alpha_2}^{t} e^{-(B_2/2B_1)\tau} \times f(t-\tau)\, V_2(r,\tau)\,d\tau \Bigg]. \tag{52}$$

5. Special case

We consider

$$f(t) = F e^{-(B_2/2B_1)t}, \tag{53}$$

where F is a constant.

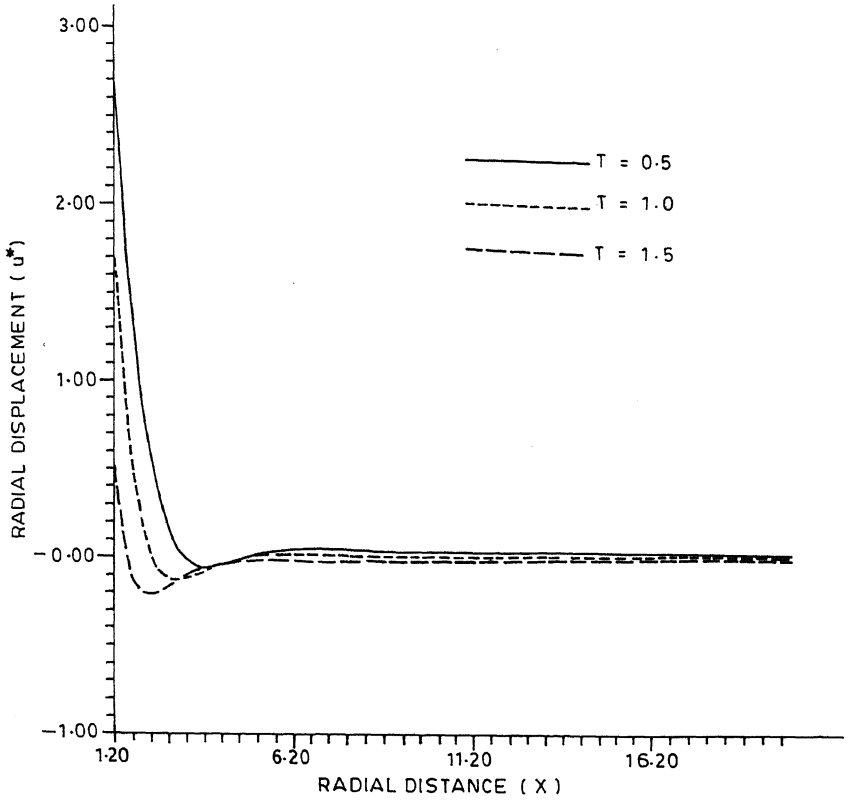

Figure 3. Variation of radial displacement ($u^* = (2Nu/Fa) \times 10$) versus radial distance ($X = r/a$).

Substituting (53) in (51) and (52), we obtain

$$u(r,t) = F\frac{a^3}{r^2 S}\Bigg[C_2 e^{-(B_2/2B_1)\{t-(r-a)/\alpha_1\}} H\left(t-\frac{r-a}{\alpha_1}\right)\int_{(r-a)/\alpha_1}^{t} V_1(r,\tau)\,d\tau$$
$$- C_1 e^{-(B_2/2B_1)\{t-(r-a)/\alpha_2\}} H\left(t-\frac{r-a}{\alpha_2}\right)\int_{(r-a)/\alpha_2}^{t} V_2(r,\tau)\,d\tau\Bigg], \qquad (54)$$

$$U(r,t) = F\frac{a^3}{r^2 S}\Bigg[m_1 C_2 e^{-(B_2/2B_1)\{t-(r-a)/\alpha_1\}} H\left(t-\frac{r-a}{\alpha_1}\right)\int_{(r-a)/\alpha_1}^{t} V_1(r,\tau)\,d\tau$$
$$- m_2 C_1 e^{-(B_2/2B_1)\{t-(r-a)/\alpha_2\}} H\left(t-\frac{r-a}{\alpha_2}\right)\int_{(r-a)/\alpha_2}^{t} V_2(r,\tau)\,d\tau\Bigg]. \qquad (55)$$

Making use of $V_1(r,t)$, $V_2(r,t)$ given by eqs (47) and (49) in eqs (54) and (55); and evaluating the integrals, we get

$$u(r,t) = F\frac{a^3}{r^2 S}\Bigg[C_2 H\left(t-\frac{r-a}{\alpha_1}\right) e^{-(B_2/2B_1)\{t-(r-a)/\alpha_1\}}$$
$$\times\left\{\frac{2r}{\alpha_1}\sin\theta_1 + \frac{2(2B_1\alpha_1 - rB_2)}{\alpha_1 S}(1-\cos\theta_1)\right\}$$

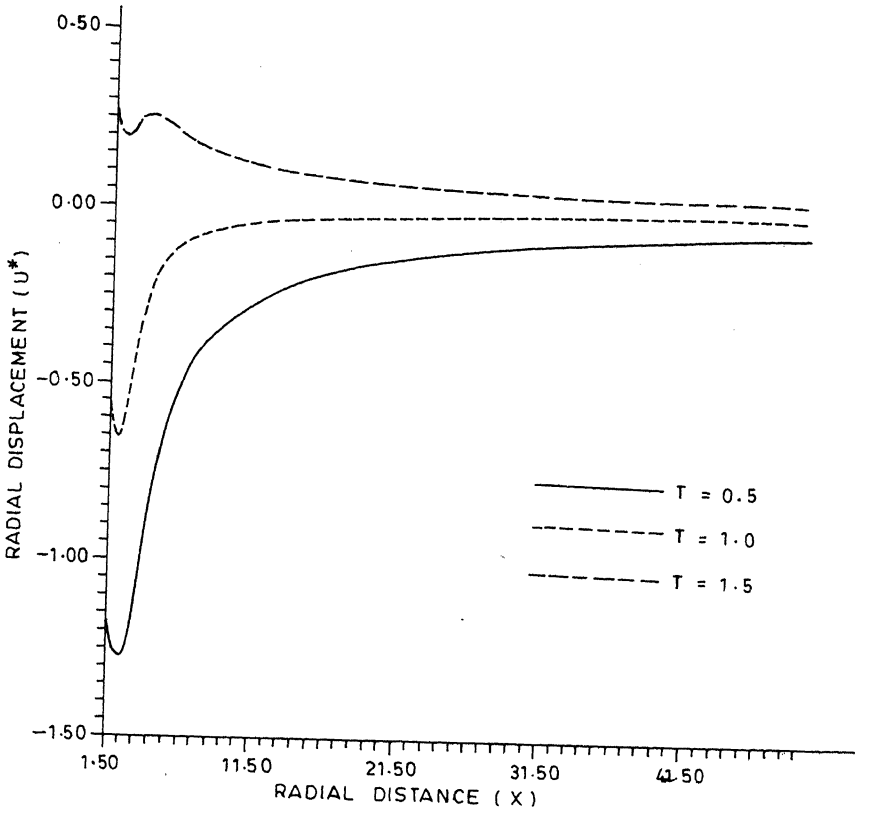

Figure 4. Variation of radial displacement $[U^* = (2Nu/Fa) \times 10]$ versus radial distance $(X = r/a)$.

$$- C_1 H\left(t - \frac{r-a}{\alpha_2}\right) e^{-(B_2/2B_1)\{t-(r-a)/\alpha_2\}} \times \left\{\frac{2r}{\alpha_2}\sin\theta_2 + \frac{2(2B_1\alpha_2 - rB_2)}{\alpha_2 S}(1-\cos\theta_2)\right\}\Bigg], \tag{56}$$

$$U(r,t) = F\frac{a^3}{r^2 S}\Bigg[m_1 C_2 H\left(t - \frac{r-a}{\alpha_1}\right) e^{-(B_2/2B_1)\{t-(r-a)/\alpha_1\}} \times \left\{\frac{2r}{\alpha_1}\sin\theta_1 + \frac{2(2B_1\alpha_1 - rB_2)}{\alpha_1 S}(1-\cos\theta_1)\right\} - m_2 C_1 H\left(t - \frac{r-a}{\alpha_2}\right) e^{-(B_2/2B_1)\{t-(r-a)/\alpha_2\}} \times \left\{\frac{2r}{\alpha_2}\sin\theta_2 + \frac{2(2B_1\alpha_2 - rB_2)}{\alpha_2 S}(1-\cos\theta_2)\right\}\Bigg]. \tag{57}$$

For $Q/R \to 0$, $\rho_{12}/\rho_{22} \to 0$, we obtain the expression for displacement in solid part, $u(r,t)$, as obtained by Vodicka [12], whereas displacement in liquid part, $U(r,t)$, vanishes.

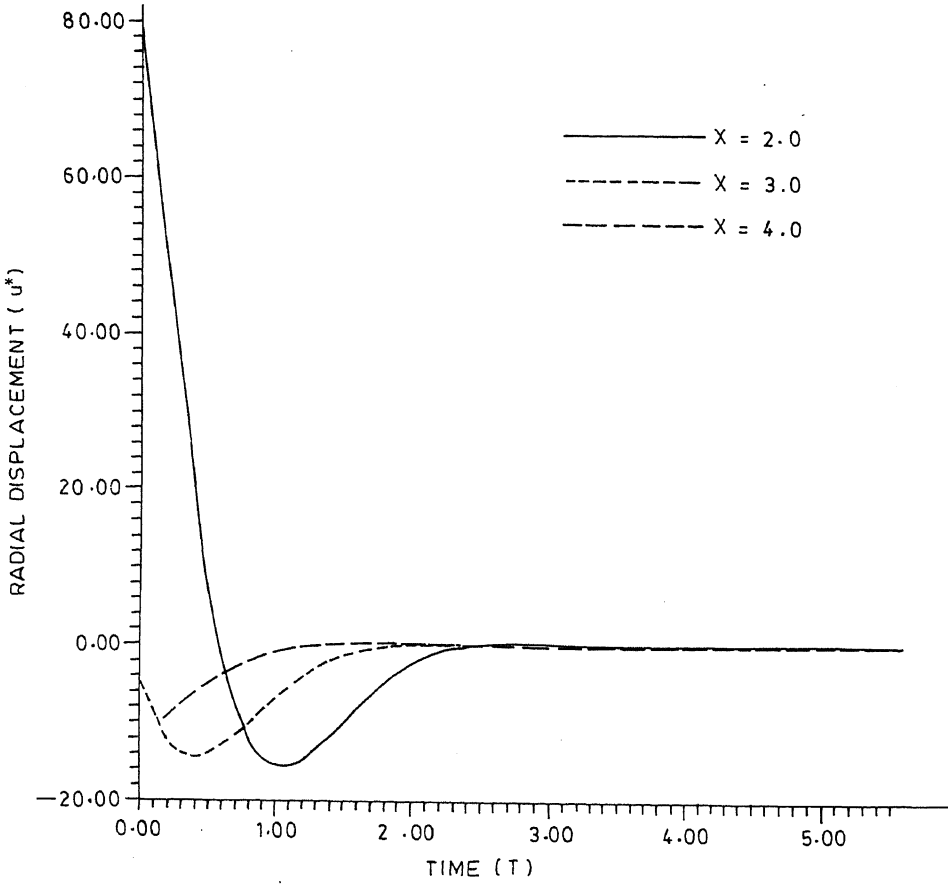

Figure 5. Variation of radial displacement ($u^* = (2Nu/Fa) \times 10^3$) versus time ($T = (\alpha_2 t/a) - (r-a)/a$)).

6. Numerical results and discussion

To study in detail, we compute the radial displacement in solid and liquid parts of a liquid-saturated porous medium, separately. For numerical calculations, we take

$$T = \frac{\alpha_2}{a}\left(t - \frac{r-a}{\alpha_2}\right)$$

and

$$X = \frac{r}{a}.$$

Keeping in view the experimental results given by Yew and Jogi [14] which differ slightly from the experimental results given by Fatt [8] for (i) water saturated sandstone and (ii) kerosene saturated sandstone, we choose the following values of the relevant parameters:

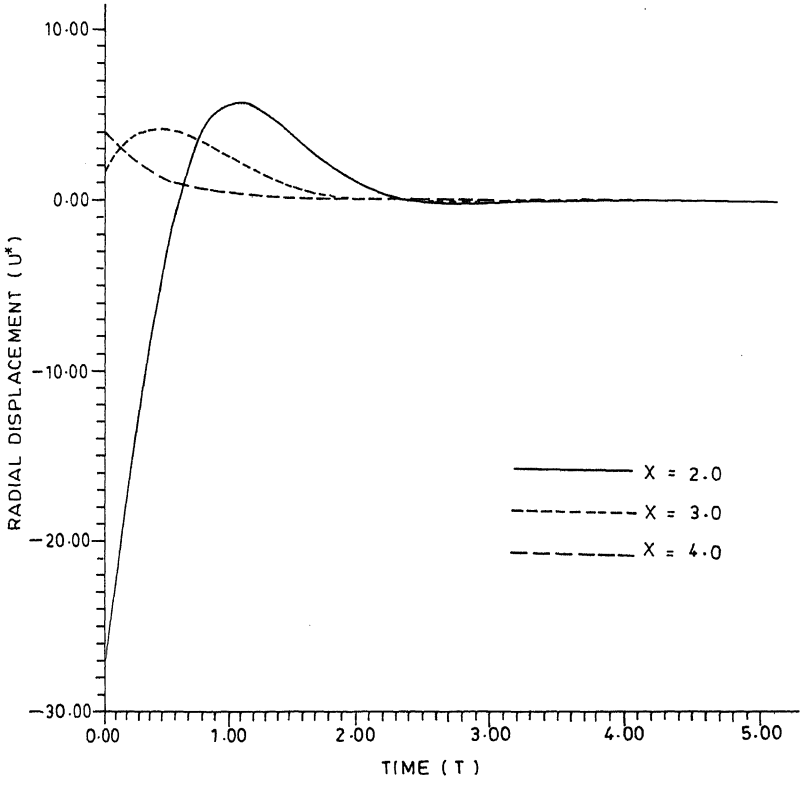

Figure 6. Variation of radial displacement ($U^* = (2Nu/Fa) \times 10^2$) versus time ($T = (\alpha_2 t/a) - (r-a)/a$).

(i) *For water saturated sandstone*:

$$P = 2{\cdot}15 \times 10^{11}\ \text{dyn cm}^{-2},$$

$$Q = 0{\cdot}013 \times 10^{11}\ \text{dyn cm}^{-2},$$

$$R = 0{\cdot}0637 \times 10^{11}\ \text{dyn cm}^{-2},$$

$$N = 0{\cdot}922 \times 10^{11}\ \text{dyn cm}^{-2},$$

$$\rho_{11} = 1{\cdot}9032\ \text{g cm}^{-3},$$

$$\rho_{12} = 0{\cdot}0\ \text{g cm}^{-3},$$

$$\rho_{22} = 0{\cdot}268\ \text{g cm}^{-3}.$$

The velocities of the P_f, P_s waves for the above constants are

$$\alpha_1 = 3{\cdot}326 \times 10^5\ \text{cm s}^{-1},$$

$$\alpha_2 = 1{\cdot}54 \times 10^5\ \text{cm s}^{-1}.$$

(ii) *For kerosene saturated sandstone*:

$$P = 0{\cdot}99663 \times 10^{11}\ \text{dyn cm}^{-2},$$

$$Q = 0{\cdot}07435 \times 10^{11}\ \text{dyn cm}^{-2},$$

$$R = 0{\cdot}03262 \times 10^{11}\ \text{dyn cm}^{-2},$$

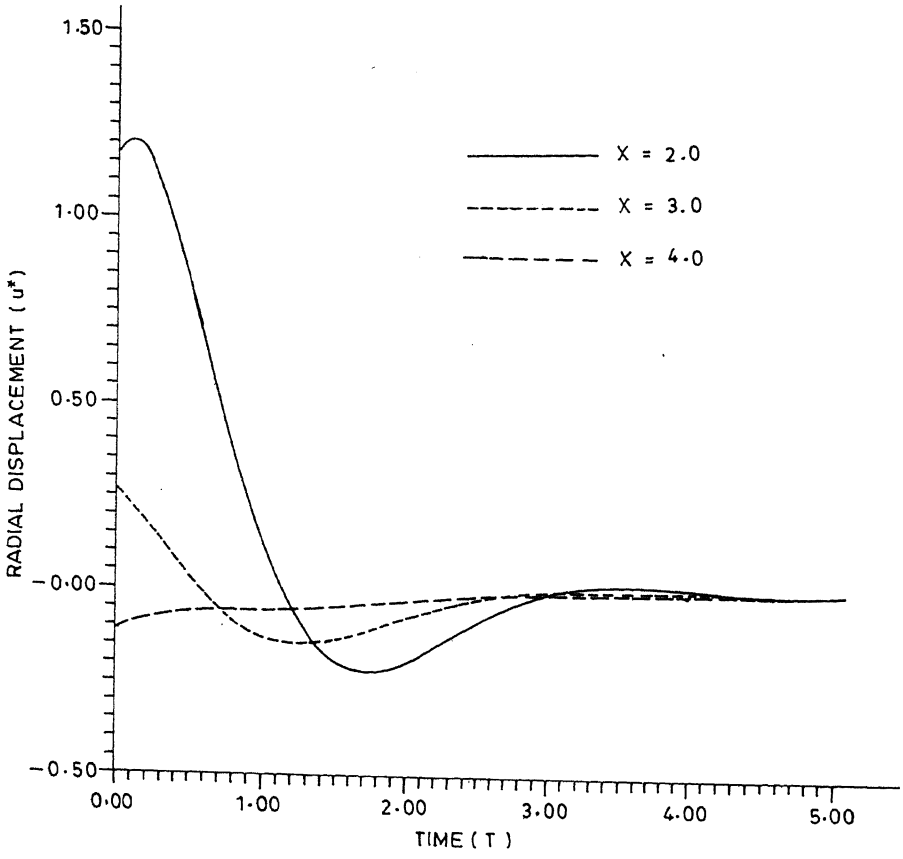

Figure 7. Variation of radial displacement ($u^* = (2Nu/Fa) \times 10$) versus time ($T = (\alpha_2 t/a) - (r-a)/a$).

$$N = 0{\cdot}2765 \times 10^{11}\ \text{dyn cm}^{-2},$$
$$\rho_{11} = 1{\cdot}926137\ \text{g cm}^{-3},$$
$$\rho_{12} = -0{\cdot}002137\ \text{g cm}^{-3},$$
$$\rho_{22} = 0{\cdot}215337\ \text{g cm}^{-3}.$$

The velocities of the P_f, P_s waves for the above constants are

$$\alpha_1 = 2{\cdot}349 \times 10^5\ \text{cm s}^{-1},$$
$$\alpha_2 = 1{\cdot}086 \times 10^5\ \text{cm s}^{-1}.$$

The relation between non-dimensional radial displacement in solid (u^*) and liquid (U^*) with non-dimensional radial distance (X) for various non-dimensional time ($T = 0{\cdot}5, 1{\cdot}0, 1{\cdot}5$) are shown in figures 1 and 2, for water saturated sandstone medium and in figures 3 and 4 for kerosene saturated sandstone.

The non-dimensional radial displacement in solid (u^*) and in liquid (U^*) have been plotted against non-dimensional time (T) for various non-dimensional radial distances ($X = 2, 3, 4$), for water saturated sandstone in figures 5 and 6 and for kerosene saturated sandstone in figures 7 and 8.

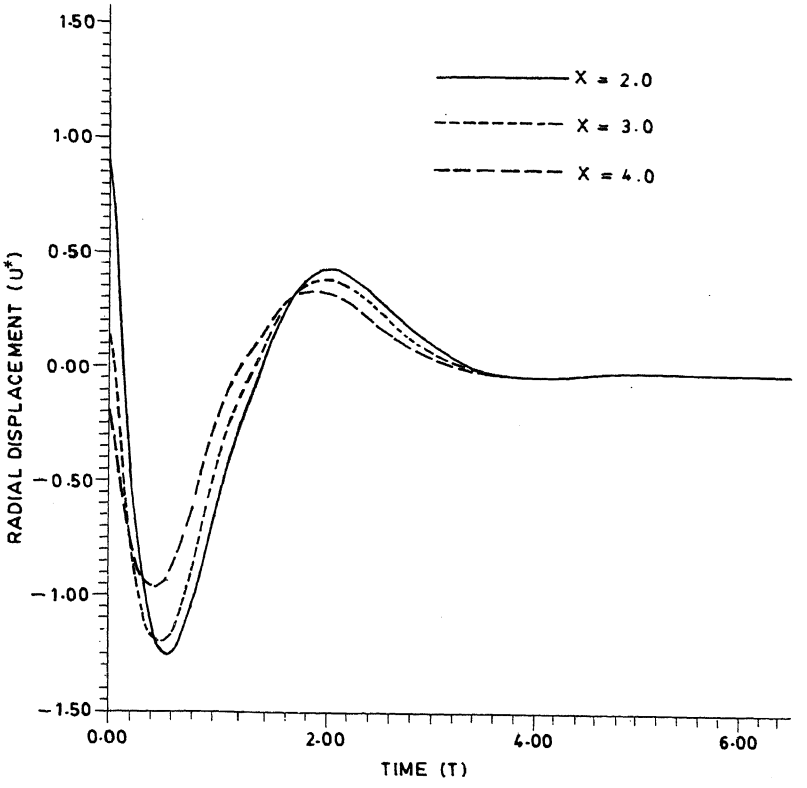

Figure 8. Variation of radial displacement ($U^* = (2Nu/Fa) \times 10$) versus time ($T = (\alpha_2 t/a) - (r-a)/a$).

It is observed that the radial displacements (u^*, U^*) depend upon the non-dimensional quantities X as well as T. In all the cases, it is evident that the radial displacement approaches to zero as the values of X and T increase, which is in accordance with the radiation condition. Also, it is noticed that the radial displacement (u^*) for solid part is large in the positive sense in comparison to the radial displacement (U^*) for liquid part, which is large in the negative sense, for all cases. If we compare the radial displacements for solid and liquid parts for similar values of X and T, it is clear that the radial displacement is almost opposite in sense.

From the comparison of figures 1, 2, 5 and 6 for water-saturated sandstone with figures 3, 4, 7 and 8 for kerosene-saturated sandstone respectively, it is clear that the radial displacement changes almost in the same way for the two media, only with the difference that for kerosene-saturated sandstone medium, the radial displacement curves shift towards the positive side in comparison to the radial displacement curves for water-saturated sandstone, with different magnitudes for different cases.

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Steady-state thermal stresses in an infinite elastic medium containing an annular crack

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Abstract. An axisymmetric steady-state thermoelastic problem of an infinite isotropic medium containing an annular crack is considered. The faces of the crack are exposed to prescribed temperature distribution. The normal components of stress and displacement on the crack plane and the stress-intensity factors at the boundaries of the crack are expressed in power series in terms of the ratio between the radii of the inner and outer boundaries. These are illustrated graphically.

Keywords. Axisymmetric; steady-state; annular crack; stress-intensity factor.

1. Introduction

Considerable effort has been devoted to two-part mixed boundary value problems in thermoelasticity by solving dual integral equations. The thermal stress distribution in the vicinity of a penny-shaped crack located in an infinite isotropic elastic medium was first carried out by Olesiak and Sneddon [12]. Assuming that the crack was opened up by the application of a prescribed axisymmetric flux of heat across its faces the problem was solved by integral transform techniques. Florence and Goodier [6] solved the problem of a uniform heat-flow in an isotropic elastic medium disturbed by a penny-shaped insulated crack. The problem of a flat external crack covering the exterior of a circle was solved by Shail [13] and Das [5]. But much less attention has been paid to the analogous three-part problems which involve solution of triple integral equations. Very recently this type of problem has been considered by Kumar and Padmshri [8]. Corresponding potential problems have been considered by various authors, namely Collins [1], Noble [10], Cooke [2], Williams [16], Clements and Love [3]. More recently, Selvadurai and Singh [15], Clements and Ang [4] have specifically considered the problem of calculating numerical values for the stress intensity factors for a circular annular crack in an infinite elastic medium.

The purpose of the present paper is to study the problem of an annular crack located in an infinite isotropic elastic medium. The faces of the crack, assumed to be free of stress, are subjected to a prescribed axisymmetric temperature distribution. The three-part mixed boundary conditions give rise to two systems of triple integral equations, one for the determination of the temperature field and the other for the resulting thermal stress field. The systems of triple integral equations are reduced to Fredholm integral equations which are solved in terms of power series. The stress intensity factors at the boundaries of the crack are calculated in power series forms involving the ratio of the radii of the inner and outer boundaries. The results are illustrated graphically. The same problem has been considered by Lal [9] who has reduced the system of triple integral equations to an infinite set of simultaneous equations which are solved numerically.

2. Basic equations

We consider the temperature and displacement fields in an isotropic elastic medium which is conducting heat. If we assume that there is symmetry about an axis, which we take to be the z-axis, then the position of a typical point of the medium may conveniently be expressed in the cylindrical polar coordinates by (r, θ, z) and the displacement vector will have the components $(u_r, 0, u_z)$. The non-vanishing components of the stress tensor will be σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz}.

In the absence of any heat source in the medium, the steady-state equations of thermoelasticity with symmetry about z-axis are (Nowacki [11], p. 11)

$$\nabla^2 u_r - \frac{1}{r^2}u_r + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} - \frac{2(1+\nu)}{1-2\nu}\alpha\frac{\partial T}{\partial r} = 0,$$

$$\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} - \frac{2(1+\nu)}{1-2\nu}\alpha\frac{\partial T}{\partial z} = 0 \qquad (1)$$

and $\nabla^2 T = 0$, where $T = T(r, z)$ denotes the deviation in the temperature from a reference temperature T_0, say, the temperature of the medium in a state of zero stress and strain, α is the co-efficient of linear thermal expansion of the medium, ν is the Poisson's ratio and

$$e = \frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}, \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \qquad (2)$$

The solution set of equations (1) can be obtained in the form

$$\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}},$$

where $\bar{u}$ is the particular solution of the non-homogeneous system of equations (1) and $\tilde{u}$ is the general solution of the homogeneous system

$$\nabla^2 \tilde{u}_r - \frac{1}{r^2}\tilde{u}_r + \frac{1}{1-2\nu}\frac{\partial \tilde{e}}{\partial r} = 0 \quad \text{and} \quad \nabla^2 \tilde{u}_z + \frac{1}{1-2\nu}\frac{\partial \tilde{e}}{\partial z} = 0, \qquad (3)$$

where

$$\tilde{e} = \frac{\partial \tilde{u}_r}{\partial r} + \frac{1}{r}\tilde{u}_r + \frac{\partial \tilde{u}_z}{\partial z}.$$

The particular solution of (1) can be derived by following the method of Goodier [7] who introduced the thermoelastic displacement potential Φ related to the displacement $\bar{\mathbf{u}}$ by means of

$$\bar{u}_r = \frac{\partial \Phi}{\partial r}, \quad \bar{u}_z = \frac{\partial \Phi}{\partial z}. \qquad (4)$$

Substituting (4) in (1) and integrating we have

$$\nabla^2 \Phi = mT, \qquad (5)$$

where

$$m=\frac{1+\nu}{1-\nu}\alpha.$$

The problem has thus been reduced to the solution of the Poisson equation (5) and the system of displacement equations of theory of elasticity (3).

The solution of equation (5) gives the function Φ and the relevant stress components corresponding to the displacements (4) are expressed by the relations

$$\bar{\sigma}_{rz}=2\mu\frac{\partial^2\Phi}{\partial r\partial z} \quad \text{and} \quad \bar{\sigma}_{zz}=2\mu\left(\frac{\partial^2\Phi}{\partial z^2}-\nabla^2\Phi\right). \tag{6}$$

For an infinite medium, (4) and (6) constitute the final solution whereas in a bounded body the function Φ can satisfy atmost a part of the conditions on the boundary and therefore an additional solution $\tilde{\mathbf{u}}$ satisfying (3) is always necessary. This can be found conveniently in terms of Love function Ψ according to the relations (Nowacki [11], p. 16)

$$\tilde{u}_r=-\frac{1}{1-2\nu}\frac{\partial^2\Psi}{\partial r\partial z} \quad \text{and} \quad \tilde{u}_z=\frac{1}{1-2\nu}\left[2(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Psi. \tag{7}$$

The complementary stresses derivable from Ψ are given by

$$\tilde{\sigma}_{rz}=\frac{2\mu}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Psi \quad \text{and}$$

$$\tilde{\sigma}_{zz}=\frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Psi. \tag{8}$$

The function Ψ satisfies the biharmonic equation

$$\nabla^2\nabla^2\Psi=0. \tag{9}$$

3. Boundary conditions

With a suitable choice of our unit of length we can assume that, the faces of the annular circular crack can be described by $a\leqslant r\leqslant b$, $z=0\pm$. The faces of the crack are exposed to a known temperature distribution $f(r)$. Further, due to symmetry about the plane $z=0$, the problem of determining the temperature and the state of stress in an infinite medium $r\geqslant 0$, $|z|\geqslant 0$ is equivalent to that for the semi-infinite medium $r\geqslant 0$, $z\geqslant 0$ of which the plane boundary $z=0$ is subjected to the following conditions:

$$\frac{\partial}{\partial z}T(r,0)=0, \quad 0\leqslant r<a, \tag{10}$$

$$T(r,0)=f(r), \quad a<r<b, \tag{11}$$

$$\frac{\partial}{\partial z}T(r,0)=0, \quad b<r<\infty, \tag{12}$$

$$\sigma_{rz}(r,0)=0, \quad 0\leqslant r<\infty, \tag{13}$$

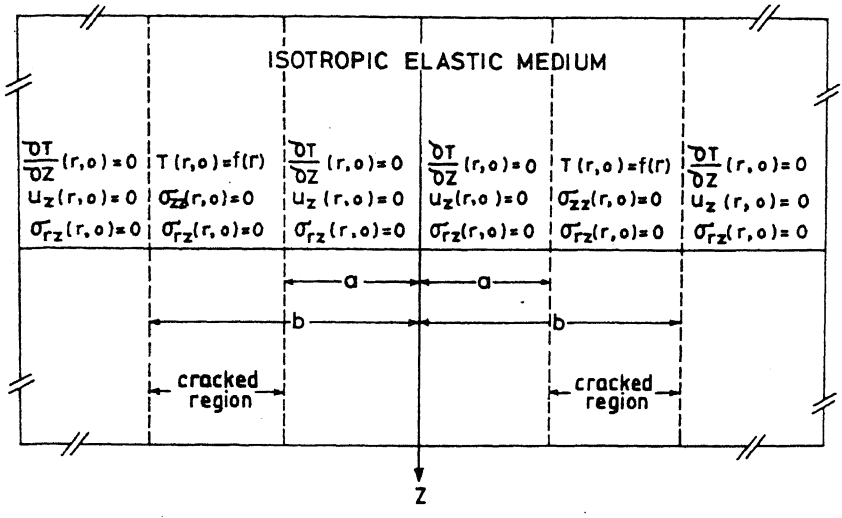

Figure 1. Mixed boundary conditions for the plane of crack $z = 0$.

$$u_z(r,0) = 0,\ 0 < r < a, \tag{14}$$

$$\sigma_{zz}(r,0) = 0,\ a < r < b, \tag{15}$$

$$u_z(r,0) = 0,\ b < r < \infty. \tag{16}$$

The boundary conditions (10) through (16) are depicted in figure 1. The temperature and the components of the stress tensor and the displacement vector tend to zero as $(r^2 + z^2)^{1/2} \to \infty$.

4. The heat conduction problem

A suitable Hankel integral representation of the temperature field $T(r, z)$ satisfying the steady-state Fourier equation

$$\nabla^2 T = 0$$

and vanishing at infinity is taken in the form

$$T(r,z) = \int_0^\infty t^{-1} A(t) \exp(-tz) J_0(tr) \mathrm{d}t, \tag{17}$$

where $A(t)$ is an unknown function to be determined from the thermal boundary conditions on $z = 0$.

Imposition of conditions (10) through (12) leads to the derivation of the system of triple integral equations

$$\int_0^\infty A(t) J_0(tr) \mathrm{d}t = 0, \quad 0 \leqslant r < a, \tag{18}$$

$$\int_0^\infty t^{-1} A(t) J_0(tr) \mathrm{d}t = f(r),\ a < r < b, \tag{19}$$

$$\int_0^\infty A(t) J_0(tr)\mathrm{d}t = 0, \quad b < r < \infty. \tag{20}$$

Employing the method of Cooke [2], the above system will be reduced to a pair of simultaneous integral equations by the following substitution:

$$\int_0^\infty t^{-1} A(t) J_0(tr)\mathrm{d}t = \begin{cases} f_1(r), & 0 \leqslant r < a \\ f_3(r), & b < r < \infty. \end{cases} \tag{21}$$

Once the functions $f_1(r)$ and $f_3(r)$ are known, the solution for $A(t)$ can be completed by Hankel inversion formula. We have the two pairs of dual integral equations

$$\int_0^\infty t^{-1} A(t) J_0(tr)\mathrm{d}t = \begin{cases} f_1(r), & 0 \leqslant r < a \\ f(r), & a < r < b, \end{cases}$$

$$\int_0^\infty A(t) J_0(tr)\mathrm{d}t = 0, \quad b < r < \infty \tag{22}$$

and

$$\int_0^\infty A(t) J_0(tr)\mathrm{d}t = 0, \quad 0 < r < a$$

$$\int_0^\infty t^{-1} A(t) J_0(tr)\mathrm{d}t = \begin{cases} f(r), & a < r < b \\ f_3(r), & b < r < \infty. \end{cases} \tag{23}$$

Employing the results given by Cooke [2] it can be shown from the first pair (22) that

$$f_3(r) = \frac{2}{\pi}(r^2 - b^2)^{1/2} \int_0^a \frac{t f_1(t)\mathrm{d}t}{(r^2 - t^2)(b^2 - t^2)^{1/2}} + h_3(r), \; b < r < \infty, \tag{24}$$

where

$$h_3(r) = \frac{2}{\pi}(r^2 - b^2)^{1/2} \int_a^b \frac{t f(t)\mathrm{d}t}{(r^2 - t^2)(b^2 - t^2)^{1/2}}. \tag{25}$$

Similarly, from the second pair (23) we get

$$f_1(r) = \frac{2}{\pi}(a^2 - r^2)^{1/2} \int_b^\infty \frac{t f_3(t)\mathrm{d}t}{(t^2 - r^2)(t^2 - a^2)^{1/2}} + h_1(r), \; 0 < r < a, \tag{26}$$

where

$$h_1(r) = \frac{2}{\pi}(a^2 - r^2)^{1/2} \int_a^b \frac{t f(t)\mathrm{d}t}{(t^2 - r^2)(t^2 - a^2)^{1/2}}. \tag{27}$$

We introduce two functions $F_1(r)$ and $F_3(r)$ such that

$$F_1(r) = \frac{\mathrm{d}}{\mathrm{d}r} \int_r^a \frac{t f_1(t)\mathrm{d}t}{(t^2 - r^2)^{1/2}}, \quad 0 < r < a \tag{28}$$

and

$$F_3(r) = \frac{\mathrm{d}}{\mathrm{d}r} \int_b^r \frac{t f_3(t)\mathrm{d}t}{(r^2 - t^2)^{1/2}}, \quad b < r < \infty. \tag{29}$$

Solutions of the above Abel integral equations take the forms

$$f_1(t) = -\frac{2}{\pi}\int_t^a \frac{F_1(s)\,ds}{(s^2-t^2)^{1/2}}, \quad 0<t<a \tag{30}$$

and

$$f_3(t) = \frac{2}{\pi}\int_b^t \frac{F_3(s)\,ds}{(t^2-s^2)^{1/2}}, \quad b<t<\infty. \tag{31}$$

Making use of (30) and (26), we have

$$\int_r^a \frac{F_1(s)\,ds}{(s^2-r^2)^{1/2}} = -\frac{\pi}{2}\left[h_1(r) + \frac{2}{\pi}(a^2-r^2)^{1/2}\int_b^\infty \frac{tf_3(t)\,dt}{(t^2-r^2)(t^2-a^2)^{1/2}}\right]. \tag{32}$$

Again, we observe that (32) is an integral equation of the Abel type. Using equation (31), solution of (32) can be written in the form

$$F_1(s) + \frac{2s}{\pi}\int_b^\infty \frac{F_3(u)\,du}{u^2-s^2} = \frac{d}{ds}\int_s^a \frac{th_1(t)\,dt}{(t^2-s^2)^{1/2}}, \quad 0<s<a. \tag{33}$$

Similarly, by making use of (31) and (24) we have

$$\int_b^r \frac{F_3(s)\,ds}{(r^2-s^2)^{1/2}} = \frac{\pi}{2}\left[h_3(r) + \frac{2}{\pi}(r^2-b^2)^{1/2}\int_0^a \frac{tf_1(t)\,dt}{(r^2-t^2)(b^2-t^2)^{1/2}}\right]. \tag{34}$$

Solution of this equation can be written in the form

$$F_3(s) + \frac{2}{\pi}\int_0^a \frac{uF_1(u)\,du}{s^2-u^2} = \frac{d}{ds}\int_b^s \frac{th_3(t)\,dt}{(s^2-t^2)^{1/2}}, \quad b<r<\infty. \tag{35}$$

Solving (33) and (35) we can find F_1 and F_3. Then (17), (21), (30), (31), (33) and (35) describe the temperature field.

5. The thermoelastic problem

The potential $\Phi(r,z)$ of the thermoelastic displacements satisfying equations (4) and (5) can be represented by the Hankel integral

$$\Phi(r,z) = -\frac{1}{2}m\int_0^\infty t^{-3}(1+tz)A(t)\exp(-tz)J_0(tr)\,dt. \tag{36}$$

To solve the isothermal elastic problem defined above, we make use of the stress function

$$\Psi(r,z) = -\int_0^\infty t^{-2}B(t)(2v+tz)\exp(-tz)J_0(tr)\,dt, \tag{37}$$

satisfying (9).

It is found that the boundary condition (13) is automatically satisfied and conditions (14)–(16) are satisfied if

$$\int_0^\infty B(t)J_0(tr)\,dt = 0, \quad 0 \leqslant r < a, \tag{38}$$

$$\int_0^\infty tB(t)J_0(tr)\mathrm{d}t = p(r), \quad a<r<b, \tag{39}$$

$$\int_0^\infty B(t)J_0(tr)\mathrm{d}t = 0, \quad b<r<\infty, \tag{40}$$

where

$$p(r) = -\frac{1}{2}m(1-2\nu)f(r). \tag{41}$$

Let

$$\int_0^\infty tB(t)J_0(tr)\mathrm{d}t = \begin{cases} g_1(r), & 0 \leqslant r < a \\ g_3(r), & b<r<\infty. \end{cases} \tag{42}$$

Once the functions $g_1(r)$ and $g_3(r)$ are known, the solution for $B(t)$ can be found by Hankel inversion formula. We have the following two pairs of dual integral equations

$$\int_0^\infty tB(t)J_0(tr)\mathrm{d}t = \begin{cases} g_1(r), & 0 \leqslant r < a \\ p(r), & a<r<b, \end{cases}$$

$$\int_0^\infty B(t)J_0(tr)\mathrm{d}t = 0, \quad b<r<\infty \tag{43}$$

and

$$\int_0^\infty B(t)J_0(tr)\mathrm{d}t = 0, \quad 0 \leqslant r < a,$$

$$\int_0^\infty tB(t)J_0(tr)\mathrm{d}t = \begin{cases} p(r), & a<r<b \\ g_3(r), & b<r<\infty. \end{cases} \tag{44}$$

Employing the results given by Cooke [2] it can be shown from the first pair (43) that

$$g_3(r) = -\frac{2}{\pi}(r^2-b^2)^{-1/2}\int_0^a \frac{t(b^2-t^2)^{1/2}g_1(t)\mathrm{d}t}{r^2-t^2} + j_3(r), \quad b<r<\infty, \tag{45}$$

where

$$j_3(r) = -\frac{2}{\pi}(r^2-b^2)^{-1/2}\int_a^b \frac{t(b^2-t^2)^{1/2}p(t)\mathrm{d}t}{r^2-t^2}. \tag{46}$$

Similarly, from the second pair (44), we find

$$g_1(r) = -\frac{2}{\pi}(a^2-r^2)^{-1/2}\int_b^\infty \frac{t(t^2-a^2)^{1/2}g_3(t)\mathrm{d}t}{t^2-r^2} + j_1(r), \quad 0<r<a, \tag{47}$$

where

$$j_1(r) = -\frac{2}{\pi}(a^2-r^2)^{-1/2}\int_a^b \frac{t(t^2-a^2)^{1/2}p(t)\mathrm{d}t}{t^2-r^2}. \tag{48}$$

Again, we introduce two functions $G_1(r)$ and $G_3(r)$ such that

$$rg_1(r) = \frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{tG_1(t)\mathrm{d}t}{(t^2-r^2)^{1/2}}, \quad 0<r<a \tag{49}$$

and

$$rg_3(r)=\frac{d}{dr}\int_b^r \frac{tG_3(t)dt}{(r^2-t^2)^{1/2}}, \quad b<r<\infty. \tag{50}$$

Solutions of the above Abel type integral equations take the forms

$$G_1(s)=-\frac{2}{\pi}\int_s^a \frac{tg_1(t)\,dt}{(t^2-s^2)^{1/2}}, \quad 0<s<a, \tag{51}$$

$$G_3(s)=\frac{2}{\pi}\int_b^s \frac{tg_3(t)\,dt}{(s^2-t^2)^{1/2}}, \quad b<s<\infty. \tag{52}$$

Equations (51), (52), (45), (47), (49) and (50) yield the following pair of simultaneous integral equations

$$G_3(s)-\frac{2s}{\pi}\int_0^a \frac{G_1(t)dt}{s^2-t^2}=\frac{2}{\pi}\int_b^s \frac{tj_3(t)dt}{(s^2-t^2)^{1/2}}, \quad b<s<\infty, \tag{53}$$

$$G_1(s)-\frac{2}{\pi}\int_b^\infty \frac{tG_3(t)dt}{(t^2-s^2)}=-\frac{2}{\pi}\int_s^a \frac{tj_1(t)dt}{(t^2-s^2)^{1/2}}, \quad 0<s<a. \tag{54}$$

Now we calculate the quantities of physical interest. Equations (17) and (21) give the temperature on the crack plane $z=0$. Thus

$$T(r,0)=\begin{cases} f_1(r), & 0\leqslant r<a \\ f_3(r), & b<r<\infty. \end{cases} \tag{55}$$

The expressions for the normal component of the displacement vector and that of the stress tensor on the plane $z=0$ are found to be

$$u_z(r,0)=\frac{2(1-\nu)}{1-2\nu}\int_0^\infty B(t)J_0(tr)dt, \quad a<r<b \tag{56}$$

and

$$\sigma_{zz}(r,0)=\begin{cases} -m\mu T(r,0)-\dfrac{2\mu}{1-2\nu}g_1(r), & 0\leqslant r<a \\ -m\mu T(r,0)-\dfrac{2\mu}{1-2\nu}g_3(r), & b<r<\infty, \end{cases} \tag{57}$$

where $T(r,0)$ is given by (55).

The stress intensity factors at the boundaries $r=a$ and $r=b$ are defined by [14]

$$K_a=\lim_{r\to a^-}[2(a-r)]^{1/2}\sigma_{zz}(r,0), \tag{58}$$

$$K_b=\lim_{r\to b^+}[2(r-b)]^{1/2}\sigma_{zz}(r,0). \tag{59}$$

6. Annular crack opened up by a constant temperature and solutions of the simultaneous integral equations

In the particular case, when the faces of the crack are maintained at a constant temperature $(-f_0)$, then

$$f(r)=-f_0, \quad a<r<b. \tag{60}$$

Then from equations (33) and (35) we have

$$F_1(s)+\frac{2s}{\pi}\int_b^\infty \frac{F_3(u)\mathrm{d}u}{(u^2-s^2)}=-sf_0\left[\frac{1}{(b^2-s^2)^{1/2}}-\frac{1}{(a^2-s^2)^{1/2}}\right],$$

$$0<s<a, \tag{61}$$

and

$$F_3(s)+\frac{2}{\pi}\int_0^a \frac{uF_1(u)\mathrm{d}u}{(s^2-u^2)}=-sf_0\left[\frac{1}{(s^2-b^2)^{1/2}}-\frac{1}{(s^2-a^2)^{1/2}}\right],$$

$$b<s<\infty. \tag{62}$$

In equation (61), putting $s_1=s/a$, $u_1=u/b$ and $c=a/b$, we get

$$F_1(as_1)+\frac{2cs_1}{\pi}\int_1^\infty \frac{F_3(bu_1)\mathrm{d}u_1}{u_1^2-c^2s_1^2}=-s_1f_0\left[\frac{c}{(1-c^2s_1^2)^{1/2}}-\frac{1}{(1-s_1^2)^{1/2}}\right],$$

$$0<s_1<1. \tag{63}$$

Putting $s_1=s/b$, $u_1=u/a$ and $c=a/b$ in equation (62) we get

$$F_3(bs_1)+\frac{2c^2}{\pi}\int_0^1 \frac{u_1F_1(au_1)\mathrm{d}u_1}{s_1^2-c^2u_1^2}=-s_1f_0\left[\frac{1}{(s_1^2-1)^{1/2}}-\frac{1}{(s_1^2-c^2)^{1/2}}\right],$$

$$1<s_1<\infty. \tag{64}$$

Assuming that $c\ll 1$, the denominators in the integrands of (63) and (64) can be expressed in the forms

$$(u_1^2-c^2s_1^2)^{-1}=\frac{1}{u_1^2}+\frac{c^2s_1^2}{u_1^4}+\frac{c^4s_1^4}{u_1^6}+\frac{c^6s_1^6}{u_1^8}+\frac{c^8s_1^8}{u_1^{10}}+O(c^{10}) \tag{65}$$

and

$$(s_1^2-c^2u_1^2)^{-1}=\frac{1}{s_1^2}+\frac{c^2u_1^2}{s_1^4}+\frac{c^4u_1^4}{s_1^6}+\frac{c^6u_1^6}{s_1^8}+\frac{c^8u_1^8}{s_1^{10}}+O(c^{10}), \tag{66}$$

where $O(c^n)$ is the Landau symbol. We further assume that the functions F_1 and F_3 can be expressed in power series of the forms

$$F_1(as_1)=\sum_{i=0}^{8} c^i m_i(s_1) \tag{67}$$

and

$$F_3(bs_1)=\sum_{i=0}^{8} c^i n_i(s_1). \tag{68}$$

Solving for m_i and n_i we get

$$\begin{aligned}F_1(as_1)=f_0\Bigg[&\frac{s_1}{(1-s_1^2)^{1/2}}-\frac{2}{\pi}cs_1-\frac{2}{3\pi}c^3s_1^3-\frac{8}{9\pi^3}c^4s_1-\frac{2}{5\pi}c^5s_1^5\\&-\frac{8}{15\pi^3}c^6s_1\left(s_1^2+\frac{14}{15}\right)-\frac{2}{\pi}c^7s_1\left(\frac{s_1^6}{7}+\frac{16}{81\pi^4}\right)\\&-\frac{8}{21\pi^3}c^8s_1\left(s_1^4+\frac{22s_1^2}{25}+\frac{29}{35}\right)\Bigg],\quad 0<s_1<1\end{aligned} \tag{69}$$

and

$$F_3(bs_1)=f_0\Bigg[1-\frac{s_1}{(s_1^2-1)^{1/2}}+\frac{4c^3}{3\pi^2 s_1^2}+\frac{4c^5}{5\pi^2}\left(\frac{1}{s_1^4}+\frac{1}{3s_1^2}\right)$$
$$+\frac{16c^6}{27\pi^4 s_1^2}+\frac{4c^7}{7\pi^2}\left(\frac{1}{s_1^6}+\frac{1}{3s_1^4}+\frac{1}{5s_1^2}\right)+\frac{16c^8}{45\pi^4}\left(\frac{1}{s_1^4}+\frac{23}{15s_1^2}\right)\Bigg],$$
$$1<s_1<\infty. \qquad (70)$$

Substituting equations (69) and (70) in (30) and (31) we can find expressions for $f_1(t)$ and $f_3(t)$. Using these in (55) we get the expression for the temperature distribution $T(r,0)$ on the plane of crack

$$T(r,0)=\begin{cases}
-\dfrac{2f_0}{\pi}\Bigg[\dfrac{\pi}{2}-\dfrac{2c}{\pi}\left(1-\dfrac{r^2}{a^2}\right)^{1/2}\Bigg\{1+\dfrac{c^2}{9}\left(1+\dfrac{2r^2}{a^2}\right)+\dfrac{4c^3}{9\pi^2}\\
+\dfrac{c^4}{25}\left(1+\dfrac{4r^2}{3a^2}+\dfrac{8r^4}{3a^4}\right)+\dfrac{4c^5}{15\pi^2}\left(\dfrac{29}{15}+\dfrac{2r^2}{a^2}\right)+c^6\left(\dfrac{1}{49}+\dfrac{16}{81\pi^4}\right.\\
\left.+\dfrac{6r^2}{245a^2}+\dfrac{8r^4}{245a^4}+\dfrac{16r^6}{245a^6}\right)+\dfrac{4c^7}{105\pi^2}\left(\dfrac{694}{105}+\dfrac{64r^2}{15a^2}+\dfrac{8r^4}{3a^4}\right)\Bigg\}\Bigg], \\
\qquad 0\leqslant r<a \qquad (71)\\
-\dfrac{2f_0}{\pi}\Bigg[\dfrac{\pi}{2}-\cos^{-1}\left(\dfrac{b}{r}\right)-\dfrac{4c^3}{\pi^2}\cdot\dfrac{b(r^2-b^2)^{1/2}}{r^2}\Bigg\{\dfrac{1}{3}+\dfrac{2c^2}{15}\left(1+\dfrac{b^2}{r^2}\right)\\
+\dfrac{4c^3}{27\pi^2}+\dfrac{c^4}{105}\left(\dfrac{23}{3}+\dfrac{22b^2}{3r^2}+\dfrac{8b^4}{r^4}\right)+\dfrac{4c^5}{135\pi^2}\left(\dfrac{28}{5}+\dfrac{2b^2}{r^2}\right)\Bigg\}\Bigg], \\
\qquad b<r<\infty. \qquad (72)
\end{cases}$$

Next, we consider the associated thermoelastic problem.

In the particular case when $f(r)=-f_0$, we assume

$$p(r)=-\tfrac{1}{2}m(1-2\nu)f_0=p_0 \quad \text{(say)}.$$

Then the pair of simultaneous integral equations (54) and (53) reduce to

$$G_1(s)-\frac{2}{\pi}\int_b^\infty \frac{tG_3(t)\,dt}{t^2-s^2}=\frac{2p_0}{\pi}[(b^2-s^2)^{1/2}-(a^2-s^2)^{1/2}],\ 0<s<a \qquad (73)$$

and

$$G_3(s)-\frac{2s}{\pi}\int_o^a \frac{G_1(t)\,dt}{s^2-t^2}=\frac{2p_0}{\pi}[(s^2-b^2)^{1/2}-(s^2-a^2)^{1/2}],\ b<s<\infty. \qquad (74)$$

Putting $s_1=s/a$, $t_1=t/b$ and $c=a/b$ in (73) we get

$$G_1(as_1)-\frac{2}{\pi}\int_1^\infty \frac{t_1G_3(bt_1)dt_1}{t_1^2-c^2s_1^2}=\frac{2p_0b}{\pi}[(1-c^2s_1^2)^{1/2}-c(1-s_1^2)^{1/2}],$$
$$0<s_1<1. \qquad (75)$$

Similarly, putting $s_1 = s/b$, $t_1 = t/a$ and $c = a/b$ in (74) we get

$$G_3(bs_1) - \frac{2cs_1}{\pi}\int_0^1 \frac{G_1(at_1)\,dt_1}{s_1^2 - c^2t_1^2} = \frac{2p_0b}{\pi}[(s_1^2-1)^{1/2} - (s_1^2-c^2)^{1/2}],$$

$$1 < s_1 < \infty. \tag{76}$$

Assuming that $c \ll 1$, the denominators of the integrands in (75) and (76) can be expressed in the forms

$$(t_1^2 - c^2s_1^2)^{-1} = \frac{1}{t_1^2} + \frac{c^2s_1^2}{t_1^4} + \frac{c^4s_1^4}{t_1^6} + \frac{c^6s_1^6}{t_1^8} + \frac{c^8s_1^8}{t_1^{10}} + O(c^{10}) \tag{77}$$

and

$$(s_1^2 - c^2t_1^2)^{-1} = \frac{1}{s_1^2} + \frac{c^2t_1^2}{s_1^4} + \frac{c^4t_1^4}{s_1^6} + \frac{c^6t_1^6}{s_1^8} + \frac{c^8t_1^8}{s_1^{10}} + O(c^{10}). \tag{78}$$

We further assume that the functions G_1 and G_3 can be expressed in power series in the forms

$$G_1(as_1) = \sum_{i=0}^{8} m_i'(s_1)c^i, \tag{79}$$

$$G_3(bs_1) = \sum_{i=0}^{8} n_i'(s_1)c^i. \tag{80}$$

By substituting (77)–(80) in equations (75) and (76) and comparing like terms in c^i, it is possible to determine the functions $m_i'(s_1)$ and $n_i'(s_1)$. We have

$$\begin{aligned} G_1(as_1) = \frac{2p_0b}{\pi}\Bigg[& 0{\cdot}63662 + c(0{\cdot}25801 - (1-s_1^2)^{1/2}) + \frac{2c^2}{\pi}(0{\cdot}16426 - s_1^2) \\ & + \frac{8c^3}{\pi^3}(-0{\cdot}05797 + 0{\cdot}33333\,s_1^2) + \frac{2c^4}{\pi}(0{\cdot}02698 + 0{\cdot}05475s_1^2 \\ & - 0{\cdot}33333\,s_1^4) + \frac{8c^5}{\pi^3}(-0{\cdot}02985 + 0{\cdot}01031\,s_1^2 + 0{\cdot}2\,s_1^4) \\ & + \frac{2c^6}{\pi}(0{\cdot}01139 + 0{\cdot}01386s_1^2 + 0{\cdot}03285\,s_1^4 - 0{\cdot}2\,s_1^6) + \frac{8c^7}{\pi^3} \\ & (-0{\cdot}01681 - 0{\cdot}00762\,s_1^2 + 0{\cdot}0138\,s_1^4 + 0{\cdot}14286\,s_1^6) \\ & + \frac{2c^8}{\pi}(0{\cdot}0063 + 0{\cdot}00699\,s_1^2 + 0{\cdot}00957\,s_1^4 + 0{\cdot}02346\,s_1^6 \\ & - 0{\cdot}14286\,s_1^8)\Bigg], \quad 0 < s_1 < 1 \end{aligned} \tag{81}$$

$$\begin{aligned} G_3(bs_1) = \frac{2p_0b}{\pi}\Bigg[& (s_1^2-1)^{1/2} - s_1 + 0{\cdot}40528\frac{c}{s_1} + 0{\cdot}16426\frac{c^2}{s_1} \\ & + \frac{4c^3}{\pi^2}\left(-\frac{0{\cdot}16908}{s_1} + \frac{0{\cdot}33333}{s_1^3}\right) + \frac{16c^4}{\pi^4}\left(\frac{0{\cdot}05314}{s_1} + \frac{0{\cdot}33333}{s_1^3}\right) \end{aligned}$$

$$+\frac{4c^5}{\pi^2}\left(-\frac{0{\cdot}02144}{s_1}-\frac{0{\cdot}14525}{s_1^3}+\frac{0{\cdot}2}{s_1^5}\right)+\frac{16c^6}{\pi^4}\left(\frac{0{\cdot}01358}{s_1}+\frac{0{\cdot}04734}{s_1^3}\right.$$

$$\left.+\frac{0{\cdot}2}{s_1^5}\right)+\frac{4c^7}{\pi^2}\left(-\frac{0{\cdot}00599}{s_1}-\frac{0{\cdot}02768}{s_1^3}-\frac{0{\cdot}11001}{s_1^5}+\frac{0{\cdot}14286}{s_1^7}\right)$$

$$+\frac{16c^8}{\pi^4}\left(\frac{0{\cdot}00382}{s_1}+\frac{0{\cdot}02068}{s_1^3}+\frac{0{\cdot}03602}{s_1^5}+\frac{0{\cdot}14286}{s_1^7}\right)\Bigg],$$

$$1<s_1<\infty. \tag{82}$$

Using (49) and (50) we get the following expressions for $g_1(r)$ and $g_3(r)$,

$$g_1(r)=\frac{2p_0b}{\pi a}\left[\frac{\pi c}{2}+\frac{2(a^2-r^2)^{1/2}}{a}\left\{-0{\cdot}42442c^2+0{\cdot}05734c^3\right.\right.$$

$$-c^4\left(0{\cdot}03335+0{\cdot}22636\frac{r^2}{a^2}\right)+c^5\left(0{\cdot}01553+0{\cdot}05504\frac{r^2}{a^2}\right)$$

$$-c^6\left(0{\cdot}01036+0{\cdot}03589\frac{r^2}{a^2}-0{\cdot}17462\frac{r^4}{a^4}\right)$$

$$+c^7\left(0{\cdot}00596+0{\cdot}02065\frac{r^2}{a^2}+0{\cdot}05055\frac{r^4}{a^4}\right)$$

$$\left.-c^8\left(0{\cdot}0044+0{\cdot}0144\frac{r^2}{a^2}+0{\cdot}03495\frac{r^4}{a^4}+0{\cdot}14782\frac{r^6}{a^6}\right)\right\}$$

$$-a(a^2-r^2)^{-1/2}\left\{0{\cdot}63662+0{\cdot}25801c-\frac{2c^2}{\pi}\left(0{\cdot}16907\right.\right.$$

$$+0{\cdot}66667\frac{r^2}{a^2}\Big)+\frac{8c^3}{\pi^3}\left(0{\cdot}05314+0{\cdot}22222\frac{r^2}{a^2}\right)$$

$$-\frac{2c^4}{\pi}\left(0{\cdot}02144+0{\cdot}05239\frac{r^2}{a^2}+0{\cdot}17778\frac{r^4}{a^4}\right)$$

$$+\frac{8c^5}{\pi^3}\left(0{\cdot}01359+0{\cdot}0602\frac{r^2}{a^2}+0{\cdot}10667\frac{r^4}{a^4}\right)$$

$$-\frac{2c^6}{\pi}\left(0{\cdot}00599+0{\cdot}01628\frac{r^2}{a^2}+0{\cdot}02819\,\frac{r^4}{a^4}-0{\cdot}09143\frac{r^6}{a^6}\right)$$

$$+\frac{8c^7}{\pi^3}\left(0{\cdot}00382+0{\cdot}02309\frac{r^2}{a^2}+0{\cdot}04001\frac{r^4}{a^4}+0{\cdot}06531\frac{r^6}{a^6}\right)$$

$$-\frac{2c^8}{\pi}\left(0{\cdot}00198+0{\cdot}00691\frac{r^2}{a^2}+0{\cdot}01131\frac{r^2}{a^4}+0{\cdot}0183\frac{r^6}{a^6}\right.$$

$$\left.\left.+0{\cdot}05805\frac{r^8}{a^8}\right)\right\}\Bigg], \quad 0\leqslant r<a \tag{83}$$

$$g_3(r)=\frac{2p_0}{\pi}\Bigg[1{\cdot}5708-\cos^{-1}\left(\frac{b}{r}\right)+b(r^2-b^2)^{-1/2}\Bigg\{-1+0{\cdot}40528c\frac{b^2}{r^2}$$
$$+0{\cdot}16426c^2\frac{b^2}{r^2}+0{\cdot}06657c^3\frac{b^2}{r^2}+0{\cdot}06348c^4\frac{b^2}{r^2}+c^5\Bigg(-0{\cdot}04054\frac{b^2}{r^2}$$
$$+0{\cdot}05404\frac{b^4}{r^4}\Bigg)+c^6\Bigg(0{\cdot}02096\frac{b^2}{r^2}+0{\cdot}0219\frac{b^4}{r^4}\Bigg)+c^7\Bigg(-0{\cdot}01693\frac{b^2}{r^2}$$
$$-0{\cdot}01428\frac{b^4}{r^4}+0{\cdot}03088\frac{b^6}{r^6}\Bigg)+c^8\Bigg(0{\cdot}01069\frac{b^2}{r^2}+0{\cdot}0102\frac{b^4}{r^4}$$
$$+0{\cdot}01251\frac{b^6}{r^6}\Bigg)\Bigg\}-\frac{(r^2-b^2)^{1/2}}{b}\Bigg\{0{\cdot}27018c^3\frac{b^4}{r^4}+0{\cdot}1095c^4\frac{b^4}{r^4}$$
$$+c^5\Bigg(-0{\cdot}0637\frac{b^4}{r^4}+0{\cdot}21616\frac{b^6}{r^6}\Bigg)+c^6\Bigg(0{\cdot}03746\frac{b^4}{r^4}+0{\cdot}0876\frac{b^6}{r^6}\Bigg)$$
$$+c^7\Bigg(-0{\cdot}029\frac{b^4}{r^4}-0{\cdot}05712\frac{b^6}{r^6}+0{\cdot}18528\frac{b^8}{r^8}\Bigg)$$
$$+c^8\Bigg(0{\cdot}02012\frac{b^4}{r^4}+0{\cdot}0408\frac{b^6}{r^6}+0{\cdot}07506\frac{b^8}{r^8}\Bigg)\Bigg\}\Bigg],\quad b<r<\infty. \tag{84}$$

From (39) and (42) we have

$$B(t)=\int_0^a rg_1(r)J_0(rt)\,\mathrm{d}r+\int_a^b rp(r)J_0(rt)\,\mathrm{d}r+\int_b^\infty rg_3(r)J_0(rt)\,\mathrm{d}r. \tag{85}$$

Using equations (56), (85), (83) and (84) the following expression for the normal component of displacement on crack plane $z=0$ is obtained.

$$u_z(r,0)=\frac{2b(1-\nu)mf_0}{\pi}\Bigg[\sin^{-1}\left(\frac{a}{r}\right)\Bigg\{-0{\cdot}63662-0{\cdot}25801c$$
$$+c^2\Bigg(-0{\cdot}10458+0{\cdot}31831\frac{r^2}{a^2}\Bigg)+c^3\Bigg(0{\cdot}01496-0{\cdot}043\frac{r^2}{a^2}\Bigg)$$
$$+c^4\Bigg(-0{\cdot}01717-0{\cdot}1742\frac{r^2}{a^2}+0{\cdot}07958\frac{r^4}{a^4}\Bigg)$$
$$+c^5\Bigg(0{\cdot}00769-0{\cdot}00132\frac{r^2}{a^2}-0{\cdot}01935\frac{r^4}{a^4}\Bigg)$$
$$+c^6\Bigg(0{\cdot}00003+0{\cdot}00649\frac{r^2}{a^2}+0{\cdot}03308\frac{r^4}{a^4}-0{\cdot}03979\frac{r^6}{a^6}\Bigg)$$
$$+c^7\Bigg(0{\cdot}00435+0{\cdot}00098\frac{r^2}{a^2}-0{\cdot}00133\frac{r^4}{a^4}-0{\cdot}01151\frac{r^6}{a^6}\Bigg)$$
$$+c^8\Bigg(0{\cdot}00733-0{\cdot}00222\frac{r^2}{a^2}-0{\cdot}00228\frac{r^4}{a^4}-0{\cdot}00467\frac{r^6}{a^6}$$

$$+0{\cdot}02486\frac{r^8}{a^8}\Bigg)\Bigg\}+\frac{(r^2-a^2)^{1/2}}{a}\Bigg\{-0{\cdot}31831c^2+0{\cdot}043c^3$$
$$+c^4\Bigg(-0{\cdot}03563-0{\cdot}07958\frac{r^2}{a^2}\Bigg)+c^5\Bigg(0{\cdot}01422+0{\cdot}01935\frac{r^2}{a^2}\Bigg)$$
$$+c^6\Bigg(-0{\cdot}00732-0{\cdot}00655\frac{r^2}{a^2}+0{\cdot}03979\frac{r^4}{a^4}\Bigg)$$
$$+c^7\Bigg(0{\cdot}00606+0{\cdot}00901\frac{r^2}{a^2}+0{\cdot}01151\frac{r^4}{a^4}\Bigg)$$
$$+c^8\Bigg(-0{\cdot}00511-0{\cdot}00787\frac{r^2}{a^2}-0{\cdot}01192\frac{r^4}{a^4}-0{\cdot}02486\frac{r^6}{a^6}\Bigg)\Bigg\}$$
$$+\frac{b}{r}\sin^{-1}\Bigg(\frac{r}{b}\Bigg)\Bigg\{0{\cdot}40528c+0{\cdot}16426c^2$$
$$+c^3\Bigg(-0{\cdot}06852+0{\cdot}06754\frac{b^2}{r^2}\Bigg)+c^4\Bigg(0{\cdot}00873+0{\cdot}02738\frac{b^2}{r^2}\Bigg)$$
$$+c^5\Bigg(-0{\cdot}00869-0{\cdot}02943\frac{b^2}{r^2}+0{\cdot}0304\frac{b^4}{r^4}\Bigg)$$
$$+c^6\Bigg(0{\cdot}00223+0{\cdot}00389\frac{b^2}{r^2}+0{\cdot}01232\frac{b^4}{r^4}\Bigg)$$
$$+c^7\Bigg(-0{\cdot}00243-0{\cdot}00561\frac{b^2}{r^2}-0{\cdot}01672\frac{b^4}{r^4}+0{\cdot}01809\frac{b^6}{r^6}\Bigg)$$
$$+c^8\Bigg(0{\cdot}00063+0{\cdot}0017\frac{b^2}{r^2}+0{\cdot}00222\frac{b^4}{r^4}+0{\cdot}00733\frac{b^6}{r^6}\Bigg)\Bigg\}$$
$$+\frac{(b^2-r^2)^{1/2}}{b}\Bigg\{1-0{\cdot}06754c^3\frac{b^2}{r^2}-0{\cdot}02738c^4\frac{b^2}{r^2}$$
$$+c^5\Bigg(0{\cdot}00917\frac{b^2}{r^2}-0{\cdot}0304\frac{b^4}{r^4}\Bigg)+c^6\Bigg(-0{\cdot}0121\frac{b^2}{r^2}-0{\cdot}01232\frac{b^4}{r^4}\Bigg)$$
$$+c^7\Bigg(0{\cdot}0071\frac{b^2}{r^2}+0{\cdot}00465\frac{b^4}{r^4}-0{\cdot}01809\frac{b^6}{r^6}\Bigg)$$
$$+c^8\Bigg(-0{\cdot}00709\frac{b^2}{r^2}-0{\cdot}00711\frac{b^4}{r^4}-0{\cdot}00733\frac{b^6}{r^6}\Bigg)\Bigg\}\Bigg],\quad a\leqslant r\leqslant b. \tag{86}$$

From (57), the expressions for the normal component of stress can be found. Using (58) and (59) the stress intensity factors at the boundaries $r=a$ and $r=b$ are found to be

$$K_a=\frac{2m\mu f_0\sqrt{a}}{\pi c}\Bigg[0{\cdot}63662+0{\cdot}25801c-0{\cdot}53205c^2+0{\cdot}07105c^3$$
$$-0{\cdot}16018c^4+0{\cdot}04656c^5+0{\cdot}02608c^6+0{\cdot}034117c^7+0{\cdot}06146c^8\Bigg], \tag{87}$$

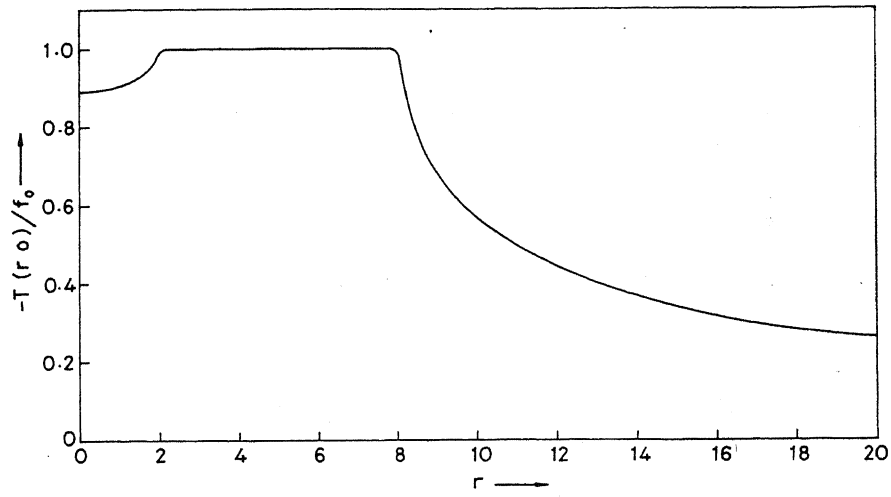

Figure 2. Variation of temperature $T(r,0)$ with r.

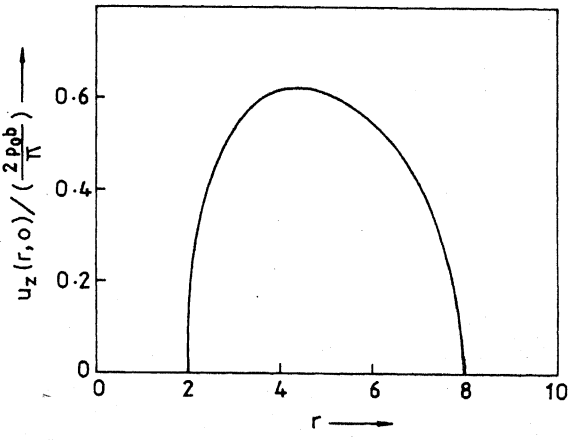

Figure 3. Variation of $u_z(r,0)$ *with* r.

$$K_b = -\frac{2m\mu f_0\sqrt{b}}{\pi}\Big[-1+0{\cdot}40528c+0{\cdot}16426c^2+0{\cdot}06657c^3$$

$$+0{\cdot}06348c^4+0{\cdot}0135c^5+0{\cdot}04286c^6-0{\cdot}00033c^7+0{\cdot}0334c^8\Big]. \quad (88)$$

Variations of temperature $T(r,0)$ and the normal components of displacement $u_z(r,0)$ and stress $\sigma_{zz}(r,0)$ with r are shown in figures 2, 3 and 4 respectively. Variation of the stress intensity factors K_a, K_b with c is shown in figure 5.

7. Conclusions

The stress intensity factors $\bar{K}_a = \sqrt{2}\,K_a/(m\mu f_0(b-a)^{1/2})$ and $\bar{K}_b = \sqrt{2}\,K_b/(m\mu f_0(b-a)^{1/2})$ are computed for various values of the ratio $c(=a/b)$. These values, given

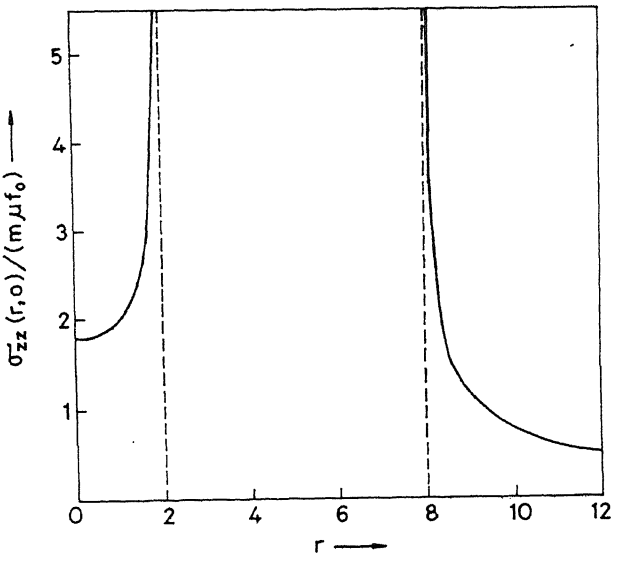

Figure 4. Variation of $\sigma_{zz}(r,0)$ with r.

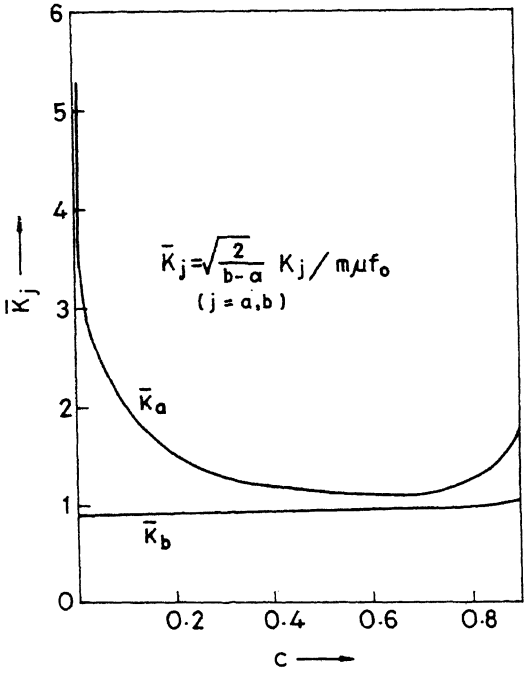

Figure 5. Variation of K_a, K_b with C.

in table 1 are in excellent agreement with those obtained by Selvadurai and Singh [15], Clements and Ang [4] for the potential problem. For the corresponding thermoelastic problem, results obtained here completely agree with those of Lal [9].

Table 1. $\bar{K}_a = \sqrt{2}\, K_a/(m\mu f_0(b-a)^{1/2})$ and $\bar{K}_b = \sqrt{2}\, K_b/(m\mu f_0\,(b-a)^{1/2})$.

$c = a/b$	$\bar{K}_a$	$\bar{K}_b$
0·01	5·7834	0·90117
0·10	1·9721	0·90892
0·20	1·5019	0·91774
0·30	1·3102	0·9268
0·40	1·20526	0·93611
0·50	1·14137	0·94575
0·60	1·10506	0·95588
0·70	1·09957	0·96741
0·80	1·15435	0·98520
0·90	1·40663	1·04501
0·99	4·00845	2·05731

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On the incoming water waves against a vertical cliff

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Abstract. The problems of obliquely incident surface water waves against a vertical cliff have been handled in both the cases of water of infinite as well as finite depth by straightforward uses of appropriate Havelock-type expansion theorems. The logarithmic singularity along the shore-line has been incorporated in a direct manner, by suitably representing the Dirac's delta function.

Keywords. Water wave; logarithmic singularity; Dirac's delta function.

1. Introduction

The problem of incoming incident surface water waves normally incident against a vertical cliff, in the linearized theory was first investigated by Stoker [6] (see also Stoker [7]) by complex variable method. Assuming that there is no reflection of these waves, a source or sink type behaviour at the shore line was introduced, to account for the fact that the wave amplitude remains finite at the origin. The problem of incoming surface water waves normally incident on a vertical cliff was reinvestigated by Mandal and Kundu [5] in the case of water of infinite depth. The problem of obliquely incident water waves against a vertical cliff was investigated by Chakrabarti [1] in the case of water of infinite depth by using the commutativity property of the differential operators appearing in the boundary conditions and the case of normally incident surface water wave was handled as a particular case.

In this note, we have employed a simple and straightforward method, by using the Havelock's expansion theorem to solve the problem of obliquely incident surface water waves by a vertical cliff in both the cases of water of infinite as well as finite depths. The method of solution of the boundary value problems under consideration is based on the exploitation of the logarithmic singularity of the velocity potential at the corner point, where the water surface meets the wall in a rather natural manner, while applying the expansion theorem of Havelock. The solutions in the case of the problems of normally incident surface water waves are reproduced as particular cases.

2. Mathematical formulation

The problems under consideration are three dimensional in nature and these problems will be studied under the assumption that the incoming waves are incident against the vertical cliff with an angle $\alpha(0 < \alpha < \pi/2)$. Assuming that the cliff occupies the position $x = 0, 0 < y < \infty$ in the case of water of infinite depth ($x = 0, 0 < y < h$ in the case of water of finite depth) in the Cartesian co-ordinate system and the fluid occupies the region $x > 0, y > 0$ in the case of water of infinite depth, whereas, in case of water of finite depth, the fluid occupies the region $x > 0, 0 < y < h$ where $y = h$ is the bottom surface in the case of water of finite depth. If we also assume that the

fluid is inviscid, incompressible, the motion irrotational and simple harmonic in time, we will have the existence of the velocity potential $\Phi(x, y, z, t)$ such that $\Phi(x, y, z, t) = \mathrm{Re}(\phi(x, y)\, e^{-i\omega t + iKz\sin\alpha})$ in the case of water of infinite depth (K is replaced by k_0 in the case of water of finite depth, where k_0 is the real and positive root of the transcendental equation $\alpha \tan h\, \alpha h = K$ in α) (see Jervis and Taylor [4], Faulkner [3]). Then under the assumption of the linearized theory, the problem of obliquely incident surface water waves against a vertical cliff can be investigated by way of determining the potential $\phi(x, y)$ [after dropping the factor $e^{-i\omega t + iKz\sin\alpha}$ from the whole mathematical formulation, (see Chakrabarti [1])] which satisfies the boundary value problem as given by

(i) the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \nu^2\phi = 0, \quad \text{in the fluid region,} \tag{2.1}$$

(ii) the linearized free surface condition

$$\frac{\partial\phi}{\partial y} + K\phi = 0, \quad \text{on} \quad y = 0 \quad K > 0, \tag{2.2}$$

where $K = \omega^2/g$, g the acceleration due to gravity,

(iii) the condition on the vertical wall

$$\frac{\partial\phi}{\partial x} = 0, \quad \text{on} \quad x = 0, \quad \text{for} \quad y > 0, \tag{2.3}$$

(iv) the condition at the corner point

$$\phi \to \ln\sqrt{x^2 + y^2} \quad \text{as} \quad \sqrt{x^2 + y^2} \to 0, \tag{2.4}$$

(v) the infinity conditions

$$\phi(x, y) \to e^{-Ky - i\mu x} \quad \text{as} \quad x \to \infty, \tag{2.5}$$

in the case of water of infinite depth. Whereas in the case of water of finite depth,

$$\phi(x, y) \to \frac{e^{-i\mu x}\cosh k_0(h - y)}{\cosh k_0 h} \quad \text{as} \quad x \to \infty, \tag{2.6}$$

and

(vi) the bottom conditions

$$\phi, |\nabla\phi| \to 0 \quad \text{as} \quad y \to \infty, \tag{2.7}$$

n the case of water of infinite depth. Whereas, in the case of water of finite depth

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = h. \tag{2.8}$$

In (2.1) and (2.5), $\nu = K\sin\alpha$ and $\mu = K\cos\alpha$ in the case of water of infinite depth, and $\nu = k_0\sin\alpha$, $\mu = k_0\cos\alpha$ in the case of water of finite depth, k_0 being the same as defined earlier.

2.1. *The method of solution for water of infinite depth*

From the condition (2.4), we have

$$\frac{\partial \phi}{\partial x} \approx \frac{x}{x^2+y^2} \quad \text{for} \quad \sqrt{x^2+y^2} \to 0. \tag{2.9}$$

But we have (see Friedman [2])

$$\lim_{x \to 0} \frac{x}{x^2+y^2} = \pi \delta(y) \text{ for all } y, \tag{2.10}$$

where $\delta(y)$ is the Dirac's delta function and the limit in (2.10) is understood in the sense of distribution.

Hence

$$\lim_{x \to 0} \frac{\partial \phi}{\partial x} \approx \pi \delta(y) \text{ for all } y, \tag{2.11}$$

which gives, after using the condition (2.3), that

$$\frac{\partial \phi}{\partial x} = C\delta(y) \text{ on } x = 0,\ y \geqslant 0, \tag{2.12}$$

where C is a suitable constant, which can be chosen as explained below.

The solution for water of infinite depth can be expressed as

$$\phi(x,y) = \mathrm{e}^{-Ky - i\mu x} + \frac{2}{\pi} \int_0^\infty \frac{A(\xi)(\xi \cos \xi y - K \sin \xi y)\, \mathrm{e}^{-\sqrt{\xi^2+\nu^2}.x}\, \mathrm{d}\xi}{\xi^2 + K^2}, \tag{2.13}$$

where $A(\xi)$ is an unknown function to be determined.

Using the condition (2.12) in the expression (2.13), we obtain

$$-i\mu \mathrm{e}^{-Ky} - \frac{2}{\pi} \int_0^\infty \frac{\sqrt{\xi^2+\nu^2}\, A(\xi)(\xi \cos \xi y - K \sin \xi y)\, \mathrm{d}\xi}{\xi^2 + K^2} = C\delta(y), \quad 0 \leqslant y \leqslant \infty. \tag{2.14}$$

By Havelock's expansion theorem, we have

$$C = \frac{-iu}{2K} \quad \text{and} \quad \xi A(\xi) = -\frac{C\xi}{\sqrt{\xi^2+\nu^2}} \tag{2.15}$$

which gives

$$A(\xi) = \frac{i\xi \cos \alpha}{2\sqrt{\xi^2+\nu^2}}. \tag{2.16}$$

Substituting for $A(\xi)$ from (2.16) in (2.13), we obtain

$$\phi(x,y) = \mathrm{e}^{-Ky - i\mu x} + \frac{i \cos \alpha}{\pi} \int_0^\infty \frac{(\xi \cos \xi y - K \sin \xi y)\, \mathrm{e}^{-\sqrt{\xi^2+\nu^2}.x}\, \mathrm{d}\xi}{\sqrt{\xi+\nu^2}\,(\xi^2+K^2)} \tag{2.17}$$

which is the full solution in this case, agreeing with the one obtained by Chakrabarti[1] earlier. In particular, when $\alpha = 0$, the velocity potential $\phi(x,y)$ for water of infinite depth agrees with the result obtained by Mandal and Kundu [5].

2.2. *Solution for water of finite depth*

Here we discuss the solution procedure in the case when the water is of finite depth. The solution is of the form as given by

$$\phi(x,y)=\frac{e^{-i\mu x}\cosh k_0(h-y)}{\cosh k_0 h}+\sum_{n=1}^{\infty} A_n e^{-\sqrt{k_n^2+\nu^2}.x}\cos k_n(h-y), \tag{2.18}$$

where A_n's are the unknowns to be determined and $k_n, n=1,2,\ldots,n$ are the roots of the transcendental equation $K+\alpha\tan\alpha h=0$ in α. As in the previous case, we have from the conditions (2.3) and (2.4)

$$\frac{\partial\phi}{\partial x}=C'\delta(y), \quad 0\leqslant y<h, \tag{2.19}$$

where C' is a suitable constant to be determined.

The expression (2.18) along with the condition (2.19) give rise to the relation

$$\frac{-i\mu\cosh k_0(h-y)}{\cosh k_0 h}-\sum_{n=1}^{\infty} A_n\sqrt{k_n^2+\nu^2}\cos k_n(h-y)=C'\delta(y), \quad 0\leqslant y<h. \tag{2.20}$$

By using the extension of Havelock's expansion theorem as required for problems involving fluids of finite depth (see Ursell [8]), we obtain from (2.20) that

$$C'=-\frac{i\mu}{4k_0}\frac{2k_0h+\sinh 2k_0h}{\cosh^2 k_0h} \quad \text{and} \quad A_n=\frac{-4C'k_n\cos k_nh}{(2k_0h+\sin 2k_nh)\sqrt{k_n^2+\nu^2}}, \tag{2.21}$$

which gives

$$A_n=\frac{2ik_n\cos\alpha(2k_0h+\sin 2k_0h)\cos k_nh}{\sqrt{k_n^2+\nu^2}(1+\cosh 2k_0h)(2k_nh+\sin 2k_nh)}. \tag{2.22}$$

Thus the final solution is obtained by substituting for A_n in the expression (2.18) for $\phi(x,y)$. In particular, when $\alpha=0$ the solution for the case of normally incident wave can be obtained.

3. Conclusion

The problems of obliquely incident water waves by a vertical wall have been investigated in a very simple and systematic manner, in both the cases of water of infinite as well as finite depths, by suitable application of Havelock's expansion theorem to the condition on the vertical cliff and the singularity behaviour at the edge.

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Discrete fourier transform computation using prime Ramanujan numbers

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Abstract. Ramanujan numbers were introduced in [2] to implement discrete fourier transform (DFT) without using any multiplication operation. Ramanujan numbers are related to π and integers which are powers of 2. If the transform size N, is a Ramanujan number, then the computational complexity of the algorithms used for computing is $O(N^2)$ addition and shift operations, and no multiplications. In these algorithms, the transform can be computed sequentially with a single adder in $O(N^2)$ addition times. Parallel implementation of the algorithm can be executed in $O(N)$ addition times, with $O(N)$ number of adders. Some of these Ramanujan numbers of order-2 are related to the Biblical and Babylonian values of π [1]. In this paper, we analytically obtain upper bounds on the degree of approximation in the computation of DFT if N is a prime Ramanujan number.

Keywords. Discrete fourier transform; Ramanujan numbers; prime numbers; computational complexity.

1. Introduction

Discrete fourier transform (DFT) is widely used in signal processing. Direct computation of DFT of size N, requires $O(N^2)$ arithmetical operations. The author proposed the use of Ramanujan numbers for the computation of DFT in [2]. The use of these numbers permits the computation of DFT in $O(N^2)$ shift and addition operations. It does not use any multiplication operations. Use of Ramanujan numbers of order-1 were outlined in [2]. The use and properties of Ramanujan numbers of order-2 are discussed in [3]. The DFT computation using Ramanujan numbers of order-2, is generally more accurate than using Ramanujan numbers of order-1, for the same range of values of N. Ramanujan numbers can also be used in the computation of other unitary trigonometric transformations like the discrete cosine [5], and the Hartley transforms [4]. Some of these Ramanujan numbers of order-2 are related to the Biblical and Babylonian values of π [1]. In this paper, we analytically obtain upper bounds on the degree of approximation in the computation of DFT if N is a prime Ramanujan number. In § 2, Ramanujan numbers are defined. A method to compute DFT using Ramanujan numbers is also described. Details of the trigonometric evaluation are outlined in § 3. The trigonometric approximations are characterized in § 4 and bounds on the approximation are computed in § 5. The results are summarized in § 6.

2. Ramanujan numbers

In this section we introduce the basic ideas. Definitions of DFT and Ramanujan numbers are given.

DEFINITION 2.1

Let $\omega = e^{2\pi i/N}$ and $i = \sqrt{-1}$. The DFT transforms a complex input sequence $\{y_k\}$ of length N, to a sequence $\{Y_k\}$ by the relationship

$$Y_k = \sum_{n=0}^{(N-1)} y_n \omega^{nk}, \quad k = 0, 1, \ldots, (N-1). \qquad \square$$

Let $y_n = p_{n1} + ip_{n2}$, where p_{n1} and p_{n2} are real numbers. Then the DFT requires computation of the form $u_n = p \cos(2\pi n/N)$ and $v_n = p \sin(2\pi n/N)$, where p is a real number and $0 \leqslant n \leqslant (N-1)$. The sequences $\{u_n\}$ and $\{v_n\}$ can be evaluated recursively by computing a sequence of the form $\{p \cos(n\rho)\}$, where ρ is any angle. Define $p\cos(n\rho) = \lambda_n$, $n = 0, 1, 2, \ldots$, and $\cos(\rho) = \lambda$, then $\lambda_0 = p$, $\lambda_1 = p\lambda$, and

$$\lambda_{n+1} = 2\lambda\lambda_n - \lambda_{n-1}, \quad n = 1, 2, 3, \ldots.$$

This is a Chebyshev type of recursion [7]. The sequence $\{\lambda_n\}$ can be evaluated by shift and addition operations, if $\lambda \cong (1 - \rho^2/2)$, and ρ has a simple representation in powers of 2. This is possible, if N is chosen to be a Ramanujan number. N is a Ramanujan number of order-1, if $2\pi/N \cong 2^{-a}$, where a is a nonnegative integer, that is $0 \leqslant a$, $a \in Z$. Z is the set of integers.

DEFINITION 2.2

Ramanujan number of order-1 is defined as follows. Let $a \in Z$, and [ ] is a function which rounds off its argument to its nearest integer.

$$\mathfrak{R}_1(a) = \left[\frac{2\pi}{\vartheta_1(a)}\right], \qquad \vartheta_1(a) = 2^{-a}, \quad 0 \leqslant a. \qquad \square$$

For example, $\mathfrak{R}_1(3) = 50$ and $\mathfrak{R}_1(5) = 201$. Ramanujan numbers of order-2 are defined next. These are defined such that $2\pi/N$ is approximated by sum or difference of two numbers which are negative powers of 2.

DEFINITION 2.3

Ramanujan numbers of order-2 are defined as follows. Let $l, m \in Z$. Then

$$\mathfrak{R}_{2j}(l, m) = \left[\frac{2\pi}{\vartheta_{2j}(l, m)}\right] \qquad j = 1, 2,$$

$$\vartheta_{21}(l, m) = 2^{-l} + 2^{-m} \qquad m > l \geqslant 0,$$

$$\vartheta_{22}(l, m) = 2^{-l} - 2^{-m} \qquad (m-1) > l \geqslant 0. \qquad \square$$

For example, $\mathfrak{R}_{21}(1, 3) = 10$ and $\mathfrak{R}_{21}(3, 5) = 40$. For the same range of values of N, DFT computation using Ramanujan numbers of order-2 is generally more accurate than using Ramanujan numbers of order-1. With the use of these numbers, the value of π is implicitly being approximated. Denote this approximated value by $\hat{\pi}$. We drop the subscript and functional dependence on $\mathfrak{R}(\ )$ and $\vartheta(\ )$ for simplicity. Then

$$\hat{\pi} = \frac{1}{2} \mathfrak{R}\, \vartheta,$$

where ε_π is the relative error in π. For example, for $N = \Re_{21}(4, 5) = 67$, $\hat{\pi} = 3{\cdot}140625$, $\varepsilon_\pi = -3{\cdot}0801 \times 10^{-4}$. For higher values of N, the absolute value of ε_π is generally smaller. This statement can be encapsulated more precisely in the following lemmas.

Lemma 2.1. *Let* $y \in R^+$, *where* R^+ *is the set of positive real numbers. If* $[y] = y(1 + \delta)$, *then* $|\delta| \leqslant 1/(2y)$.

Proof. $[y] = y + y\delta$. Then $|y\delta| \leqslant 1/2$ by definition of the rounding operation. The result follows. □

Lemma 2.2. $|\varepsilon_\pi| \leqslant \vartheta/(4\pi)$.

Proof. Let N be a Ramanujan number. Then using the above equations we obtain $N = [2\pi/\vartheta] = 2\hat{\pi}/\vartheta$. Using the definition of $\hat{\pi}$ and the result of Lemma 2.1, we get the desired result. □

From Lemma 2.2, it is readily observed that the upper bound on $|\varepsilon_\pi|$ can be approximated by $1/(2N)$. Therefore ε_π is $O(N^{-1})$. The result of Lemma 2.2 will be used subsequently to establish analytically the degree of approximation in the computation of DFT of size, a prime Ramanujan number. It can be noted parenthetically that for $N = \Re_{21}(1, 2) = 8$, $\hat{\pi} = 3{\cdot}0$. This is the Biblical value of π. Also for $N = \Re_{21}(0, 2) = 5$, $\hat{\pi} = 3{\cdot}125$. This is the Babylonian value of π [1]. We outline an algorithm to do trigonometric evaluation in the next section.

3. Trigonometric evaluation

Details of the trigonometric evaluation via a Chebyshev-type of recursion are briefly outlined. This algorithm is identified as algorithm A in [2]. To compute the DFT, an algorithm is developed to evaluate sequences of the type

$$\{u_n | u_n = p \cos(2\pi n/N), 0 \leqslant n \leqslant (N-1),\ p \in R\},$$

$$\{v_n | v_n = p \sin(2\pi n/N), 0 \leqslant n \leqslant (N-1),\ p \in R\},$$

where R is the set of real numbers. Evaluations of these type of sequences are needed in the computation of the DFT. The members of the finite sequences $\{u_n\}$ and $\{v_n\}$ can be computed as follows. Define the sequence $W(M, p)$ as

$$\{w_n | w_n = p \cos(2\pi n/M), 0 \leqslant n \leqslant \Psi,\ p \in R\},$$

where $\Psi = (M/4 - 1)$ and $M = \beta N$. β is equal to 1, if N is divisible by 4. It is equal to 2, if N is divisible by 2, but not by 4. Otherwise it is equal to 4 (N is not divisible by 2). Note that M is divisible by 4. The use of the variable β facilitates the computation of the sequence $W(M, p)$ by considering cosine values from only the first quadrant of the circle. Then $\{u_n\}$ and $\{v_n\}$ can be evaluated by computing the members of the sequence $W(M, p)$. This $W(M, p)$ sequence can be estimated as follows. Approximate w_n's by z_n's, $0 \leqslant n \leqslant \Psi$. Then

$$z_0 = p,$$

$$z_1 = (1 - \alpha)\, p,$$

$$z_{n+1} = 2(1 - \alpha)\, z_n - z_{n-1},\ 1 \leqslant n \leqslant (\Psi - 1),$$

where $\alpha = (2\hat{\pi}/M)^2/2 = (\vartheta/\beta)^2/2$. α is of the form 2^{-d}, $d \in Z$ for N, a Ramanujan number of order-1. If N is a Ramanujan number of order-2, then α has a simple binary representation. Notice that z_n's can be computed by shift and addition operations. If N is a Ramanujan number of order-1, the computation of z_n's would require $(2\Psi - 1)$ equal to $(M/2 - 3)$ additions. Shift operations take negligible time. The DFT can be evaluated by computing the sequences $W(M, p_{nj}), j = 1, 2$, where $y_n = p_{n1} + ip_{n2}$; and $0 \leqslant n \leqslant (N-1)$. It should now be clear that, a sequential evaluation of DFT, with a single adder takes up $O(N^2)$ addition times. With the use of $O(N)$ adders, a parallel implementation of DFT will require $O(N)$ addition times. Two more numerically superior algorithms are outlined in [2]. Next, we characterize the approximation in the computation of the DFT.

4. Characterization of trigonometric approximation

The sequence $C(M) = \{c_n | c_n = \cos(nx), x = 2\pi/M, 0 \leqslant n \leqslant \Psi\}$ is implicitly involved in the computation of the sequence $W(M, p)$. The sequence $C(M)$ in turn is implicitly approximated by the sequence $\hat{C}(M) = \{\hat{c}_n | \hat{c}_n \cong c_n, 0 \leqslant n \leqslant \Psi\}$. In this sequence, $\hat{c}_0 = 1$, $\hat{c}_1 = 1 - \hat{x}^2/2$, $\hat{c}_{n+1} = 2\hat{c}_1 \hat{c}_n - \hat{c}_{n-1}$, $1 \leqslant n \leqslant (\Psi - 1)$; $\hat{x} = 2\hat{\pi}/M = \vartheta/\beta$. Also $z_n = p\hat{c}_n$, $0 \leqslant n \leqslant \Psi$. Recall that the variable α, used in the computation of the z_n's is equal to $\hat{x}^2/2$. We next estimate the error $\Delta c_n = (\hat{c}_n - c_n)$, $0 \leqslant n \leqslant \Psi$ via the following two lemmas.

Lemma 4.1. $\Delta c_1 = (\hat{c}_1 - c_1) \cong -x^2 \varepsilon_\pi$.

Proof. $\Delta x = (\hat{x} - x) = x\varepsilon_\pi$, $c_1 = 1 - x^2/2 + O(x^{-4})$, $x < 1$, $\hat{c}_1 = 1 - \hat{x}^2/2$. Then $\Delta c_1 \cong -(\hat{x}^2 - x^2)/2 = -(\hat{x} - x)(\hat{x} + x)/2 \cong -x^2\varepsilon_\pi$. Error in the approximation of c_1 by $\hat{c}_1$ is dominated by the approximation of x by $\hat{x}$, and not by the truncation of the cos x series. □

Lemma 4.2. $\Delta c_n = (n \sin nx/\sin x)\, \Delta c_1 \cong -nx\varepsilon_\pi \sin(nx)$, $0 \leqslant n \leqslant \Psi$.

Proof. $c_n = \cos(n x) = \cos(n \cos^{-1} c_1)$. Then $dc_n/dc_1 = n \sin(nx)/\sin x$. The result follows by using Lemma 4.1. □

Lemma 4.2 is used in the next section to evaluate the degree of approximation in the computation of the DFT.

5. Approximation bound

The degree of approximation in the computation of the transform can be characterized by the ratio of the average magnitude squared error in the transformed sequence, and the average magnitude square of the transformed sequence. Indicate this ratio by A_N. This value A_N can be bounded as in [2]. Let $c_{(nk)} = \cos(2\pi nk/N)$, $s_{(nk)} = \sin(2\pi nk/N)$ for $n, k = 0, 1, 2, \ldots, (N-1)$. Also, let $\hat{c}_{(nk)}$'s and $\hat{s}_{(nk)}$'s be the computed values of $c_{(nk)}$'s and $s_{(nk)}$'s respectively, that is, $\hat{c}_{(nk)} \cong c_{(nk)}$, $\hat{s}_{(nk)} \cong s_{(nk)}$ for $i, k = 0, 1, 2, \ldots, (N-1)$. Define $\Delta c_{(nk)} = (\hat{c}_{(nk)} - c_{(nk)})$ and $\Delta s_{(nk)} = (\hat{s}_{(nk)} - s_{(nk)})$ for $i, k = 0, 1, 2, \ldots, (N-1)$. Then using the Cauchy–Schwarz–Bunyakovski inequality 6], it can be shown as in [2] that A_N is bounded by D_N/N. Where

$$D_N = \sum_{k=0}^{(N-1)} \sum_{n=0}^{(N-1)} \{\Delta c^2_{(nk)} + \Delta s^2_{(nk)}\}.$$

If a Ramanujan number is prime, for example $\Re_1(1) = 13$, $\Re_{21}(2, 3) = 17$ and $\Re_{22}(2, 5) = 29$, then a simple expression can be obtained for the upper bound on A_N via the following theorem.

Theorem 5.1. *If N is a prime number, then $A_N \leqslant N^3 \Delta c_1^2/(3 \sin^2 x)$.*

Proof. Since N is a prime number, the c_k's, $1 \leqslant k \leqslant (N-1)$ are implicitly used equally in the computation of D_N. Each of these is used $2(N-1)$ times, while $c_0 = 1$ and $c_N = 0$ is used $(2N-1)$ times each. Therefore

$$A_N \leqslant \frac{1}{N}\left[2(N-1)\sum_{k=0}^{N} \Delta c_k^2 + \Delta c_0^2 + \Delta c_N^2\right].$$

But Δc_0 and Δc_N, the error in the computation of c_0 and c_N are each zero, because c_0 and c_N do not require any computation. Also, if N is a prime number $\Psi = (N-1)$, and $x = \pi/2N$. Using Lemma 4.2, we obtain

$$A_N \leqslant \frac{2(N-1)}{N}\sum_{k=1}^{(N-1)} k^2 \Delta c_1^2 \sin^2(kx)/\sin^2 x.$$

It can be shown that

$$\sum_{k=1}^{N} k^2 \sin^2(kx) = \frac{N(N+1)(2N+1)}{12}.$$

The theorem follows. □

Using Lemma 4.1, the upper bound on A_N given by Theorem 5.1 can be approximated by $\pi^2 \varepsilon_\pi^2 N/12$. Notice that the upper bound on A_N has a squared dependency on ε_π. Therefore smaller the value of ε_π, smaller is the upper bound on A_N. It can also be observed that, since ε_π is $O(N^{-1})$, the upper bound on A_N is $O(N^{-1})$. Consider the following examples. If $N = \Re_{21}(0, 2) = 5$, $\hat{\pi} = 3{\cdot}125$, $\varepsilon_\pi = -5{\cdot}2816 \times 10^{-3}$, $\Delta c_1 = 1{\cdot}1536 \times 10^{-4}$, then the upper bound on A_N from Theorem 5.1 is equal to $5{\cdot}80664 \times 10^{-6}$. And if, $N = \Re_{22}(1, 6) = 13$, $\hat{\pi} = 3.1484375$, $\varepsilon_\pi = 2{\cdot}1788 \times 10^{-3}$, $\Delta c_1 = -4{\cdot}0722 \times 10^{-5}$, the upper bound on A_N from Theorem 5.1 is equal to $8{\cdot}3586 \times 10^{-5}$. Notice the low values of these upper bound estimates.

6. Summary

Use of Ramanujan numbers facilitates the computation of DFT by addition and shift operations. The computational complexity of the algorithms used, for computing a transform of size N is $O(N^2)$ addition and shift operations. In these algorithms, the transform can be computed sequentially with a single adder in $O(N^2)$ addition times. Parallel implementation of the algorithm can be executed in $O(N)$ addition times, with $O(N)$ number of adders. Further, Ramanujan numbers can be used for computing other trigonometric transforms like the discrete cosine and Hartley transforms. Some of the Ramanujan numbers of order-2 are related to the Biblical and Babylonian values of π. In this paper, we analytically obtained upper bounds on the degree of approximation in the computation of DFT if N is a prime Ramanujan number.

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Degenerations of the moduli spaces of vector bundles on curves I

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Abstract. Let Y be a smooth projective curve degenerating to a reducible curve X with two components meeting transversally at one point. We show that the moduli space of vector bundles of rank two and odd determinant on Y degenerates to a moduli space on X which has nice properties, in particular, it has normal crossings. We also show that a nice degeneration exists when we fix the determinant. We give some conjectures concerning the degeneration of moduli space of vector bundles on Y with fixed determinant and arbitrary rank.

Keywords. Nodal curve; torsion free sheaf; moduli.

1. Introduction

Let Y be a smooth projective curve specializing to a projective curve X with nodes as the only singularities. One knows that the theory of moduli spaces of vector bundles on a smooth projective curve extends in a nice manner to the case of X, using torsion free sheaves; further these moduli spaces have good specialization properties, namely the moduli space $M(Y)$ on Y (fixing certain invariants) specializes to a corresponding space $M(X)$ on X (see [11, 8, 7, 13]). Now $M(X)$ is closely related to a corresponding moduli space $M(\tilde{X})$ on the normalization $\tilde{X}$ of X, though working out this relationship is a non-trivial exercise. These facts serve as a tool for an indepth study of the moduli space $M(Y)$, based on induction on the genus of Y, for the genus of any irreducible component of $\tilde{X}$ is strictly less than that of Y. For example, the work of Narasimhan and Ramadas concerning the Verlinde formulae is motivated by these considerations (see [7]).

The work of Gieseker proving a conjecture of Newstead and Ramanan on the vanishing of certain Chern classes of the moduli space $M(Y)$ of rank two and odd degree, is also motivated by similar considerations (see [4]). However, Gieseker does *not* use torsion free sheaves. He takes X to be *irreducible* with only *one node* and gives a specialization $M'(X)$ of $M(Y)$, where $M'(X)$ is a scheme intrinsically attached to X with its singularities as normal crossings. Further the total space which gives this specialization is regular, if the total space which gives the specialization of Y to X is regular. Gieseker obtains $M'(X)$ by using only *vector bundles* on some curves semi-stably equivalent to X.

The aim of this work is to show that if we take X to be *reducible with only one node p*, the moduli spaces using torsion free sheaves give all the good properties given by Gieseker. To be more precise, a suitable moduli space $M(X)$ of rank two torsion free sheaves on X has only two smooth projective irreducible components M_1, M_2 which intersect transversally on a (smooth) irreducible variety N. One can describe M_1, M_2 by a certain moduli problem of vector bundles of rank two on the normalization $\tilde{X}$ of X,

which is the disjoint union of the irreducible components X_1, X_2 of X. The variety N turns out to be a product of certain parabolic moduli spaces on X_1, X_2 respectively. Further, the moduli space $M(Y)$ of vector bundles on Y of rank two and odd degree specializes to $M(X)$ such that the total space which gives this specialization is regular if the total space which gives the specialization of Y to X is regular (see § 4). The crucial point which leads to the above nice properties is that if $M(X)$ is chosen as the moduli space of semi-stable, rank two, torsion free sheaves for a *generic polarization* on X, then if $V \in M(X)$, V *cannot* be of the form $(m_{X,p} \oplus m_{X,p})$ locally at the node p of X ($m_{X,p}$ = maximal ideal of the stalk $O_{X,p}$ of the structure sheaf of X at p), i.e. either (a) V is locally free at p, or (b) V is of the form $m_{X,p} \oplus \mathcal{O}_{X,p}$ locally at p. By deformation arguments as in [3], it is not difficult to see that the functor defining $M(X)$ at V of the form as in (b), is *formally smooth* over the local ring $\mathcal{O}_{X,p}$, which leads to the required normal crossing.

That torsion free sheaves of the form $m_{X,p} \oplus m_{X,p}$ locally at p, may not appear in a well-chosen $M(X)$, could be seen intuitively as follows. The moduli problem corresponding to $M(X)$ can be translated as one on vector bundles on the normalization $\tilde{X}$ of X. A vector bundle V of rank two on $\tilde{X}$ gives rise to a pair $V = (V_1, V_2)$, where V_i is a rank two vector bundle on $X_i, i = 1, 2$. Then the moduli problem on $\tilde{X}$ is the moduli problem of vector bundles $V = (V_1, V_2)$ together with a homomorphism A of the fibre of V_1 into that of V_2 at p, i.e. the moduli problem of *triples* (V_1, V_2, A) (see § 2). An element of $\mathbb{G}_m \times \mathbb{G}_m$ gives a canonical automorphism of V (scalar multiplication on each component). Thus a triple (V_1, V_2, A) is isomorphic to $(V_1, V_2, \lambda A)$, λ a non-zero scalar. Thus the moduli problem of triples (V_1, V_2, A), $A \neq 0$, is equivalent to the moduli problem of triples (V_1, V_2, A') where A' is an element of the *projective space* associated to the vector space of homomorphisms of this fibre of V_1 into that of V_2 at p. Now this defines a functor which is *proper* over the moduli functor of vector bundles on $\tilde{X}$. Hence one could expect a *proper scheme* to solve this moduli problem by using GIT techniques. Now the triples of the form $(V_1, V_2, 0)$, correspond precisely to torsion free sheaves, which are of the form $m_{X,p} \oplus m_{X,p}$, locally at p, and the foregoing argument shows that we could expect to avoid them.

The notion of a triple is an improvement on the considerations in [11] trying to relate the moduli problem of torsion free sheaves on X (with a fixed Hilbert polynomial) to one on the normalization $\tilde{X}$ of X. The crucial point is that this can be translated as a moduli problem of vector bundles on X with a *fixed Hilbert polynomial* (this does not figure in [11] and is inspired by the GPB's of Usha Bhosle [2]).

It seems likely that the degeneration $M(X)$ of $M(Y)$ that we give, could also be used to prove the above mentioned conjecture of Newstead and Ramanan on the vanishing of certain Chern classes of $M(Y)$ (moduli space of rank two and odd degree) as well as the Verlinde formulae (for the rank two case). To do this, one has to generalize our moduli space $M(X)$ by taking parabolic structures at a finite number of smooth points on X (as is done in the work of Narasimhan and Ramadas, cited above) to have correct induction machinery.

Consider now the general case of a smooth projective curve Y specializing to X with only nodes. Let $\mathcal{L}$ be a line bundle on Y of degree d specializing to a torsion free sheaf $\mathcal{L}$ of rank one and degree d on X. Let $M(Y, n, d)$ denote the moduli of semi-stable vector bundles V of rank n and degree d and $M(Y, n, \mathcal{L})$ denote the subvariety of $M(Y, n, d)$ such that $\det V = \bigwedge^n V = \mathcal{L}$. Then the question how $M(Y, n, \mathcal{L})$ specializes when $\mathcal{L}$ specializes to X, has not been properly investigated. In particular, one could ask

whether $M(Y, n, \mathscr{L})$ specializes to an intrinsically defined subscheme of a moduli space $M(X, n, d)$ to which $M(Y, n, d)$ specializes, which we could then denote by $M(X, n, L)$. The difficulty in this problem is that, in general, there does not seem to be a well-defined morphism det: $M(X, n, d) \to M(X, 1, d)$.

An interesting corollary of our method is that in our case (X reducible with one node) we get a well-defined morphism det: $M(X, 2, d) \to M(X, 1, d)$, which shows that $M(Y, 2, \mathscr{L})$ specializes to a nice subscheme $M(X, 2, L)$ of $M(X, 2, d)$ (see § 7).

Take now X to be irreducible with only one node. Then isomorphism classes of torsion free sheaves of rank one with a fixed degree d, is a compactification $M(X, 1, d)$ of the generalized Jacobian $J^d(X)$ of isomorphism classes of line bundles on X of degree d. Let $M(X, n, d)$ be the moduli space of semi-stable torsion-free sheaves of rank n and degree d and $M(X, n, d)^0$ the open subscheme consisting of vector bundles. Then we have a morphism

$$\det: M(X, n, d)^0 \to J^d(X),$$

$$V \mapsto \overset{n}{\wedge} V.$$

This does not seem to extend to a morphism of $M(X, n, d)$ into $M(X, 1, d)$. However, we give a precise conjecture which gives the specialization of $M(Y, n, \mathscr{L})$ as an intrinsically defined subscheme $M(X, n, L)$ of $M(X, n, d)$. In case L is a line bundle, $M(X, n, L)$ is just the closure of the fibre $(\det)^{-1}(L)$ in $M(X, n, d)$.

After we submitted the manuscript to the Journal, Balaji pointed out the paper of Huashi Xia [14]. Some of the results in our paper are also contained in [14] but the methods are different.

2. Triples associated to a torsion free sheaf on a reducible nodal curve

Let X be the nodal curve which is a union of two smooth curves X_1 of genus g_1 and X_2 of genus g_2 meeting at one point. Let p be the node of X.

Locally at the node p of X we have the following: If $m_{X,p}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$ then there exists $x_1, x_2 \in m_{X,p}$ such that $X_1 = \{x_2 = 0\}$ and $X_2 = \{x_1 = 0\}$.

PROPOSITION 2.1

Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathscr{F}$ has depth 1 *(i.e. depth* 1 *at every $x \in X$) if and only if $\mathscr{F}$ is pure of dimension* 1 *i.e. for all $\mathcal{O}_X$-submodules $\mathscr{G} \subsetneq \mathscr{F}$, $\mathscr{G} \neq 0$, dimension of* Supp *$(\mathscr{G}) = 1$.*

Proof. Since depth is a local property and proposition is clear at the smooth points of X, we need to prove the proposition at the node p of X. Hence we assume $\mathscr{F}$ is a finitely generated $\mathcal{O}_{X,p}$-module. Suppose $\mathscr{F}$ is not pure. Then there exists a subsheaf $T \subsetneq \mathscr{F}$, $(T \neq 0)$ such that T is supported at p. This implies $m_{X,p}^r T = 0$. Take r minimal; so that there exists a $t \in T$, $t \neq 0$ such that $a \cdot t \neq 0$ for some $a \in m_{X,p}^{r-1}$. Set $b = a \cdot t \in T$. Then $x \cdot b = 0$ for every $x \in m_{X,p}$. This implies that the depth of $\mathscr{F}$ is zero, which contradicts the assumption that the depth of $\mathscr{F}$ is one. This proves the only if part of the proposition. To prove the if part of the proposition, assume that $\mathscr{F}$ is not of depth one. i.e. for all

$a \in m_{X,p}$, $a \neq 0$, m_a: $\mathscr{F} \to \mathscr{F}$ (multiplication by a) is not injective. Then consider

$$\mathrm{Ann}(a) = \{f \in \mathscr{F} \mid a \cdot f = 0\},$$

where a is such that its restriction to every X_i is not zero (i.e. a is a part of an $m_{X,p}$ sequence). Then the dimension of the support of Ann(a) is zero and $\mathrm{Ann}(a) \subsetneq \mathscr{F}$, and $\mathrm{Ann}(a) \neq 0$. This shows that $\mathscr{F}$ is not pure.

DEFINITION 2.1

A depth 1 sheaf $\mathscr{F}$ on X is called a torsion free sheaf. We say that $\mathscr{F}$ is of rank (r_1, r_2) on X if $\mathscr{F}$ restricted to X_i is of rank $r_i, i = 1, 2$. We say $\mathscr{F}$ is of rank r if $r_1 = r_2 = r$.

Note. One of the r_i (not both) could be zero for a torison free sheaf.

Notations. Let $\mathscr{F}$ be a torsion free sheaf on X. Let $\bar{\mathscr{F}}_1$ be the restriction of $\mathscr{F}$ to X_1 and $\bar{\mathscr{F}}_2$ be the restriction of $\mathscr{F}$ to X_2. Note that locally at p we have $\bar{\mathscr{F}} = \mathscr{F}/x_2.\mathscr{F}$ and $\bar{\mathscr{F}}_2 = \mathscr{F}/x_1.\mathscr{F}$. Let $\mathscr{F}_1 = \bar{\mathscr{F}}_1/(\text{torsion})$ and $\mathscr{F}'_2 = \bar{\mathscr{F}}_2/(\text{torsion})$.

PROPOSITION 2.2

Let $\mathscr{F}$ be a torsion free sheaf of rank (r, s) on X. The canonical homomorphisms

$$\mathscr{F} \to \bar{\mathscr{F}}_1 \oplus \bar{\mathscr{F}}_2$$

and

$$\mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}'_2$$

are injective. Besides if we set

$$0 \to \mathscr{F} \to \bar{\mathscr{F}}_1 \oplus \bar{\mathscr{F}}_2 \to \bar{T} \to 0,$$

$$0 \to \mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}'_2 \to T \to 0,$$

the supports of $\bar{T}$ and T are at p and in fact

$$m_{X,p}\, T = m_{X,p}\, \bar{T} = 0.$$

Proof. The injectivity assertions are immediate, for the canonical homomorphisms

$$\mathscr{F} \to \bar{\mathscr{F}}_1 \oplus \bar{\mathscr{F}}_2$$

and

$$\mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}'_2$$

are isomorphisms outside p. Hence the kernels of these homomorphisms are supported at p and $\mathscr{F}$ being pure, the kernels reduce to zero.

To prove the remaining assertions, note that they are really local at the mode p of X and hence we assume that $\mathscr{F}$ is a finitely generated $\mathscr{O}_{X,p}$-module. Note that we have the following exact sequence of $\mathscr{O}_{X,p}$-modules

$$0 \to x_2\mathscr{F} \to \mathscr{F} \to \bar{\mathscr{F}}_1 \to 0,$$

$$0 \to x_1\mathscr{F} \to \mathscr{F} \to \bar{\mathscr{F}}_2 \to 0.$$

Consider the canonical homomorphism $x_2\mathscr{F} \to \mathscr{F}'_2$ given by composite

$$x_2\mathscr{F} \subsetneq \mathscr{F} \to \bar{\mathscr{F}}_2 \to \mathscr{F}'_2.$$

This is an isomorphism outside p and since $x_2\mathscr{F}$ (being a subsheaf of $\mathscr{F}$) is pure, it follows that $x_2\mathscr{F} \to \mathscr{F}'_2$ is injective. Besides, we see that $x_2\mathscr{F}$ maps onto $x_2\mathscr{F}'_2$ and hence $x_2\mathscr{F}$ identifies with $x_2\mathscr{F}'_2$ (this is immediate, for $\mathscr{F}$ maps onto $\bar{\mathscr{F}}_2$ and hence maps onto $\mathscr{F}'_2$). Similarly $x_1\mathscr{F} \to \mathscr{F}_1$ is injective and $x_1\mathscr{F}$ identifies with $x_1\mathscr{F}_1$. By the same argument we see that the canonical homomorphisms

$$x_1\mathscr{F} \to \bar{\mathscr{F}}_1, x_2\mathscr{F} \to \bar{\mathscr{F}}_2$$

are injective and hence we get the following natural isomorphisms

$$x_1\mathscr{F} \simeq x_1\bar{\mathscr{F}}_1, x_2\mathscr{F} \simeq x_2\bar{\mathscr{F}}_2.$$

(In particular it follows that x_2 annihilates the torsion of $\bar{\mathscr{F}}_2$ and since x_1 annihilates $\bar{\mathscr{F}}_2$, it follows that $m_{X,p}$ annihilates the torsion of $\bar{\mathscr{F}}_2$.)

Now to show that, say

$$m_{X,p}\, T = 0,$$

we have only to show that

$$x_i(\mathscr{F}_1 \oplus \mathscr{F}'_2) \subset \mathscr{F} \ (i = 1, 2).$$

Take say x_2, then $x_2\bar{\mathscr{F}}_1 = 0$ by definition and hence $x_2\mathscr{F}_1 = 0$. Besides we have seen $x_2\mathscr{F}'_2 \simeq x_2\mathscr{F}$ thus

$$x_2(\mathscr{F}_1 \oplus \mathscr{F}'_2) \subset \mathscr{F}.$$

Similarly

$$x_1(\mathscr{F}_1 \oplus \mathscr{F}'_2) \subset \mathscr{F}.$$

(The injectivity of say

$$\mathscr{F} \to \bar{\mathscr{F}}_1 \oplus \bar{\mathscr{F}}_2$$

also follows in an explicit way, wince the kernel of this map is $x_2\mathscr{F} \cap x_1\mathscr{F}$, which is annihilated by $m_{X,p}$, and since $\mathscr{F}$ is pure, this kernel is zero.) This proves the proposition.

Notations. Let $\mathscr{F}(p), \mathscr{F}_1(p), \mathscr{F}'_2(p)$ etc. denote the fibres at p i.e., $\mathscr{F}(p) = \mathscr{F}/m_{X,p}\mathscr{F}$ etc.

Remark 2.1. Set

$$\mathbf{N} = \mathrm{Ker}(\mathscr{F}_1(p) \oplus \mathscr{F}'_2(p) \to T).$$

Then the canonical projections

$$\mathbf{N} \to \mathscr{F}_1(p) \text{ and } \mathbf{N} \to \mathscr{F}'_2(p)$$

are surjective.

Proof. Since $\mathscr{F} \to \mathscr{F}_1$ and $\mathscr{F} \to \mathscr{F}'_2$ are surjective, we see that the canonical homomorphisms $\mathscr{F}(p) \to \mathscr{F}_1(p)$ and $\mathscr{F}(p) \to \mathscr{F}'_2(p)$ are surjective.

Note that the sequence

$$\mathscr{F}(p) \to \mathscr{F}_1(p) \oplus \mathscr{F}'_2(p) \to T \to 0$$

is exact and hence $\mathbf{N}$ = image of the canonical homomorphism $(\mathscr{F}(p) \to \mathscr{F}_1(p) \oplus \mathscr{F}'_2(p))$.

Therefore the surjections

$$\mathscr{F}(p) \to \mathscr{F}_1(p) \text{ and } \mathscr{F}(p) \to \mathscr{F}'_2(p)$$

pass through $\mathbf{N}$ and hence

$$\mathbf{N} \to \mathscr{F}_1(p) \text{ and } \mathbf{N} \to \mathscr{F}'_2(p)$$

are surjective.

Remark 2.2. From the above remark it follows that $\mathscr{F}$ can be identified with the subsheaf of $\mathscr{F}_1 \oplus \mathscr{F}'_2$ consisting of all $f=(f_1, f_2)$ such that the evaluation of f at p is in $\mathbf{N}$. More generally if $\mathscr{G}_i$ is a vector bundle on X_i for $i=1,2$ and $\mathbf{M}$ is a linear subspace of $\mathscr{G}_1(p) \oplus \mathscr{G}_2(p)$ then the subsheaf

$$\mathscr{G} = \{g=(g_1, g_2) | g_i \in \mathscr{G}_i, i=1,2 \text{ and } g(p)=(g_1(p), g_2(p)) \in \mathbf{M}\}$$

of $\mathscr{G}_1 \oplus \mathscr{G}_2$ is a torsion free $\mathscr{O}_X$-submodule.

Remark 2.3. Let $K = \text{Ker}\{\text{canonical homomorphism } \mathbf{N} \to \mathscr{F}_1(p)\}$. Then we see $K \subset \mathscr{F}'_2(p)$. Let $i: \mathscr{F}'_2 \to \mathscr{F}_2$ be the Hecke modification with $\text{Ker}(i_p) = K$ (see below for the definition of a Hecke modification.) Set $\mathbf{N}^1 = \text{Image } (\mathbf{N})$ under the canonical homomorphism

$$\mathscr{F}_1(p) \oplus \mathscr{F}'_2(p) \xrightarrow{\Theta} \mathscr{F}_1(p) \oplus \mathscr{F}_2(p).$$

Then we see that

a) the canonical homomorphism of $\mathbf{N}^1$ onto $\mathscr{F}_1(p)$ is surjective and since $\dim(\mathbf{N}^1) = \dim(\mathscr{F}_1(p))$, the map $\mathbf{N}^1 \to \mathscr{F}_1(p)$ is in fact an isomorphism.
b) $\mathbf{N} = \Theta^{-1}(\mathbf{N}^1)$.

From the following commutative diagram of $\mathscr{O}_X$ modules

$$\begin{array}{ccc} \mathscr{F} & \subsetneq & \mathscr{F}_1 \oplus \mathscr{F}'_2 \\ \| & & \downarrow \\ \mathscr{F} & \subsetneq & \mathscr{F}_1 \oplus \mathscr{F}_2 \end{array}$$

we see that

$$\mathscr{F} = \{f=(f_1, f_2) \in \mathscr{F}_1 \oplus \mathscr{F}_2 | \text{ evaluation of } f \text{ at } p \text{ is in } \mathbf{N}^1\}.$$

Now giving $\mathbf{N}^1$ is equivalent to defining a homomorphism

$$A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)$$

and hence we can also define $\mathscr{F}$ as

$$\mathscr{F} = \{f=(f_1, f_2) \in \mathscr{F}_1 \oplus \mathscr{F}_2 | A(f_1(p)) = f_2(p)\}.$$

Remark 2.4. Let V be a vector bundle on a smooth curve Y and K be a subspace of V_p (p be a point of Y). Then there are two canonical constructions called Hecke modifications defined as follows:

I) $W \xrightarrow{\phi} V$, $\text{Im}(W_p) = K$, where W is a vector bundle and ϕ is a homomorphism of vector bundles, which is an isomorphism outside p.

(II) $V \xrightarrow{\phi} W$, $\mathrm{Ker}(\phi_p) = K$, where W is a vector bundle and ϕ is a homomorphism of vector bundles, which is an isomorphism outside p.

W and ϕ in (I) are defined as follows:
Let $T = V_p/K$ and $j: V \to T$ be the canonical $\mathcal{O}_X$-module homomorphism.
Then $W = \mathrm{Ker}(j)$ and ϕ be the natural homomorphism.

To define W and ϕ of (II) we proceed as follows:
Let $V_p \times V_p^* \to k (= k(p))$ be the canonical dual pairing and $K^\perp$ denote the orthogonal of K under the dual pairing. Let V^* be the dual of V. Define W^* and ϕ^* as in (I) such that

$$W^* \xrightarrow{\phi^*} V^* \quad \mathrm{Im}(W_p^*) = K^\perp.$$

Let ϕ be the dual of ϕ^* and W be the dual of W^*. Then $W \xrightarrow{\phi} V$ is a homomorphism of vector bundles and ϕ_p is the dual of ϕ_p^*. Also

$$\mathrm{Ker}(\phi) = (\mathrm{Im}(\phi^*))^\perp = (K^\perp)^\perp = K.$$

Thus we get the W and ϕ of (II) with the required properties.

Note. a) If $K = 0$ in the above remark, then in both the constructions we have $W = V$.
b) If $K = V_p$ in the above remark, then in case (I) we get $W = V \otimes \mathcal{O}_Y(-p)$ and in case (II) we get $W = V \otimes \mathcal{O}_Y(p)$.

DEFINITION 2.2

Let $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) be a locally free sheaf on X_1 (resp. on X_2). Let $A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)$ be a linear map. Then we call $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ a triple, where $\vec{A}$ denotes that A is a homomorphism from $\mathscr{F}_1(p)$ to $\mathscr{F}_2(p)$.

Remark 2.5. As we have seen above every torsion free sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ gives rise to a triple.

Note that we could have similarly defined a canonical Hecke modification $\mathscr{F}_1 \to \mathscr{F}'_1$ and a homomorphism $B: \mathscr{F}'_2(p) \to \mathscr{F}'_1(p)$ (here $\mathscr{F}_1, \mathscr{F}'_2$ are as in the notations after the proof of the Proposition 2.1) so that

$$\mathscr{F} = \{f = (f_1, f_2) \in \mathscr{F}'_1 \oplus \mathscr{F}'_2 \mid B(f_2(p)) = f_1(p)\}.$$

We write this triple as $(\mathscr{F}'_1, \mathscr{F}'_2, \vec{B})$. Thus to $\mathscr{F}$ we can associate canonically two triples.

We denote the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ or $(\mathscr{F}'_1, \mathscr{F}'_2, \vec{B})$ by $\mathscr{F}$ and often by abuse of notation $\mathscr{F}$ also denotes the $\mathcal{O}_X$-module defined by the triples.

Note that given a triple $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, the other triple associated to the $\mathcal{O}_X$-module $\mathscr{F}$ can be obtained directly as follows: Consider the diagram

$$\begin{array}{ccc} \mathscr{F}_1(p) & \xrightarrow{i_p} & \mathscr{F}'_1(p) \\ & \downarrow A & \\ \mathscr{F}_1(p) & \xleftarrow{j_p} & \mathscr{F}'_2(p) \end{array}$$

where $\mathscr{F}_1 \xrightarrow{i} \mathscr{F}'_1$ (resp. $\mathscr{F}_2 \xleftarrow{j} \mathscr{F}'_2$) is the canonical Hecke modification such that $\mathrm{Ker}(i_p) = \mathrm{Ker}(A)$ (resp. $\mathrm{Im}(j_p) = \mathrm{Im}(A)$). Then there is a canonically defined homomorphism

$$B: \mathscr{F}'_2(p) \to \mathscr{F}'_1(p).$$

To see this, for $x \in \mathscr{F}'_2(p)$ let $f \in \mathscr{F}_1(p)$ be such that $A(f) = j_p(x)$. Now if we set $B(x) = i_p(f)$ then it is clearly well defined.

Remark 2.6. The above construction of the triple associated to a torsion free $\mathcal{O}_X$-module $\mathscr{F}$ gives the local structure of $\mathscr{F}$ at p, namely locally at p

$$\mathscr{F}_p \simeq \mathcal{O}^a_{X,p} \oplus \mathcal{O}^b_{X_1,p} \oplus \mathcal{O}^c_{X_2,p}$$

(see [11], Huitiême Partie; Proposition 3).) In particular, if $\mathscr{F}$ is a torsion free sheaf of rank r then

$$\mathscr{F}_p \simeq \mathcal{O}^a_{X,p} \oplus m^b_{X,p}, \quad a+b=r.$$

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple associated to $\mathscr{F}$. We can choose a basis of the free $\mathcal{O}_{X_1,p}$-module $\mathscr{F}_{1,p}$ (resp. a basis of the free $\mathcal{O}_{X_2,p}$-module $\mathscr{F}_{2,p}$) $e_1, \ldots, e_{r_1}$ (resp. $f_1, \ldots, f_{r_2}$) such that $A(e_i) = f_i$, $(1 \leqslant i \leqslant s)$ and $A(e_i) = 0$, $(s \leqslant i \leqslant r_1)$. Then by setting $a = s, b = r_1 - s$ and $c = r_2 - s$ we get the desired result.

Remark 2.7. Let $\mathscr{F}$ be a torsion free $\mathcal{O}_X$-module. Let $\bar{\mathscr{F}}_1, \bar{\mathscr{F}}_2, \mathscr{F}_1, \mathscr{F}'_2$ be as in the notation before Proposition 2.2 and $K \subset \mathscr{F}'_2(p)$ be as in Remark 2.3. Let $T_{1,p}$ = torsion of $\bar{\mathscr{F}}_1$. Then $T_{1,p}$ maps isomorphically onto the subspace K under the canonical map $T_{1,p} \to \mathscr{F}'_2(p)$.

Proof. To prove this note that (see Remark 2.6)

$$\mathscr{F}_p \simeq \mathcal{O}^a_{X,p} \oplus \mathcal{O}^b_{X_1,p} \oplus \mathcal{O}^c_{X_2,p} \text{ and } \bar{\mathscr{F}}_{1,p} \simeq \mathcal{O}^{a+b}_{X_1,p} \oplus (\mathcal{O}_{X,p}/m_{X,p})^c.$$

Now if $\mathbf{N}$ is as in Remark 2.1, we clearly see that $\mathbf{N}$ maps onto $\mathscr{F}_1(p)$ with kernel K which is isomorphic to $T_{1,p}$ as required.

Remark 2.8. Given a directed arrow, say $\to$, the association to a torsion free $\mathcal{O}_X$-module $\mathscr{F}$ the canonical triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, is functorial (note that triples form a category in an obvious way).

If $\mathscr{F} \xrightarrow{\phi} \mathscr{G}$ is given, and $\mathscr{F}$ and $\mathscr{G}$ are represented by triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ respectively, it then give rise to in a canonical way an $\mathcal{O}_{X_1}$-module (resp. $\mathcal{O}_{X_2}$-module) map $\phi_1 : \mathscr{F}_1 \to \mathscr{G}_1$ (resp. $\phi_2 : \mathscr{F}_2 \to \mathscr{G}_2$), and a commutative diagram

$$\begin{array}{ccc} \mathscr{F}_1(p) & \xrightarrow{\phi_1 \otimes k(p)} & \mathscr{G}_1(p) \\ \downarrow A & & \downarrow B \\ \mathscr{F}_2(p) & \xrightarrow{\phi_2 \otimes k(p)} & \mathscr{G}_2(p) \end{array}.$$

Proof. This follows because in the construction of the triple from a torsion free $\mathcal{O}_X$-module $\mathscr{F}$, the constructions $\mathscr{F}_i, i = 1, 2$ and A are all functorial.

Lemma 2.3. *Let $\vec{C}$ be the category of triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Then the functor*

$$\mathscr{F} \mapsto (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$$

from the category of torsion free $\mathcal{O}_X$-modules to $\vec{C}$ is an equivalence.

Proof. The inverse of

$$\mathscr{F} \mapsto (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$$

is given by

$$(\mathscr{F}_1, \mathscr{F}_2, \vec{A}) \mapsto \mathscr{F} = \{f = (f_1, f_2) \in \mathscr{F}_1 \oplus \mathscr{F}_2 \mid A(f_1(p)) = f_2(p)\}.$$

Now the lemma follows from the Remark 2.8 above.

Remark 2.9. If $\overline{C}$ is the category of triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ then again the functor

$$\mathscr{F} \mapsto (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$$

is an equivalence of categories.

2.1 *Subtriple*

DEFINITION 2.3

Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple as before. A triple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ is said to be a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ if $\mathscr{G}_1 \overset{i_1}{\subsetneq} \mathscr{F}_1$ (resp. $\mathscr{G}_2 \overset{i_2}{\subsetneq} \mathscr{F}_2$) is an inclusion of $\mathcal{O}_{X_1}$-modules (resp. $\mathcal{O}_{X_2}$-modules) and the following diagram

$$\begin{array}{ccc} \mathscr{G}_1(p) & \xrightarrow{i_1 \otimes k(p)} & \mathscr{F}_1(p) \\ \downarrow B & & \downarrow A \\ \mathscr{G}_2(p) & \xrightarrow{i_2 \otimes k(p)} & \mathscr{F}_2(p) \end{array}$$

is commutative.

Note. If $\mathscr{G} \subsetneq \mathscr{F}$ is an $\mathcal{O}_X$-submodule of $\mathscr{F}$, then there is a $\mathscr{G}'$, $\mathscr{G} \subsetneq \mathscr{G}' \subsetneq \mathscr{F}$ such that $\mathscr{F}/\mathscr{G}'$ is torsion free and $\mathscr{G}'/\mathscr{G}$ is torsion. One has only to take the inverse image in $\mathscr{F}$ of the torsion subsheaf of $\mathscr{F}/\mathscr{G}$. Note if $\mathscr{G} \subsetneq \mathscr{F}$ is such that $\mathscr{F}/\mathscr{G}$ is torsion free, then if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$) is the triple associated to $\mathscr{F}$ (resp. $\mathscr{G}$), then for $\mathscr{G}_i \subsetneq \mathscr{F}_i$, it need not be true that $\mathscr{F}_i/\mathscr{G}_i$ is torsion free. For example, let $\mathscr{F}$ be a vector bundle on X and let $\mathscr{F}'(= x_1 . \mathscr{F})$ be the subsheaf of $\mathscr{F}$ vanishing on X_2. Then $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ where $\mathscr{F}_i = \mathscr{F}|_{X_i}$ and $\mathscr{F}' = (\mathscr{G}, 0, \vec{0})$, where $\mathscr{G} = \mathscr{F}'$ considered as a sheaf (vector bundle) on X_1. Now $\mathscr{F}/\mathscr{F}'$ is torsion free; however for $\mathscr{G} \subsetneq \mathscr{F}_1, \mathscr{F}_1/\mathscr{G}$ is not torsion free.

Remark 2.10. If $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is a triple and $\mathscr{F}$ is the corresponding depth one $\mathcal{O}_X$-module, then is a 1–1 correspondence between subtriples of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $\mathcal{O}_X$-submodules of $\mathscr{F}$.

Proof. This follows from Lemma 2.3.

2.2 *Euler characterstic of a triple*

DEFINITION 2.4

Given a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ we define its Euler characteristic by

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - \mathrm{rk}(\mathscr{F}_2),$$

where $\chi_{X_i}(\mathscr{F}_i)$ is the usual Euler characteristics of the sheaf of $\mathscr{O}_{X_i}$- module $\mathscr{F}_i (i = 1, 2)$ and $\mathrm{rk}(\mathscr{F}_2)$ is the rank of the $\mathscr{O}_{X_2}$- module $\mathscr{F}_2$. Similarly, given a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we define its Euler characteristic by

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - \mathrm{rk}(\mathscr{F}_1).$$

Remark 2.11. If $\mathscr{F}$ is the $\mathscr{O}_X$-module associated to a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \vec{B})$) then

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X(\mathscr{F}) = \chi_X((\mathscr{F}'_1, \mathscr{F}'_2, \vec{B})).$$

Proof. Given a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ we get an exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}_2 \to T_A \to 0$$

where $\mathscr{F}$ is the $\mathscr{O}_X$-module associated to the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (see Lemma 2.3) and T_A is a torsion sheaf supported at the node p of X. In fact T_A is a vector space over the residue field $k(p)$ at p and its dimension is equal to the rank of $\mathscr{F}_2$. Thus we see that

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X(\mathscr{F}).$$

Similarly,

$$\chi_X(\mathscr{F}'_1, \mathscr{F}'_2, \vec{B})) = \chi_X(\mathscr{F}).$$

Remark 2.12. If $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ are two triples such that $\chi_{X_1}(\mathscr{F}_1) = \chi_{X_1}(\mathscr{G}_1), \chi_{X_2}(\mathscr{F}_2) = \chi_{X_2}(\mathscr{G}_2)$ and rank $(\mathscr{F}_2)$ = rank $(\mathscr{G}_2)$, then

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X((\mathscr{G}_1, \mathscr{G}_2, \vec{B})).$$

A similar result holds for triples of type $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$.

2.3 *Stability of triples*

Fix a polarization (a_1, a_2) on X with $a_i > 0$ rational and $a_1 + a_2 = 1$, i.e. we take an ample line bundle L on X such that if $L_i = L|_{X_i}$, then

$$\frac{\deg(L_1)}{\deg(L_2)} = \frac{a_1}{a_2}.$$

DEFINITION 2.5

Let$(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple. Then we define

$$\mu((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \mu_{(X, a_1, a_2)}((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) := \frac{\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{a_1 \mathrm{rk}(\mathscr{F}_1) + a_2 \mathrm{rk}(\mathscr{F}_2)}.$$

DEFINITION 2.6

Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple. We say that the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is stable (resp. semi-stable) if for every proper subtriple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$

$$\mu((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < \mu((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))$$

(resp. $\leqslant$).

Note. In the definition of semi-stability (resp. stability) $\mathscr{G}_i$ are only subsheaves of $\mathscr{F}_i$ and we cannot suppose that (unlike the case of a smooth projective curve) it suffices to take $\mathscr{G}_i$ as subbundles of $\mathscr{F}_i$ (see the note before Remark (2.10)).

Let g be the genus of X and $g_i (i = 1,2)$ be the genus of $X_i (i = 1,2)$. Note that $g = g_1 + g_2$. Let d be an integer and let $S((a_1, a_2), (2,2), \chi = d + 2(1-g))$ denote the set of all isomorphism classes of semi-stable $\mathcal{O}_X$-modules which are torsion free of type $(2, 2)$ and Euler characteristic $\chi = d + 2(1-g)$ (here type $(2, 2)$ means the $\mathcal{O}_X$ -module when restricted to each component has rank 2) (see [11], ch. (VII.II)).

Remark 2.13. Let $\mathscr{F}$ be an $\mathcal{O}_X$ -module such that

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), \chi = d + 2(1-g)).$$

Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple which corresponds to $\mathscr{F}$ (see Lemma 2.3). Then $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is semi-stable. Conversely, if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is a semi-stable triple such that $\mathrm{rk}\mathscr{F}_1 = 2 = \mathrm{rk}\mathscr{F}_2$ and Euler characteristic $d + 2(1-g)$, then if $\mathscr{F}$ is the corresponding torsion free sheaf, we have $[\mathscr{F}] \in S((a_1, a_2), (2,2), \chi = d + 2(1-g))$. Moreover, $\mathscr{F}$ is stable iff the corresponding triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is stable.

Proof. This follows from Remark 2.10, Remark 2.11 and Lemma 2.3.

3. Euler characteristic of the components of a semi-stable triple

Notations are as in the previous section. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple. Recall that by definition

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - \mathrm{rk}(\mathscr{F}_2).$$

Now if

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), \chi = d + 2(1-g))$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ the triple which corresponds to $\mathscr{F}$, then

$$\chi := \chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X(\mathscr{F}) = d + 2(1-g))$$

is fixed. We show below that if

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), d + 2(1-g))$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is the triple which corresponds to $\mathscr{F}$, then the following inequalities hold; $a_1 \cdot \chi \leqslant \chi_{X_1}(\mathscr{F}_1) \leqslant a_1 \cdot \chi + 2$ and $a_2 \cdot \chi \leqslant \chi_{X_2}(\mathscr{F}_2) \leqslant a_2 \cdot \chi + 2$. This has some interesting consequences.

Let

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), d + 2(1-g))$$

and let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple which corresponds to $\mathscr{F}$.

Case 1. $\vec{A}$ is invertible.

Since the canonical map $(\mathscr{F}_1 \otimes \mathcal{O}_{X_1}(-p))(p) \to \mathscr{F}_1(p)$ is zero, $(\mathscr{F}_1 \otimes \mathcal{O}_{X_1}(-p), 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Hence by the definition of semi-stability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (see Remark 2.13), we get

$$\frac{\chi_{X_1}(\mathscr{F}_1) - 2}{2a_1} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 2. \tag{1}$$

Similarly, since $(0, \mathscr{F}_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we get

$$\frac{\chi_{X_2}(\mathscr{F}_2) - 2}{2a_2} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2. \tag{2}$$

On the other hand

$$\chi = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2.$$

Using this in (2) we get

$$\chi - \chi_{X_1}(\mathscr{F}_1) \leqslant a_2\chi.$$

Since $a_1 + a_2 = 1$, the last equation together with (1) gives

$$a_1\chi \leqslant \chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 2.$$

Similarly we see that

$$a_2\chi \leqslant \chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2.$$

Case 2. $\mathrm{rk}(\vec{A}) = 1$.

Let $\mathscr{F}'_1 \to \mathscr{F}_1$ be the Hecke modification such that $\mathrm{Im}(\mathscr{F}'_1(p)) = \mathrm{Ker}(A)$ (see Remark 2.4). Then clearly $(\mathscr{F}'_1, 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Again by the semi-stability of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we get

$$\frac{\chi_{X_1}(\mathscr{F}'_1)}{2a_1} \leqslant \frac{\chi}{2}.$$

But $\chi_{X_1}(\mathscr{F}'_1) = \chi_{X_1}(\mathscr{F}_1) - 1$, hence

$$\chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 1. \tag{3}$$

Again, since $(0, \mathscr{F}_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we get

$$\frac{\chi_{X_2}(\mathscr{F}_2) - 2}{2a_2} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2. \tag{4}$$

Since

$$\chi = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2,$$

by (4) we get

$$\chi - \chi_{X_1}(\mathscr{F}_1) \leqslant a_2\chi.$$

Also, since $a_1 + a_2 = 1$, the last equation together with (3) gives

$$a_1\chi \leqslant \chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 1.$$

Similarly

$$a_2\chi + 1 \leqslant \chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2.$$

Case 3. $\mathrm{rk}(\vec{A}) = 0$.

In this case note that $A = 0$. Then clearly $(\mathscr{F}_1, 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{0})$. Again by the semi-stability of $(\mathscr{F}_1, \mathscr{F}_2, \vec{0})$ we get

$$\frac{\chi_{X_1}(\mathscr{F}_1)}{2a_1} \leqslant \frac{\chi}{2}.$$

Hence

$$\chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi. \tag{5}$$

Again, since $(0, \mathscr{F}_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{0})$, we get

$$\frac{\chi_{X_2}(\mathscr{F}_2) - 2}{2a_2} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2. \tag{6}$$

Now, as above, from (5) and (6) we get

$$\chi_{X_1}(\mathscr{F}_1) = a_1\chi.$$

and

$$\chi_{X_2}(\mathscr{F}_2) = a_2\chi + 2.$$

Thus we have proved the following:

Theorem 3.1. (a) *Let* $\chi \neq 0$ *and let* (a_1, a_2) *be a polarization (i.e.* $a_i > 0$ *rational and* $a_1 + a_2 = 1$*). Assume that* $a_1\chi$ *is not an integer. Let* (d_1, d_2) *be the unique tuple satisfying the inequalities*

$$a_1 \cdot \chi < d_1 < a_1 \cdot \chi + 1, \quad a_2 \cdot \chi + 1 < d_2 < a_2 \cdot \chi + 2$$

and $d_1 + d_2 = \chi + 2$. *If* $[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = d + 2(1-g))$ *and* $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ *is the triple which corresponds to* $\mathscr{F}$, *then* $\mathrm{rank}(A) \geqslant 1$ *and we must have either* $\chi(\mathscr{F}_1) = d_1, \chi(\mathscr{F}_2) = d_2$ *or* $\chi(\mathscr{F}_1) = d_1 + 1, \chi(\mathscr{F}_2) = d_2 - 1$.

Moreover, if $\mathrm{rank}\,(A) = 1$, *then we must have* $\chi(\mathscr{F}_1) = d_1$ *and* $\chi(\mathscr{F}_2) = d_2$. (b) *Let* $\chi = 0$ *and let* (a_1, a_2) *be any polarization (i.e.* $a_i > 0$ *rational and* $a_1 + a_2 = 1$*). If*

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = 0)$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ *is the triple which corresponds to* $\mathscr{F}$, *then*

(i) $\chi(\mathscr{F}_1) = 0$ *and* $\chi(\mathscr{F}_2) = 2$, *or*

(ii) $\chi(\mathscr{F}_1) = 1$ *and* $\chi(\mathscr{F}_2) = 1$, *or*

(iii) $\chi(\mathscr{F}_1) = 2$ *and* $\chi(\mathscr{F}_2) = 0$.

Moreover, if $\mathrm{rk}(A) = 1$ (*resp.* $\mathrm{rk}(A) = 0$), *then either* $\chi(\mathscr{F}_1) = 0$ *and* $\chi(\mathscr{F}_2) = 2$ *or* $\chi(\mathscr{F}_1) = 1$ *and* $\chi(\mathscr{F}_2) = 1$ (*resp.* $\chi(\mathscr{F}_1) = 2$ *and* $\chi(\mathscr{F}_2) = 0$).

COROLLARY 3.1

With the notations of Theorem 3.1 (*a*), *if*

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = d + 2(1-g))$$

then either $\mathscr{F}$ *is locally free or a torsion free* $\mathcal{O}_X$*-module with* $\mathscr{F}_p \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$.

Remark 3.1. Let $\mathscr{G}_s (s \in S)$ be a family of (a_1, a_2) semi-stable torsion free sheaves on X of rank two and Euler characteristic $\chi \neq 0$ (with $a_1\chi$ is not an integer), parametrized by an irreducible variety S. Then by Theorem 3.1 we can choose a direction of the arrow such that $\chi_{X_1}(\mathscr{G}_{s,1})$ (resp. $\chi_{X_2}(\mathscr{G}_{s,2})$) is independent of $s \in S$, where $\mathscr{G}_{s,1}$ (resp. $\mathscr{G}_{s,2}$) is the bundle on X_1 (resp. X_2) in the triple associated to $\mathscr{G}_s$.

4. Properties of the moduli space of rank two torsion free sheaves on X and its deformations

4.1 *Infinitesimal deformations of rank two torsion free sheaves of type* $\mathcal{O}_{X,p} \oplus m_{X,p}$ *on* X

Let $\mathscr{F}_0$ be a torsion free $\mathcal{O}_X$-module. Assume that $\mathscr{F}_0 \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathscr{F}_0 \otimes \mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$, where $m_{X,p}$ is the maximal ideal at the node p.

Let $\underline{C}$ be the category of Artin local k algebras. Let $\mathbf{F}$: $\underline{C} \to (\underline{\text{Sets}})$ be the functor defined by

$$\mathbf{F}(A) = \left\{ \begin{array}{l} \text{Isomorphism class of coherent } \mathcal{O}_{\text{Spec}(A) \times X}\text{-modules } \mathscr{F} \\ \text{flat over Spec}(A) \text{ and } \mathscr{F}|_{\text{Spec}(A/m_A) \times X} \simeq \mathscr{F}_0 \end{array} \right\}.$$

Note that $\mathbf{F}$ has a versal deformation space.

Let $\mathbf{FM}$: $\underline{C} \to (\underline{\text{Sets}})$ be the functor defined by

$$\mathbf{FM}(A) = \left\{ \begin{array}{l} \text{Isomorphism class of } \mathcal{O}_{X,p} \otimes A\text{-modules } M \\ \text{flat over } A \text{ and } M \otimes A/m_A \simeq \mathcal{O}_{X,p} \oplus m_{X,p} \end{array} \right\}.$$

There is a natural morphism of functors

$$\Phi: \mathbf{F} \to \mathbf{FM}$$

defined by

$$\mathscr{F} \mapsto \mathscr{F} \otimes (\mathcal{O}_{X,p} \otimes A),$$

$$\mathbf{F}(A) \to \mathbf{FM}(A).$$

PROPOSITION 4.1

The morphism $\Phi: \mathbf{F} \to \mathbf{FM}$ *is formally smooth.*

Proof. From Grothendieck's criterion [10] it is enough to verify the following. If

$$A' \to A \to 0$$

is a small extension in $\underline{C}$, then the natural map

$$\mathbf{F}(A') \to \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$$

is surjective.

Let $A' \to A \to 0$ be a small extension in $\underline{C}$, and $\mathscr{F} \in \mathbf{F}(A)$ and $N \in \mathbf{FM}(A')$ be such that

$$\mathscr{F} \otimes (\mathscr{O}_{X,p} \otimes A) \simeq N \otimes_{A'} A.$$

We show that there is an $\mathscr{F}' \in \mathbf{F}(A')$ such that $\mathscr{F}' \otimes A \simeq \mathscr{F}$ and $\mathscr{F}' \otimes (\mathscr{O}_{X,p} \otimes A') \simeq N$. Let $U_i (i = 1, 2)$ be affine open subschemes of X (U_i need not be connected) with

(i) $X = U_1 \cup U_2$

(ii) $p \in U_1$

(iii) $\mathscr{F}|_{(U_1 - p) \times \mathrm{Spec}(A)}$ and $\mathscr{F}|_{(U_2) \times \mathrm{Spec}(A)}$ are free.

(Note that the above assumption implies $p \notin U_2$). Then $\mathscr{F}$ is given by the patching data

$$(a_{ij}) \in GL_2(\mathscr{O}_{(U_1 \cap U_2) \times \mathrm{Spec}(A)}) = GL_2(\mathscr{O}_{(U_1 \cap U_2)} \otimes A).$$

Since $A' \to A$ is a small extension there exists

$$(a'_{ij}) \in GL_2(\mathscr{O}_{(U_1 \cap U_2) \times \mathrm{Spec}(A')}) = GL_2(\mathscr{O}_{(U_1 \cap U_2)} \otimes A')$$

such that under the natural map

$$GL_2(\mathscr{O}_{(U_1 \cap U_2)} \otimes A') \to GL_2(\mathscr{O}_{(U_1 \cap U_2)} \otimes A),$$

(a'_{ij}) maps to (a_{ij}). Let (a'_{ij}) be one such element and let $\mathscr{G}$ be a sheaf of $\mathscr{O}_{(U_1) \times \mathrm{Spec}(A')}$-modules such that $\mathscr{G}$ is free on $(U_1 - p) \times \mathrm{Spec}(A)$ and $\mathscr{G} \otimes (\mathscr{O}_{X,p} \otimes A') \simeq N$. Let $\mathscr{F}'$ be the sheaf of $\mathscr{O}_{X \times \mathrm{Spec}(A')}$-modules obtained by patching $\mathscr{G}$ and $\mathscr{O}^2_{U_2 \times \mathrm{Spec}(A')}$ along $(U_1 \cap U_2) \times \mathrm{Spec}(A')$ using (a'_{ij}). Now it is clear that $\mathscr{F}' \in \mathbf{F}(A')$ and maps to $(\mathscr{F}, N) \in \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$ under the natural map

$$\mathbf{F}(A') \to \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A').$$

Since $(\mathscr{F}, N) \in \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$ was arbitrary, we get the desired surjection. Thus $\Phi: \mathbf{F} \to \mathbf{FM}$ is formally smooth.

Next consider the following functors from $\underline{C}$ to (Sets).

$$\mathbf{EFM}(A) = \left\{ \begin{array}{l} \text{Sub } (\mathscr{O}_{X,p} \otimes A)\text{-modules } \mathscr{M} \text{ of } (\mathscr{O}_{X,p} \otimes A)^2 \\ \text{such that } (\mathscr{O}_{X,p} \otimes A)^2/\mathscr{M} \text{ is flat over } A \\ \text{and is a deformation of } \mathscr{O}_{X,p}/m_{X,p} \end{array} \right\},$$

$$\mathbf{M}(A) = \left\{ \begin{array}{l} \text{Isomorphism classes of } (\mathscr{O}_{X,p} \otimes A)\text{-modules which are} \\ \text{flat over } A \text{ and are deformations of } \mathscr{O}_{X,p}/m_{X,p} \end{array} \right\}.$$

Then we get the following morphisms between functors:

$$\Psi: \mathbf{EFM} \to \mathbf{FM}$$

defined by $\mathbf{EFM}(A) \ni \mathscr{M} \mapsto [\mathscr{M}] \in \mathbf{FM}(A)$, where $[\mathscr{M}]$ denotes the isomorphism class of $\mathscr{M}$, and

$$\Lambda: \mathbf{EFM} \to \mathbf{M}$$

defined by $\mathbf{EFM}(A) \ni \mathscr{M} \mapsto [(\mathscr{O}_{X,p} \otimes A)^2/\mathscr{M}] \in \mathbf{M}(A)$ where $[(\mathscr{O}_{X,p} \otimes A)^2/\mathscr{M}]$ denotes the isomorphism class of $(\mathscr{O}_{X,p} \otimes A)^2/\mathscr{M}$.

We claim that Ψ and Λ are formally smooth (in the sense of Schlessinger [10]). That Λ is formally smooth is immediate (see [1]). To show that Ψ is formally smooth, we need the following.

Lemma 4.2. *Let $A \to B$ be a local homomorphism of noetherian local rings. Let N and L be B modules of finite type with L flat over A. Then a B homomorphism $f: N \to L$ is injective with an A flat cokernel if and only if $f \otimes_A k : N \otimes_A k \to L \otimes_A k$ is injective, where k is the residue field of A.*

Proof. See [9], Appendix.

Lemma 4.3. *Let $A \to B$ be a local homomorphism of noetherian local rings. Let N be a B module of finite type which is flat over A and satisfies* $\mathrm{Ex}(1/B)$ $(\bar{N}, \bar{B}) = 0$, *where $\bar{B} = B \otimes_A k$ and $\bar{N} = N \otimes_A k$ and k is the residue field of A. Then* $\mathrm{Hom}_B(N, B)$ *is A-flat and* $\mathrm{Hom}_B(N, B) \otimes_A k = \mathrm{Hom}_{\bar{B}}(\bar{N}, \bar{B})$.

Proof. See [9], corollary in the Appendix.

PROPOSITION 4.4.

*The morphism Ψ: **EFM** $\to$ **FM** is formally smooth.*

Proof. From Grothendieck's criterion [10] it is enough to verify the following. If $A' \to A \to 0$ is small extension in $\underline{C}$, then the natural map

$$\mathbf{EFM}(A') \to \mathbf{EFM}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$$

is surjective.

Let $A' \to A \to 0$ be a small extension in $\underline{C}$, and $\mathscr{M} \in \mathbf{EFM}(A)$ and $[\mathscr{M}'] \in \mathbf{FM}(A')$ be such that

$$[\mathscr{M}' \otimes_{A'} \otimes A] = [\mathscr{M}].$$

Now applying Lemma 4.3 to the local homomorphism $A' \to (\mathscr{O}_{X,p} \otimes A')$ and to the $(\mathscr{O}_{X,p} \otimes A')$-module $\mathscr{M}'$ (note that the hypothesis of the lemma $\mathrm{Ext}^1_{\mathscr{O}_{X,p}}(m_{X,p}, \mathscr{O}_{X,p}) = 0$, is true in this case (see [3])), we conclude that $(\mathscr{M}')^* = \mathrm{Hom}_{\mathscr{O}_{X,p} \otimes A'}(\mathscr{M}', \mathscr{O}_{X,p} \otimes A')$ is A' flat and hence $(\mathscr{M}')^* \otimes_{A'} A = (\mathscr{M})^*$. *A priori*, we can lift the inclusion $\mathscr{M} \subsetneq (\mathscr{O}_{X,p} \otimes A)^2$ to $\phi: \mathscr{M}' \to (\mathscr{O}_{X,p} \otimes A')^2$. Using Lemma 4.2, we conclude that ϕ is injective and $(\mathscr{O}_{X,p} \otimes A')^2 / \mathrm{Im}(\phi)$ is flat over A'. Hence $\mathscr{M}' \in \mathbf{EFM}(A')$ as required. This proves the proposition.

Remark 4.1. Let (a_1, a_2) be as in Theorem 3.1 and let $\mathscr{O}_X(1)$ be a line bundle of type (a_1, a_2) on X. Let **Quot** be the quot scheme of rank two quotients of $\mathscr{O}_X(1)^N$ with a fixed Hibert polynomial. Now using the above results we conclude that if $v \in \mathbf{Quot}$ corresponds to a torsion free coherent sheaf of $\mathscr{O}_X$ module $\mathscr{F}_0$ such that $\mathscr{F}_0 \otimes \mathscr{O}_{X,p} \simeq \mathscr{O}_{X,p} \oplus m_{X,p}$ and $\mathscr{F}_0 \otimes \mathscr{O}_{X,x} \simeq \mathscr{O}_{X,x} \oplus \mathscr{O}_{X,x}$ for all $x \in X - p$, then there is an integer l such that the local ring $\hat{\mathscr{O}}_{\mathbf{Quot},v}[[u_1, \ldots, u_l]]$ is isomorphic to $\hat{\mathscr{O}}_{X,p}[[t_1, \ldots, t_s]]$ for some s, where $\hat{\mathscr{O}}_{X,p}$ is the completion of the local ring $\mathscr{O}_{X,p}$ with respect to its maximal ideal.

Theorem 4.1. *Let $\chi \neq 0$ be an integer. Let (a_1, a_2) be a polarization. Assume that $a_1.\chi$ is not an integer. Then the moduli space $M(2,(a_1,a_2),\chi)$ of (a_1,a_2)-semi-stable rank two torsion free sheaves on X with Euler characteristics χ is a reduced, connected projective scheme and with exactly two irreducible components. Moreover, when χ is odd the moduli space is a union of two smooth varieties intersecting transversally.*

Proof. Existence of the moduli space in question as a projective scheme follows from the very general result of ([11], septième partie, III, Théorèm 15). By Theorem 3.1(a) we see that a (a_1, a_2)-semistable rank two torsion free sheaf is either a locally free sheaf or a torsion free sheaf of type $\mathcal{O}_{X,p} \oplus m_{X,p}$. Remaining assertions of the theorem now follows from Remark 4.1 and the general results of GIT quotients (see [6]). This proves the theorem.

For a different proof of the above theorem see Theorem 6.1.

4.2 *Infinitesimal deformation of sheaves of type $\mathcal{O}_{X,p} \oplus m_{X,p}$ on X along a smooth deformation of X*

Let R be a complete discrete valuation ring with residue field $\mathbb{C}$. Let χ be a regular two dimensional scheme proper and flat over $S := \mathrm{Spec}(R)$ and the closed fibre of the characteristic morphism $\chi \to S$ is isomorphic to X (= union of two smooth curves meeting transversally along a point p). Let $\mathcal{F}_0$ be a coherent sheaf $\mathcal{O}_X$ -modules such that $\mathcal{F}_0 \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathcal{F}_0 \otimes \mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$.

We have the following functors:

1) Let $\underline{\mathbf{F}}: \underline{C} \to (\underline{\mathrm{Sets}})$ be the functor defined by

$$\underline{\mathbf{F}}(A) = \left\{ (f, [\mathcal{F}]) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } [\mathcal{F}] \text{ is an} \\ \text{isomorphism class of coherent } \mathcal{O}_{\mathrm{Spec}(A) \times_f \chi}\text{-modules} \\ \mathcal{F} \text{ flat over } \mathrm{Spec}(A) \text{ and } \mathcal{F}|_{\mathrm{Spec}(A/m_A) \times \chi} \simeq \mathcal{F}_0 \end{array} \right\}.$$

2) Let $x_0 \in \chi$ be the closed point which corresponds to p on X. Note that $R \subsetneq \mathcal{O}_{x_0}, \chi$. Let $\underline{\mathbf{FM}}: \underline{C} \to (\underline{\mathrm{Sets}})$ be the functor defined by

$$\underline{\mathbf{FM}}(A) = \left\{ (f, [M]) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } [M] \text{ is an} \\ \text{isomorphism class } (\mathcal{O}\chi_{x_0} \otimes_R A) \text{ modules} \\ M \text{ flat over } A \text{ and } M \otimes_R R/m_R \simeq \mathcal{O}_{X,p} \oplus m_{p,X} \end{array} \right\}.$$

3) Let $\underline{\mathbf{EFM}}: \underline{C} \to (\underline{\mathrm{Sets}})$ be the functor defined by

$$\underline{\mathbf{EFM}}(A) = \left\{ (f, M) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } M \text{ is a} \\ \mathcal{O}\chi_{x_0} \otimes_R A) \text{ submodule of } \mathcal{O}\chi_{x_0} \otimes_R A)^2 \\ \text{such that } \mathcal{O}\chi_{x_0} \otimes_R A)^2/M \text{ is flat over } A \\ \text{and is a deformation of } \mathcal{O}_{X,p}/m_{X,p} \end{array} \right\}.$$

4) Let $\underline{\mathbf{M}}: \underline{C} \to (\underline{\mathrm{Sets}})$ be the functor defined by

$$\underline{\mathbf{M}}(A) = \left\{ (f, N) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } N \text{ is a} \\ (\mathcal{O}_{x_0}, \chi \otimes_R A) \text{ module flat over } A \\ \text{and is a deformation of } \mathcal{O}_{X,p}/m_{X,p} \end{array} \right\}.$$

Now arguments similar to the one used in the case of 'infinitesimal deformations of rank two torsion free sheaves of type $\mathcal{O}_X \oplus m_{X,p}$ on X' shows that the natural morphisms of functors

$$\begin{array}{rcl} \underline{\mathbf{F}} & \to & \underline{\mathbf{FM}} \\ \underline{\mathbf{EFM}} & \to & \underline{\mathbf{FM}} \\ \underline{\mathbf{EFM}} & \to & \underline{\mathbf{M}} \end{array}$$

are all formally smooth.

Remark 4.2. Let (a_1, a_2), S and χ be as above. Let $\mathcal{O}_\chi(1)$ be a relative ample line bundle such that $\mathcal{O}_\chi(1)|_X$ is a polarization of type (a_1, a_2). Let $\mathbf{Quot}^P/S$ be the quot scheme of quotients of $\mathcal{O}_\chi(1)^N$ which are of rank two and with a fixed Hilbert polynomial P along the fibres. Now using the above results, we conclude that if $v \in \mathbf{Quot}^P/S$ corresponds to a torsion free coherent $\mathcal{O}_\chi$-module $\mathcal{F}_0$ such that $\mathcal{F}_0 \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathcal{F}_0 \otimes \mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$, then there is an integer l such that the local ring $\hat{\mathcal{O}}_{\mathbf{Quot}^P/S,v}[[u_1, \ldots, u_l]]$ is isomorphic to $\hat{\mathcal{O}}_{\chi,x_0}[[t_1, \ldots, t_s]]$ for some s, where $\mathcal{O}_{\chi,\underline{x_0}}$ is the completion of the local ring $\mathcal{O}_{\chi,\underline{x_0}}$ with respect to the maximal ideal $\underline{x_0}$ of χ.

Remark 4.3. From Remark 4.2, it follows that if $v \in \mathbf{Quot}^P/S$ corresponds to a torsion free coherent $\mathcal{O}_\chi$-module $\mathcal{F}_0$ such that $(\mathcal{F}_0 \otimes \mathcal{O}_{X,p}) \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $(\mathcal{F}_0 \otimes \mathcal{O}_{X,x}) \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$, then v is a smooth point of $\mathbf{Quot}^P/S$.

Theorem 4.2. *Notations are as above. Then there exists a family*

$$\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0) \to S$$

of moduli spaces of rank two torsion free sheaves along the fibers of $\chi \to S$ with Euler characteristic χ and semi-stable with respect to $\mathcal{O}(1)$. Moreover,

$$\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0) \to S$$

is an integral scheme which is proper and flat over S. If χ is odd $\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0)$ is a regular scheme, if $\chi \neq 0$ is even (in particular degree of the bundles under consideration is zero, i.e., $\chi = 2 - 2g$) the scheme $\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0)$ gives a degeneration of the general fibre.

Proof. Using the results of geometric invariant theory (GIT) over an arbitrary base (see [13]) one can construct a family

$$\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0) \to S$$

of moduli spaces of rank two torsion free sheaves along the fibers $\mathcal{X} \to S$ with Euler characteristic χ and semi-stable with respect to $\mathcal{O}(1)$, as a GIT quotient of the appropriate Quot scheme $\mathbf{Quot}^P/S$. Moreover,

$$\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0) \to S$$

is an integral scheme which is proper and flat over S. Then by the results proved above, we see that $\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0)$ is a regular scheme if χ is odd and if $\chi \neq 0$ is even (in particular degrees of the underlying bundle is zero, i.e., $\chi = 2 - 2g$) the scheme $\mathcal{M}(2, \mathcal{O}_\mathcal{X}, \chi \neq 0)$ gives a degeneration of the general fibre, which 'essentially' has all the nice properties of the general fiber.

5. Direct construction of the moduli space of triples

From now on, for simplicity, we assume that the genus of $X_i (i = 1, 2)$ is $\geqslant 2$ even though the results still hold if the genus of $X_i (i = 1, 2)$ is $\geqslant 0$. Let $\chi \neq 0$ be an integer. Let $0 < a_1 < 1$ be a rational number such that $a_1\chi$ is not an integer. Set $a_2 = 1 - a_1$.

It follows, from the existence of moduli space of rank two torsion free sheaves on X (see Theorem 4.1) and the results of § 2 of this paper, that the moduli space of triples exists. In this section we indicate a direct construct of the moduli space of (a_1, a_2)-semistable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ where $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is a rank two vector bundle on X_1 (resp. X_2) with $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1$ (resp. $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$) and

$$\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi.$$

Using this we constuct (see § 6) moduli space of torsion free sheaves on X. This shows that the moduli space of torsion free sheaves of rank two on X can be constructed directly from the space of vector bundles with some extra data on the normalization. We believe that this could be useful for applications (for example, as in [4, 7]).

First some preliminaries.

Let

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = d + 2(1 - g))$$

and let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple which corresponds to $\mathscr{F}$. By Theorem 3.1 we have the following inequalities;

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2,$$

$$a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

and $\chi = \chi_X(\mathscr{F}) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2$. Moreover, if $\mathscr{F}$ is not locally free, then we have the following sharper inequalities;

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1,$$

$$a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

with $\chi = \chi_X(\mathscr{F}) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2$ and $\mathrm{rk}(A) = 1$.

Write $a_1 = a/b$ with a, b relatively prime positive integers. Let $\mathscr{L}_1$ be a line bundle on X_1 of degree a and let $\mathscr{L}_2$ be a line bundle on X_2 of degree $b - a$. Let $\mathscr{O}_X(1)$ be an ample line bundle on X such that $\mathscr{O}_X(1)|_{X_1} = \mathscr{L}_1$ and $\mathscr{O}_X(1)|_{X_2} = \mathscr{L}_2$. Since $a/(b - a) = a_1/a_2$, observe that $\mathscr{O}_X(1)$ defines a polarization of type (a_1, a_2) on X. If $\mathscr{F}$ is a torsion free sheaf of $\mathscr{O}_X$-modules of rank 2 and if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is the triple corresponding to it, then the triple $(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes n}, \mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes n}, \overrightarrow{A \otimes \lambda^{\otimes n}})$ corresponds to the $\mathscr{O}_X$-module $\mathscr{F} \otimes \mathscr{O}_X(n)$, where $\lambda\colon \mathscr{L}_1(p) \to \mathscr{L}_2(p)$ is the isomorphism defining $\mathscr{O}_X(1)$. Since the bi-degree of the line bundle $\mathscr{O}_X(1)$ is $(a, b - a)$ we have

$$\chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) = \chi_{X_1}(\mathscr{F}_1) + 2am,$$

$$\chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) = \chi_{X_2}(\mathscr{F}_2) + 2(b - a)m,$$

$$\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) = \chi(\mathscr{F}) + 2bm.$$

Thus, if

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi \neq 0)$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ the triple which corresponds to $\mathscr{F}$, then

$$[\mathscr{F} \otimes \mathscr{O}_X(m)] \in S((a_1, a_2), (2,2), \chi + 2mb \neq 0)$$

and the corresponding triple $(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}, \mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes n}, \overrightarrow{A \otimes \lambda^{\otimes n}})$ satisfies the following inequalities;

$$a_1\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) < \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) < a_1\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) + 2,$$

$$a_2\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) < \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) < a_2\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) + 2$$

and $\chi_X(\mathscr{F} \otimes \mathscr{O}_X(m)) = \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) + \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) - 2$. Moreover if $\mathscr{F}$ is not locally free then $\mathscr{F} \otimes \mathscr{O}_X(m)$ is not locally free and we have the following sharper inequalities;

$$a_1\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) < \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) < a_1\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) + 1,$$

$$a_2\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) + 1 < \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) < a_2\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) + 2$$

with $\chi_X(\mathscr{F} \otimes \mathscr{O}_X(m)) = \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) + \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) - 2$ and $\mathrm{rk}(A \otimes \lambda^{\otimes m}) = 1$.

Remark 5.1. Let $\chi \neq 0$ and $0 < a_1 < 1$ be such that $a_1\chi$ is not an integer. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a $(a_1, a_2 = 1 - a_1)$ semistable triple of rank $(2,2)$ with $\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi$ and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1,$$

$$a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2.$$

Then $\mathscr{F}_1$ is a semistable $\mathscr{O}_{X_1}$-module and $\mathscr{F}_2$ is a semi-stable $\mathscr{O}_{X_2}$-module.

Proof. Let $L_2 \subset \mathscr{F}_2$ be a line subbundle. Then $(0, L_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Now by the semi-stability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we conclude that

$$\chi(L_2) - 1 \leqslant a_2\frac{\chi}{2}.$$

Then from the assumption on $\chi(\mathscr{F}_2)$, we get

$$\chi(L_2) \leqslant a_2\frac{\chi}{2} + 1 < \frac{\chi_{X_2}(\mathscr{F}_2) - 1}{2} + 1.$$

Thus

$$\chi(L_2) < \frac{\chi_{X_2}(\mathscr{F}_2)}{2} + \frac{1}{2}.$$

Since $\chi(L_2)$ is an integer, we get

$$\chi(L_2) \leqslant \frac{\chi_{X_2}(\mathscr{F}_2)}{2}.$$

This proves that $\mathscr{F}_2$ is a semi-stable $\mathscr{O}_{X_2}$-module.

Let $L_1 \subset \mathscr{F}_1$ be a line subbundle. Then $L_1 \otimes \mathscr{O}_{X_1}(-p) \subset \mathscr{F}_1 \otimes \mathscr{O}_{X_1}(-p)$ hence $(L_1 \otimes \mathscr{O}_{X_1}(-p), 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Now by the semistability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we conclude that

$$\chi(L_1) - 1 \leqslant a_1\frac{\chi}{2}.$$

Thus

$$\chi_{X_1}(L_1) \leqslant a_1 \frac{\chi}{2} + 1 < \frac{\chi_{X_1}(\mathscr{F}_1)}{2} + 1. \tag{1}$$

Note that, if $\chi_{X_1}(\mathscr{F}_1)$ is even, then (1) gives

$$\chi_{X_1}(L_1) \leqslant \frac{\chi_{X_1}(\mathscr{F}_1)}{2}$$

and hence $\mathscr{F}_1$ is semistable. Thus from now on we assume $\chi_{X_1}(\mathscr{F}_1)$ is odd. Then from (1) we get

$$\chi_{X_1}(L_1) \leqslant \frac{\chi_{X_1}(\mathscr{F}_1)}{2} + \frac{1}{2}. \tag{2}$$

Claim.

$$\chi_{X_1}(L_1) \leqslant \frac{\chi_{X_1}(\mathscr{F}_1)}{2}$$

for every line subbundle L_1 of $\mathscr{F}_1$

To prove the claim we assume the contrary and get a contradiction. Assume that there is a line subbundle L_1 of $\mathscr{F}_1$ such that

$$\chi_{X_1}(L_1) > \frac{\chi_{X_1}(\mathscr{F}_1)}{2}.$$

Then for any such subbundle L_1 of $\mathscr{F}_1$ we have (by (2) above)

$$\chi_{X_1}(L_1) = \frac{\chi_{X_1}(\mathscr{F}_1)}{2} + \frac{1}{2}. \tag{3}$$

Let L_1 be such a line bundle. Note that $(L_1, \mathscr{F}_2, \overrightarrow{A|_{L_1}})$ is a subtriple of the semistable triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, hence we get

$$\frac{\chi(L_1) + \chi(\mathscr{F}_2) - 2}{a_1 + 2a_2} \leqslant \frac{\chi}{2}.$$

Since $a_1 + a_2 = 1$, the last inequality gives

$$\chi(L_1) + \chi(\mathscr{F}_2) - 2 \leqslant \frac{\chi}{2} + a_2 \frac{\chi}{2}.$$

Using (3) in this equation we get

$$\frac{\chi_{X_1}(\mathscr{F}_1)}{2} + \frac{1}{2} + \chi(\mathscr{F}_2) - 2 \leqslant \frac{\chi}{2} + a_2 \frac{\chi}{2}.$$

Hence

$$\chi_{X_1}(\mathscr{F}_1) + 2\chi(\mathscr{F}_2) - 3 \leqslant \chi + a_2\chi.$$

Since $\chi = \chi_X(\mathscr{F}) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2$, we get

$$\chi + \chi(\mathscr{F}_2) - 1 \leqslant \chi + a_2\chi.$$

Hence

$$\chi(\mathscr{F}_2) \leqslant a_2\chi + 1,$$

which is a contradiction to the assumption that

$$a_2\chi + 1 < \chi(\mathscr{F}_2).$$

This proves the claim, and completes the proof of the remark.

Theorem 5.1. *Let $\chi \neq 0$ and $0 < a_1 < 1$ be such that $a_1\chi$ is not an integer. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a $(a_1, a_2 = 1 - a_1)$ semistable triple of rank (2,2) with $\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi$. Then $\mathscr{F}_1$ is a semi-stable $\mathcal{O}_{X_1}$-module and $\mathscr{F}_2$ is a semi-stable $\mathcal{O}_{X_2}$-module.*

Proof. If the semi-stable triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is such that rank $(\vec{A}) = 1$, then by Theorem 3.1 the inequality in Remark 5.1 is the only possibility, hence $\mathscr{F}_1$ is a semi-stable $\mathcal{O}_{X_1}$-module and $\mathscr{F}_2$ is a semi-stable $\mathcal{O}_{X_2}$-module. For the other cases one can use the arrow in the other direction and the fact that the nature of the problem is symmetric in X_1 and X_2. This proves the theorem.

Let χ and (a_1, a_2) as above. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a rank (2,2) triple such that $\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi$ and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1,$$

$$a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2.$$

Let

$$S_1 = \left\{ \mathscr{G}_1 \;\middle|\; \begin{array}{l} \mathscr{G}_1 \subset \mathscr{F}_1 \text{ is a } \mathcal{O}_{X_1}\text{-submodule of one of the following types:} \\ \text{i) } \mathscr{G}_1 = 0, \\ \text{ii) } \mathscr{G}_1 \text{ is a subbundle,} \\ \text{iii) } \mathscr{F}_1 \otimes \mathcal{O}_{X_1}(-p) \subset \mathscr{G}_1 \subset \mathscr{F}_1. \end{array} \right\}$$

and

$$S_2 = \left\{ \mathscr{G}_2 \;\middle|\; \begin{array}{l} \mathscr{G}_2 \subset \mathscr{F}_2 \text{ is a } \mathcal{O}_{X_2}\text{-submodule of one of the following types:} \\ \text{i) } \mathscr{G}_2 = 0, \\ \text{ii) } \mathscr{G}_2 \text{ is a subbundle,} \\ \text{iii) } \mathscr{F}_2 \otimes \mathcal{O}_{X_2}(-p) \subset \mathscr{G}_2 \subset \mathscr{F}_2. \end{array} \right\}.$$

Finally, let

$$S = \left\{ (\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \;\middle|\; \begin{array}{l} (\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \text{ is a subtriple of } (\mathscr{F}_1, \mathscr{F}_2, \vec{A}) \\ \text{with } \mathscr{G}_1 \in S_1 \text{ and } \mathscr{G}_2 \in S_2 \end{array} \right\}.$$

Lemma 5.2. The triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is (a_1, a_2)-stable (resp. semi-stable) if and only if for all the subtriples $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \in S$

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < (a_1 \mathrm{rk}(\mathscr{G}_1) + a_2 \mathrm{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

$$(\text{resp.} \leqslant).$$

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be (a_1, a_2) stable (resp. semi-stable). Then

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < (a_1 \mathrm{rk}(\mathscr{G}_1) + a_2 \mathrm{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

$$(\text{resp.} \leqslant)$$

for all subtriples $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ and hence in particular for the subtriples in S. This proves the only if part of the lemma.

To prove the if part of the lemma, assume that the above strict inequalities (resp. inequalities) hold for all subtriples $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \in S$ of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Let $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \notin S$ be a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Then we see that one of the following holds:

(i) $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) = (0, \mathscr{G}_2, \vec{0})$ and $\mathscr{G}_2$ is neither a subbundle of $\mathscr{F}_2$ nor $\mathscr{F}_2 \otimes \mathscr{O}_{X_2}(-p) \subset \mathscr{G}_2$.
(ii) rank $(\mathscr{G}_1) = 1$ and $\mathscr{G}_1$ is a subbundle of $\mathscr{F}_1$ but $\mathscr{G}_2$ is neither a subbundle of $\mathscr{F}_2$ nor $\mathscr{F}_2 \otimes \mathscr{O}_{X_2}(-p) \subset \mathscr{G}_2$.
(iii) rank $(\mathscr{G}_1) = 1$ and $\mathscr{G}_1$ is a not subbundle of $\mathscr{F}_1$.
(iv) rank $(\mathscr{G}_1) = 2$ and $\mathscr{F}_1 \otimes \mathscr{O}_{X_1}(-p)$ is not contained in $\mathscr{G}_1$.

In the cases (i) and (ii), if we take $\mathscr{L}_1 = \mathscr{G}_1$ and $\mathscr{L}_2$ to be the smallest $\mathscr{O}_{X_2}$-subbundle of $\mathscr{F}_2$ containing $\mathscr{G}_2$ and $C = B$, then $(\mathscr{L}_1, \mathscr{L}_2, \vec{C}) \in S$. Since rank $(\mathscr{G}_1) =$ rank $(\mathscr{L}_1)$, rank $(\mathscr{G}_2) =$ rank $(\mathscr{L}_2)$, clearly we get

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < \chi((\mathscr{L}_1, \mathscr{L}_2, \vec{C})) < (a_1 \operatorname{rk}(\mathscr{G}_1) + a_2 \operatorname{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

(resp. $\leqslant$).

In the case (iii), take $\mathscr{L}_1$ to be the smallest $\mathscr{O}_{X_1}$-subbundle of $\mathscr{F}_1$ generated by $\mathscr{G}_1$. Then rank $(\mathscr{L}_1) = 1$ and $\chi(\mathscr{G}_1) \leqslant \chi(\mathscr{L}_1) - 1$. Now if rank $(\mathscr{G}_2) = 2$, then set $\mathscr{L}_2 = \mathscr{F}_2$ and $C = A|_{\mathscr{L}_1(p)}$. Clearly $(\mathscr{L}_1, \mathscr{L}_2, \vec{C}) \in S$ and

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < \chi((\mathscr{L}_1, \mathscr{L}_2, \vec{C})) < (a_1 \operatorname{rk}(\mathscr{G}_1) + a_2 \operatorname{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

(resp. $\leqslant$)

as required. On the other hand if rank $(\mathscr{G}_2) = 1$, let $\mathscr{L}_2$ be the smallest $\mathscr{O}_{X_2}$-subbundle of $\mathscr{F}_2$ generated by $\mathscr{G}_2$. Then rank $(\mathscr{L}_2) = 1$, $\chi(\mathscr{G}_2) \leqslant \chi(\mathscr{L}_2)$ and $(0, \mathscr{L}_2, \vec{0}) \in S$ and hence

$$\chi(\mathscr{G}_2) - 1 \leqslant \chi(\mathscr{L}_2) - 1 < a_2 \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2} < \frac{\chi(\mathscr{F}_2))}{2} - \frac{1}{2}$$

(resp. $\leqslant$).

Thus

$$\chi(\mathscr{G}_1) + \chi(\mathscr{G}_2) - 1 \leqslant \chi(\mathscr{L}_1) + \chi(\mathscr{L}_2) - 2 < \chi(\mathscr{L}_1) - 1 + \frac{\chi((\mathscr{F}_2))}{2} - \frac{1}{2}. \quad (1)$$

But, if $C = A|_{\mathscr{L}_1(p)}$, then $(\mathscr{L}_1, \mathscr{F}_2, \vec{C}) \in S$ and hence

$$\chi(\mathscr{L}_1, \mathscr{F}_2, \vec{C}) = \chi(\mathscr{L}_1) + \chi(\mathscr{F}_2) - 2 < \frac{\chi}{2} + a_2 \frac{\chi}{2}$$

(resp. $\leqslant$).

Thus

$$\chi(\mathscr{L}_1) - 1 \leqslant \frac{\chi}{2} + a_2 \frac{\chi}{2} - (\chi(\mathscr{F}_2) - 1).$$

Using this in the inequality (1) we get

$$\chi(\mathscr{G}_1) + \chi(\mathscr{G}_2) - 1 \leqslant \frac{(1 + a_2)\chi + 1 - \chi(\mathscr{F}_2)}{2}.$$

Now, since by our assumption $a_2\chi - \chi(\mathscr{F}_2) < -1$, we get

$$\chi(\mathscr{G}_1) + \chi(\mathscr{G}_2) - 1 < \frac{\chi}{2}.$$

This proves the required inequality in this case.

This proof in case (iv) is divided into following three subcases:

(a) $\mathscr{G}_2 = 0$. Consider the canonical surjection

$$\mathscr{F}_1 \to \mathscr{F}_1(p)$$

and let $\mathscr{L}_1$ be the inverse image of $\ker(A)$ under this surjection. Then $(\mathscr{L}_1, 0, \vec{0}) \in S$ and $\mathscr{G}_1 \subset \mathscr{L}_1$ with $\chi(\mathscr{G}_1) \leqslant \chi(\mathscr{L}_1)$. Hence again we get the desired result.

(b) Rank $(\mathscr{G}_2) = 1$. Let $\mathscr{L}_2$ be the $\mathscr{O}_{X_2}$-subbundle of $\mathscr{F}_2$ generated by $\mathscr{G}_2$. Let V denote the image of $\mathscr{L}_2$ in $\mathscr{F}_2(p)$ under the natural map $\mathscr{F}_2 \to \mathscr{F}_2(p)$ and V' be the inverse image of V under the map A. Let $\mathscr{L}_1$ be the inverse image of V' under the natural surjection

$$\mathscr{F}_1 \to \mathscr{F}_1(p)$$

and set $C = A|_{V'}$. Now, clearly $(\mathscr{L}_1, \mathscr{L}_2, \vec{C}) \in S$, and $\mathscr{G}_i \subsetneq \mathscr{L}_i (i = 1, 2)$. Also, since rank $(\mathscr{L}_i) =$ rank $(\mathscr{G}_i)$ $(i = 1, 2)$, we see that $\chi(\mathscr{G}_i) \leqslant \chi(\mathscr{L}_i)$ $(i = 1, 2)$. Thus we get the desired result.

(c) Rank $(\mathscr{G}_2) = 2$. In this case, since $\mathrm{rk}(\mathscr{G}_i) = \mathrm{rk}(\mathscr{F}_i) = 2$ $(i = 1, 2)$ and $\mathscr{G}_i \subsetneq \mathscr{F}_i$ $(i = 1, 2)$, the required inequality is obvious.

This completes the proof of the lemma.

Theroem 5.3. *Let $\chi \neq 0$ be an integer and $0 < a_1 < 1$ be a rational number. Assume that $a_1\chi$ is not an integer. Then there is an irreducible projective variety $M(2, a_1, \vec{\chi})$ which is a course moduli space of $(a_1, a_2 = 1 - a_1)$-semistable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$* where *$\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is a rank two vector bundle on X_1 (resp. X_2), A: $\mathscr{F}_1(p)$ $\mathscr{F}_2(p)$* is a linear map and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2,$$
$$\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}) = \chi.$$

Moreover, when χ is odd $M(2, a_1, \vec{\chi})$ is smooth.

Proof. Let T be the set of isomorphism classes of (a_1, a_2)-semi-stable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, where $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is a rank two vector bundle on X_1 (resp. X_2), $A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)$ is a linear map and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2,$$
$$\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi.$$

Let

$$T_1 = \{[\mathscr{F}_1] | [(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in T \quad \text{for some} \quad \mathscr{F}_2 \text{ and } A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)\}$$

and

$$T_2 = \{[\mathscr{F}_2^{\,1}] | [(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in T \quad \text{for some} \quad \mathscr{F}_1 \text{ and } A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)\},$$

where $[\cdot]$ denote the isomorphism class of the object. Then by Theorem 5.1, T_1 (resp. T_2) consists of isomorphism classes of semistable vector bundles of rank two on X_1 (resp. X_2) of fixed Euler characteristics and hence T_1 (resp. T_2) is bounded (see [8], ch. (5.3)).

Hence there is an integer m_0 such that for all $m \geqslant m_0$ and for all $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$), $\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}$ (resp. $\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}$) is generated by global sections and the first cohomology group $H^1(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) = 0$ (resp. $H^1(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) = 0$). Similarly, we see that given any real number θ_1 (resp. θ_2), there is an integer $m(\theta_1)$ (resp. $m(\theta_2)$) such that for all $m \geqslant m(\theta_1)$ (resp. $m \geqslant m(\theta_2)$) and for any subbundle $\mathscr{G}_1$ of $\mathscr{F}_1$ (resp. $\mathscr{G}_2$ of $\mathscr{F}_2$) with $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$), degree of $\mathscr{G}_1$ (resp. $\mathscr{G}_2$) $\geqslant m(\theta_1)$ (resp. $\geqslant m(\theta_2)$), $\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}$ (resp. $\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}$) is generated by global sections and the first cohomology group $H^1(\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}) = 0$ (resp. $H^1(\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}) = 0$).

Set $\theta_1 = -2(g-8)$ (resp. $\theta_2 = -2(g-8)$), where g is the genus of X. Note that, since we have assumed that the genus of X_i, $i = 1, 2$ is atleast 2, $g \geqslant 4$. Let m be a fixed integer which is $\geqslant \max\{m(\theta_1), m(\theta_2)\}$, where $m(\theta_1)$ (resp. $m(\theta_2)$) is as above for the given choice of θ_1 (resp. θ_2). Let k_1 (resp. k_2) be the common dimension of all the vector spaces $H^0(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m})$ (resp. $H^0(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m})$) with $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$) Let E_1 (resp. E_2) be a vector space of dimension k_1 (resp. k_2). Let $\mathbf{Q}_1$ (resp. $\mathbf{Q}_2$) be the 'Quot' scheme of coherent sheaves on X_1 (resp. X_2), which are quotients of $E_1 \otimes \mathscr{O}_{X_1}$ (resp. $E_2 \otimes \mathscr{O}_{X_2}$) and whose Hilbert polynomial is that of $\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}$ (resp. $\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}$) for $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$). Denote by R_1 (resp. R_2) the open subset of $\mathbf{Q}_1$ (resp. $\mathbf{Q}_2$) consisting of those points $q_1 \in \mathbf{Q}_1$ (resp. $q_2 \in \mathbf{Q}_2$) such that if

$$E_1 \otimes \mathscr{O}_{X_1} \to \mathscr{F}_{q_1} \to 0 \quad (\text{resp. } E_2 \otimes \mathscr{O}_{X_2} \to \mathscr{F}_{q_2} \to 0)$$

is the corresponding quotient, then $H^1(\mathscr{F}_{q_1}) = 0$ (resp. $H^1(\mathscr{F}_{q_2}) = 0$), the natural map induces an isomorphism $H^0(\mathscr{F}_{q_1}) \simeq E_1$ (resp. $H^0(\mathscr{F}_{q_2}) \simeq E_2$) and $\mathscr{F}_{q_1}$ (resp. $\mathscr{F}_{q_2}$) is locally free. It is known that (see [12]) R_1 (resp. R_2) is a non-singular variety of dimension $k_1^2 + 4(g_1 - 1)$ (resp. $k_2^2 + 4(g_2 - 1)$).

Let

$$E_1 \otimes \mathscr{O}_{\mathbf{Q}_1 \times X_1} \to \mathbf{F}_1 \to 0 \quad (\text{resp. } E_2 \otimes \mathscr{O}_{\mathbf{Q}_2 \times X_2} \to \mathbf{F}_2 \to 0)$$

be the universal quotient sheaf on $\mathbf{Q}_1 \times X_1$ (resp. $\mathbf{Q}_2 \times X_2$). Set $F_1 = p_1^*(\mathbf{F}_1|_{R_1 \times p})$) (resp. $F_2 = p_2^*(\mathbf{F}_2|_{R_2 \times p})$) on $R_1 \times R_2$. Note that $F_i (i = 1, 2)$, are locally free of rank two. Thus $\mathrm{Hom}(F_1, F_2)$ is a rank four vector bundle on $R_1 \times R_2$. Let n be a large positive integer (which we will choose later). Let $r_1, \ldots, r_{na}$ (resp. $r_{na+1}, \ldots, r_{nb}$) be distinct points on X_1 (resp. X_2), where a and b are integers chosen at the beginning of this section. Define a morphism

$$\tau_n : \mathrm{Hom}(F_1, F_2) \to H_2(E_1 \oplus E_2)^{nb+1},$$

$$\phi_{(q_1, q_2)} \mapsto (\mathbf{F}_1|_{q_1 \times r_1}, \ldots, \mathbf{F}_1|_{q_1 \times r_{na}}, \mathbf{F}_2|_{q_2 \times r_{na+1}}, \ldots, \mathbf{F}_1|_{q_2 \times r_{nb}}, \Delta(\phi)),$$

where $H_2(E_1 \oplus E_2)$ is the Grassmanian of two-dimensional quotients of $E_1 \oplus E_2$ and $\Delta(\phi)$ is the two-dimensional quotient given by the composite of the surjections

$$E_1 \oplus E_2 \to \mathbf{F}_1|_{q_1 \times p} \oplus \mathbf{F}_2|_{q_2 \times p} \to \frac{\mathbf{F}_1|_{q_1 \times p} \oplus \mathbf{F}_2|_{q_2 \times p}}{\mathrm{Graph}(\phi)}.$$

Let

$$G = (GL(E_1) \times GL(E_2)) \cap SL(E_1 \oplus E_2).$$

Note that G acts on $\mathrm{Hom}(F_1, F_2)$ and on $H_2(E_1 \oplus E_2)^{nb+1}$ and τ_n is equivariant for this action.

Lemma 5.4. *A point*

$$(x_1, \ldots, x_{nb+1}) \in H_2(E_1 \oplus E_2)^{nb+1}$$

is semi-stable (resp. stable) for the action of G with respect to the linearization $(\varepsilon_1,\ldots,\varepsilon_1,\varepsilon)$ if and only if for any subvector space of the form $F_1\oplus F_2$ with $F_1\subset E_1, F_2\subset E_2$ and $F_1\oplus F_2\neq E_1\oplus E_2$, we have

$$-(\dim F_1+\dim F_2)(2nb\varepsilon_1+2\varepsilon)+(\dim E_1+\dim E_2)\left(\varepsilon_1\sum_{i=1}^{nb}\dim V_i+\varepsilon\dim V_{nb+1}\right)$$

$$\geqslant 0\ (\textit{resp.}\ >0),\ \textit{where}\ V_i=\mathrm{Im}(F_1\oplus F_2\to x_i), i=1,\ldots,nb+1.$$

Proof. Let D be the maximal torus

$$\left\{(t_1,\ldots,t_{n_1},l_1,\ldots,l_{n_2})\left|\left(\prod_{i=1}^{n_1}t_i\prod_{i=1}^{n_2}l_i\right)=1\right.\right\}$$

of $SL_{n_1+n_2}(\simeq SL(E_1\oplus E_2))$, where $n_1=\dim E_1$ and $n_2=\dim E_2$. Note that $D\subset(GL_{n_1}\times GL_{n_2})\cap SL_{n_1+n_2}(\simeq G)$. Thus D is also a maximal torus of G. Note that any one-parameter subgroup of D is of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n_1}},t^{b_1}\ldots,t^{b_{n_2}})|t\in k^*,a_i,b_i\in\mathbb{Z}\quad\text{and}\quad\left(\sum_{i=1}^{n_1}a_i+\sum_{i=1}^{n_2}b_i\right)=0\right\}.$$

Now since any one-parameter subgroup of GL_{n1+n2} can be conjugated to a one-parameter subgroup of the diagonal subgroup of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n_1}},t^{b_1},\ldots,t^{b_{n_2}})\left|\begin{array}{l}t\in k^*,a_i,b_i\in\mathbb{Z},\left(\sum_{i=1}^{n1}a_i+\sum_{i=1}^{n2}b_i\right)=0\\ \text{and}\ \ a_1\geqslant\cdots\geqslant a_{n1}\geqslant b_1\cdots\geqslant b_{n2}\end{array}\right.\right\},$$

it follows that any one-parameter subgroup of G can be conjugated to a one-parameter subgroup of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n_1}},t^{b_1},\ldots,t^{b_{n_2}})\left|\begin{array}{l}t\in k^*,a_i,b_i\in\mathbb{Z},\left(\sum_{i=1}^{n1}a_i+\sum_{i=1}^{n2}b_i\right)=0\\ \text{and}\ \ a_1\geqslant\cdots\geqslant a_{n_1},b_1\geqslant\cdots\geqslant b_{n2}\end{array}\right.\right\}.$$

On the other hand, we clearly see that a point

$$(x_1,\ldots,x_{nb+1})\in H_2(E_1\oplus E_2)^{nb+1}$$

is semi-stable (resp. stable) for the action of G with respect to the linearization $(\varepsilon_1,\ldots,\varepsilon_1,\varepsilon)$ if and only if for every one-parameter c of G

$$\mu(x_1,\ldots,x_{nb+1};c)=\varepsilon_1\sum_{i=1}^{nb}\mu(x_i;c)+\varepsilon\mu(x_{nb+1};c)\geqslant 0\quad(\text{resp.}>0)$$

(see[11], Proposition (28), Premiére partie. III).

Let c be a one-parameter subgroup of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n_1}},t^{b_1},\ldots,t^{b_{n_2}})\left|\begin{array}{l}t\in k^*,a_i,b_i\in\mathbb{Z},\left(\sum_{i=1}^{n_1}a_i+\sum_{i=1}^{n_2}b_i\right)=0\\ \text{and}\ \ a_1\geqslant\cdots\geqslant a_{n_1},\ b_1\geqslant\cdots\geqslant b_{n_2}\end{array}\right.\right\}.$$

Then

$$\mu(x_1,\ldots,x_{nb+1};c)\geqslant 0 \quad (\text{resp.}>0)$$

f and only if for all natural numbers p,p_1,p_2 such that $1\leqslant p<n_1+n_2, 0\leqslant p_1\leqslant n_1$, $)\leqslant p_2\leqslant n_2$ and $p_1+p_2=p$,

$$\mu(x_1,\ldots,x_{nb+1};c_{p,p_1,p_2})\geqslant 0 \quad (\text{resp.}>0),$$

where

$$c_{p,p_1,p_2}=\left\{\left(\underbrace{t^{n_1+n_2-p},\ldots,t^{n_1+n_2-p}}_{p_1},\ \underbrace{t^{-p},\ldots,t^{-p}}_{n_1-p_1},\right.\right.$$
$$\left.\left.\underbrace{t^{n_1+n_2-p},\ldots,t^{n_1+n_2-p}}_{p_2},\ \underbrace{t^{-p},\ldots,t^{-p}}_{n_2-p_2}\right)\middle| t\in k^*\right\}.$$

That is, every c of the above form is a positive linear combination of $\{c_{p,p_1,p_2}\}$ and μ is linear on he set of all c of the above form.

Note that if $E_1\oplus E_2\to V$ is a quotient of dimension two, then

$$\mu(V;c_{p,p_1,p_2})=-2(p_1+p_2)+(n_1+n_2)\dim(V'),$$

where $V'=\mathrm{Im}(F_1\oplus F_2\to V)$ with F_1 as the subspace of E_1 spanned by the eigenvectors of $_{p,p_1,p_2}$ with eigenvalue $t^{n_1+n_2-p}$ and F_2 as the subspace of E_2 spanned by the eigenvectors of $_{p,p_1,p_2}$ with eigenvalue $t^{n_1+n_2-p}$. Hence

$$\mu(x_1,\ldots,x_{nb+1};c_{p,p_1,p_2})=\varepsilon_1\sum_{i=1}^{nb}(\dim(V_i)(n_1+n_2)-2p)$$
$$+\varepsilon(\dim(V_{nb+1})(n_1+n_2)-2p),$$

where $V_i=\mathrm{Im}(F_1\oplus F_2\to x_i), i=1,\ldots,nb+1$. From this the Lemma follows immediately.

Continuing the proof of Theorem 5.3, note that each point of $\mathrm{Hom}(F_1,F_2)$ gives rise to a triple as follows; if $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)_{(q_1,q_2)}$, then it defines a homomorphism

$$A:\mathscr{F}|_{q_1\times X_1}(p)\to\mathscr{F}|_{q_2\times X_2}(p)$$

and thus gives rise to a triple $(\mathscr{F}|_{q_1\times X_1},\mathscr{F}|_{q_2\times X_2}\vec{A})$.

DEFINITION 5.1

A point $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)$ is said to be semistable (resp. stable) if the corresponding triple $\mathscr{F}|_{q_1\times X_1},\mathscr{F}|_{q_2\times X_2},\vec{A})$ is semi-stable (resp. stable).

Lemma 5.5. *If n is sufficiently large and $H_2(E_1\oplus E_2)^{nb+1}$ is G linearized with respect to the polarization $(\varepsilon_1,\ldots,\varepsilon_1,\varepsilon)$, then*

(1) *A point $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)$ is semi-stable $\Rightarrow\tau_n(\phi_{(q_1,q_2)})$ is semi-stable for the G action.*
(2) *A point $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)$ is not semi-stable $\Rightarrow\tau_n(\phi_{(q_1,q_2)})$ is not semi-stable for the G action.*
(3) *$\tau_n(\phi_{(q_1,q_2)})$ is stable for the G action if and only if $\phi_{(q_1,q_2)}$ is stable.*

Proof. Let $\phi_{(q_1,q_2)}$ be stable. Then to show $\tau_n(\phi_{(q_1,q_2)})=(x_1,\ldots,x_{nb+1})$ is stable for the G action, by Lemma 5.4, we need to show that for any subvector space of the form $K_1\oplus K_2$ with

$K_1 \subset E_1, K_2 \subset E_2$ and $K_1 \oplus K_2 \neq E_1 \oplus E_2$, the real number

$$-(\dim K_1 + \dim K_2)(2nb\varepsilon_1 + 2\varepsilon)$$
$$+(\dim E_1 + \dim E_2)(\varepsilon_1 \textstyle\sum_{i=1}^{nb} \dim V_i + \varepsilon \dim V_{nb+1}) > 0, \qquad (*)$$

where $V_i = \mathrm{Im}(K_1 \oplus K_2 \to x_i)\, i = 1, \ldots, nb+1$.

Let $K_1 \oplus K_2$ with $K_1 \subset E_1, K_2 \subset E_2$ and $K_1 \oplus K_2 \neq E_1 \oplus E_2$ be given. Assume that there exists a subtriple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ of $(\mathscr{F}|_{q_1 \times X_1} \otimes \mathscr{L}_1^{\otimes -m}, \mathscr{F}|_{q_2 \times X_2} \otimes \mathscr{L}_2^{\otimes -m}, \overrightarrow{A \otimes \lambda^{\otimes -m}})$ such that

(i) $\mathscr{G}_i \subset \mathscr{F}|_{q_i \times X_i} \otimes \mathscr{L}_i^{\otimes -m}$ is a subbundle for $i = 1, 2$.

(ii) $H^0(\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}) = K_1$, $H^0(\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}) = K_2$.

(iii) $H^1(\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}) = 0$, $H^1(\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}) = 0$.

By assumption $(\mathscr{F}|_{q_1 \times X_1}, \mathscr{F}|_{q_2 \times X_2}, \vec{A})$ is stable, hence we get

$$\dim K_1 + \dim K_2 - \mathrm{rk}\mathscr{G}_2 < (a_1 \mathrm{rk}\mathscr{G}_1 + a_2 \mathrm{rk}\mathscr{G}_2)\left(\frac{\dim E_1 + \dim E_2}{2} - 1\right). \qquad (**)$$

On the other hand, we see that $V_i = \mathrm{rk}\mathscr{G}_1\, (1 \leqslant i \leqslant na)$ and $V_i = \mathrm{rk}\mathscr{G}_2 (na+1 \leqslant i \leqslant nb)$. Hence the L.H.S. of $(*)$ is equal to

$$-(\dim K_1 + \dim K_2)(2nb\varepsilon_1 + 2\varepsilon)$$
$$+(n_1 + n_2)(na\varepsilon_1\, \mathrm{rk}\mathscr{G}_1 + \varepsilon_1(nb - na)\mathrm{rk}\mathscr{G}_2 + \varepsilon \dim V_{nb+1}).$$

Dividing this by $2nb$ and recalling that $a_1 = a/b$ and $a_2 = 1 - a_1 = (b-a)/b$, we get that inequality $(*)$ which is equivalent to the inequality

$$-(\dim K_1 + \dim K_2)\left(\varepsilon_1 + \frac{\varepsilon}{nb}\right)$$
$$+\frac{n_1 + n_2}{2}\left(\varepsilon_1 a_1 \mathrm{rk}\mathscr{G}_1 + \varepsilon_1 a_2 \mathrm{rk}\mathscr{G}_2 + \frac{\varepsilon}{nb}\dim V_{nb+1}\right) > 0.$$

Now set $\varepsilon = 2nb/(n_1 + n_2)$. Then the above inequality becomes

$$-(\dim K_1 + \dim K_2)\varepsilon_1 - \frac{\dim K_1 + \dim K_2}{(n_1 + n_2)/2}$$
$$+\frac{n_1 + n_2}{2}(\varepsilon_1 a_1 \mathrm{rk}\mathscr{G}_1 + \varepsilon_1 a_2 \mathrm{rk}\mathscr{G}_2 + \dim V_{nb+1}) > 0. \qquad (***)$$

Note that

$$\frac{\dim K_1 + \dim K_2}{(n_1 + n_2)/2}$$
$$= \frac{d(\mathscr{G}_1) + (\mathrm{rk}\mathscr{G}_1)ma + (\mathrm{rk}\mathscr{G}_1)(1 - g_1) + d(\mathscr{G}_2) + (\mathrm{rk}\mathscr{G}_2)m(b-a) + (\mathrm{rk}\mathscr{G}_2)(1 - g_2)}{(d(\mathscr{G}_1)/2) + ma + (1 - g_1) + (d(\mathscr{G}_2)/2) + m(b-a) + (1 - g_2)}$$

$$= \frac{\dfrac{d(\mathscr{G}_1) + (\mathrm{rk}\mathscr{G}_1)(1 - g_1)}{mb} + \dfrac{d(\mathscr{G}_2) + (\mathrm{rk}\mathscr{G}_2)(1 - g_2)}{mb} + (\mathrm{rk}\mathscr{G}_1)a_1 + (\mathrm{rk}\mathscr{G}_2)a_2}{\dfrac{d(\mathscr{F}_1) + 2(1 - g_1) + d(\mathscr{F}_2) + 2(1 - g_2)}{2mb} + 1},$$

where $\mathscr{F}_i = \mathscr{F}|_{q_i \times X_i} \otimes \mathscr{L}_i^{\otimes -m} (i = 1, 2), d(\mathscr{F}_i) = \text{degree } (\mathscr{F}_i) (i = 1, 2)$ and $d(\mathscr{G}_i) = \text{degree } (\mathscr{G}_i) (i = 1, 2)$.

Now set

$$\frac{1}{\varepsilon_1} = \frac{d(\mathscr{F}_1) + 2(1 - g_i) + d(\mathscr{F}_2) + 2(1 - g_2)}{2mb} + 1.$$

Now dividing the inequality (***) by ε_1 we get

$$\left(\frac{n_1 + n_2}{2} - 1\right)(a_1 \text{rk}\mathscr{G}_1 + a_2 \text{rk}\mathscr{G}_2) + \dim V_{nb+1} - (\dim K_1 + \dim K_2)$$

$$+ \dim V_{nb+1} \left(\frac{d(\mathscr{F}_1) + 2(1 - g_1) + d(\mathscr{F}_2) + 2(1 - g_2)}{2mb}\right)$$

$$- \left(\frac{d(\mathscr{G}_1) + (\text{rk}\mathscr{G}_1)(1 - g_1)}{mb} + \frac{d(\mathscr{G}_2) + (\text{rk}\mathscr{G}_2)(1 - g_2)}{mb}\right) > 0.$$

Now our assumptions on the subtriple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ of

$$(\mathscr{F}|_{q_1 \times X_1} \otimes \mathscr{L}_1^{\otimes -m}, \mathscr{F}|_{q_2 \times X_2} \otimes \mathscr{L}_2^{\otimes -m}, \overrightarrow{A \otimes \lambda^{\otimes -m}})$$

imply that $\dim V_{nb+1} = \text{rk}\mathscr{G}_2$, hence the above inequality is a consequence of the inequality (**) and Theorem 5.1.

The remaining part of the proof of the lemma is similar to the proof of the Proposition 4.2 of [5].

Proof of Theorem 5.3 (*continued*). For sufficiently large n it can be proved that (proof is similar to the proof of Theorem (5.6) of [8] and also see [11])

$$\tau_n : \text{Hom}(F_1, F_2) \to H_2(E_1 \oplus E_2)^{nb+1}$$

is injective and by Lemma 5.5, it follows that the restriction to semistable points induces a morphism

$$\tau_n^{ss} : \text{Hom}(F_1, F_2)^{ss} \to (H_2(E_1 \oplus E_2)^{nb+1})^{ss},$$

which maps the set of stable points $\text{Hom}(F_1, F_2)^s$ into the set of stable points $(H_2(E_1 \oplus E_2)^{nb+1})^s$. Thus $\text{Hom}(F_1, F_2)//G = \text{Hom}(F_1, F_2)^{ss}/G$ exists by [6]. Since the normalization of this quotient can be identified with an irreducible component of the moduli space of rank 2 torsion free sheaves on X, it is irreducible and projective by ([11], septième partie, III, Théorèm 15). This is the required variety $M(2, a_1, \vec{\chi})$. Moreover, when χ is odd, we see that "semistable" = "stable", hence the variety $M(2, a_1, \vec{\chi})$ is smooth.

Note. In the proof of the above theorem, we used the fact that the moduli space of torsion free sheaves is proper to conclude that the moduli space of triples $M(2, a_1, \vec{\chi})$ is proper. We believe that, using methods similar to the Langton's proof of properness of moduli of semi-stable vector bundles on a smooth curve, one can also directly prove the properness of the moduli space of triples.

6. Another proof of Theorem 4.1

Note that by Theorem 3.1, if $\mathscr{F}$ is a torsion free (a_1, a_2) semi-stable sheaf on X of rank two and Euler characteristic χ, then there exists a unique triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$) such that $\mathscr{F}_i$ (resp. $\mathscr{F}'_i$) is locally free of rank 2 on $X_i (i = 1, 2)$, satisfying

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

$$(\text{resp. } a_1\chi < \chi_{X_1}(\mathscr{F}'_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}'_2) < a_2\chi + 2)$$

and

$$\chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_1}(\mathscr{F}_2) \,(\text{resp. } \chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}'_1) + \chi_{X_1}(\mathscr{F}'_2)$$

with

$$\vec{A} : \mathscr{F}_1(p) \to \mathscr{F}_2(p) \,(\text{resp. } \overleftarrow{B} : \mathscr{F}'_2(p) \to F'_1(p)),$$

a nonzero linear map. Moreover,

(a) if $\mathscr{F}$ is locally free, then $\vec{A}$ is invertible and $\mathscr{F}_i = \mathscr{F}'_i (i = 1, 2)$ with $\overleftarrow{B} = (\vec{A})^{-1}$,
(b) if $\mathscr{F}$ is not locally free, then $\vec{A}$ (resp. $\overleftarrow{B}$) has rank one and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1 \text{ and } a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

$$(\text{resp. } a_1\chi + 1 < \chi_{X_1}(\mathscr{F}'_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}'_2) < a_2\chi + 1).$$

The triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$ are related by the following diagram:

$$\begin{array}{ccc} \mathscr{F}_1(p) & \xrightarrow{i(p)} & \mathscr{F}'_1(p) \\ \downarrow \vec{A} & & \overleftarrow{B} \uparrow \\ \mathscr{F}_2(p) & \xleftarrow{j(p)} & \mathscr{F}'_2(p) \end{array},$$

where $j : \mathscr{F}'_2 \to \mathscr{F}_2$ (resp. $i : \mathscr{F}_1 \to \mathscr{F}'_1$) is the canonical Hecke modification such that $\mathrm{Im}(j(p)) = \mathrm{Im}(\vec{A})$ (resp. $\mathrm{Ker}(i(p)) = \mathrm{Ker}(\vec{A})$) and $\mathrm{Ker}(j(p)) = \mathrm{Ker}(\overleftarrow{B})$ (resp. $\mathrm{Im}(i(p)) = \mathrm{Im}(\overleftarrow{B})$ (see Remark 2.4).

Let $M_1 = M(2, a_1, \overrightarrow{\chi})$ be the moduli space of (a_1, a_2)-semi-stable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ of rank two with

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1 \text{ and } a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

and $\chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2)$ (this space is constructed in Theorem 5.3). Let $M_2 = M(2, a_1, \overleftarrow{\chi})$ be the moduli of (a_1, a_2) semi-stable triples $(\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{B})$ of rank two with

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 1$$

and $\chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2)$ (the construction of M_2 is exactly similar to that of M_1). Let

$$M'_1 = \{[(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in M_1 \,|\, \mathrm{rk}(\vec{A}) = 1\}$$

and

$$M'_2 = \{[(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})] \in M_2 \,|\, \mathrm{rk}(\overleftarrow{B}) = 1\}.$$

From (b) above, we observe that there is a natural isomorphism $M'_1 \to M'_2$. Now by lemma 2.3 and Remark 2.9, it follows that $M(2, (a_1, a_2), \chi)$ is isomorphic to $M_1 \cup M_2$ with the natural identification of the closed subscheme M'_1 of M_1 with the closed subscheme M'_2 of M_2. Moreover, when χ is odd, both M_1 and M_2 are smooth (see Theorem 5.3) and M'_1 (resp. M'_2) is a smooth closed subvariety of M_1 (resp. M_2) (see Theorem 6.1 below). Thus, if χ is odd, $M(2, (a_1, a_2), \chi)$ is a normal crossing variety. This proves the theorem.

Theorem 6.1. *Notations are as above. Let P_1 be the moduli space of parabolic semi-stable (= parabolic stable) bundles $(\mathscr{F}_1, 0 \subset F^2\mathscr{F}_1(p) \subset \mathscr{F}_1(p))$ of rank two on X_1 with parabolic weight $(a_1/2, a_2/2)$ (if $a_1 < a_2$, otherwise $(a_2/2, a_1/2)$) and degree of $\mathscr{F}_1$ equal to χ_1, where χ_1, is an integer satisfying $a_1\chi < \chi_1 < a_1\chi + 1$ with χ as in the theorem above. Similarly let P_2 be the moduli space of parabolic semi-stable (= parabolic stable) bundles $(\mathscr{F}_2, 0 \subset F^2\mathscr{F}_2(p) \subset \mathscr{F}_2(p))$ of rank two on X_2 with parabolic weight $(a_1/2, a_2/2)$ if $a_1 < a_2$, otherwise $(a_2/2, a_1/2)$) and degree of $\mathscr{F}_2$ equal to χ_2, where χ_2, is an integer satisfying $a_2\chi + 1 < \chi_2 < a_2\chi + 2$ with χ as above. Moreover, we see that P_1 and P_2 are smooth. By sending $[(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in M'_1$ to $(\mathscr{F}_1, 0 \subset F^2\mathscr{F}_1(p) \subset \mathscr{F}_1(p)) \times (\mathscr{F}_2, 0 \subset F^2\mathscr{F}_2(p) \subset \mathscr{F}_2(p)) \in P_1 \times P_2$, with $F^2\mathscr{F}_1(p) = \mathrm{Ker}(\vec{A})$ and $F^2\mathscr{F}'_1(p) = \mathrm{Im}(\vec{A})$, we can identify M'_1 with $P_1 \times P_2$. Similarly we can identify M'_2 also with $P_1 \times P_2$.*

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a semi-stable triple such that $[(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in M'_1$. Then we prove that $(\mathscr{F}_1, 0 \subset F^2\mathscr{F}_1(p) \subset \mathscr{F}_1(p))$(resp.$(\mathscr{F}_2, 0 \subset F^2\mathscr{F}_2(p) \subset \mathscr{F}_2(p))$) is parabolic stable on X_1 (resp. X_2) with respect to the weights (a_1, a_2). By Theorem (5.1) we know that $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is semi-stable on X_1 (resp. X_2). Thus for any line subbundle $\mathscr{L}_1$ (resp. $\mathscr{L}_1$) of $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) we have (assuming $a_1 < a_2$)

$$\chi(\mathscr{L}_1) + a_1/2 < \chi(\mathscr{F}_1)/2 + 1/4 \quad (\text{resp. } \chi(\mathscr{L}_2) + a_1/2 < \chi(\mathscr{F}_2)/2 + 1/4).$$

Thus, if either $\mathscr{L}_1(p) \neq F^2\mathscr{F}_1(p)$ or $\chi(\mathscr{L}_1) < \frac{1}{2}\chi(\mathscr{F}_1)$ (resp. if either $\mathscr{L}_2(p) \neq F^2\mathscr{F}_2(p)$ or $\chi(\mathscr{L}_2) < \frac{1}{2}\chi(\mathscr{F}_2)$) we see that par deg $(\mathscr{L}_1) <$ par $\mu(\mathscr{F}_1)$(resp. par deg$(\mathscr{L}_2) <$ par $\mu(\mathscr{F}_2)$). We next show that there is no subbundle $\mathscr{L}_1$ of $\mathscr{F}_1$ such that $\chi(\mathscr{L}_1) = \chi(\mathscr{F}_1)/2$ and $\mathscr{L}_1(p) = F^2\mathscr{F}_1(p)$ (resp. $\mathscr{L}_2$ of $\mathscr{F}_2$ such that $\chi(\mathscr{L}_2) = \chi(\mathscr{F}_2)/2$ and $\mathscr{L}_2(p) = F^2\mathscr{F}_2(p)$). If such a line-bundle $\mathscr{L}_1$ exists, then $(\mathscr{L}_1, 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ but semi-stability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ implies $\chi(\mathscr{L}_1) \leqslant a_1\chi/2$. Thus $\chi(\mathscr{F}_1) \leqslant a_1\chi$, which contradicts the fact $a_1\chi < \chi(\mathscr{F}_1)$. (The proof that there is no line subbundle $\mathscr{L}_2$ of $\mathscr{F}_2$ such that $\chi(\mathscr{L}_2) = \chi(\mathscr{F}_2)/2$ and $\mathscr{L}_2(p) = F^2\mathscr{F}_2(p)$ is similar.)

This implies that we get a morphism from the moduli space of triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ with $\mathrm{rk}(\vec{A}) = 1$ into the product of parabolic moduli spaces under consideration. This moduli space of triples is proper, being a closed subset of the moduli space of all semi-stable triples. Further the above morphism is injective and both the spaces have the same dimension. Hence the morphism is bijective and the required implication follows. In fact, the required assertion can be checked directly by a case-by-case analysis.

7. 'det' of semi-stable torsion free sheaf of rank 2 on X

Notations are as in the previous sections. Let $\mathscr{F}$ be a rank two, (a_1, a_2)-semi-stable torsion free sheaf on X. Assume that $\chi = \chi(\mathscr{F}) \neq 0$ and $a_1\chi$ is not an integer. If $\mathscr{F}$ is a vector bundle of rank two, then its second exterior power $\overset{2}{\wedge}(\mathscr{F})$ is a line bundle denoted by $\det(\mathscr{F})$. Moreover, if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \vec{B})$) is the triple representing $\mathscr{F}$, then we see that $\det(\mathscr{F})$ is represented by the triple $(\overset{2}{\wedge}(\mathscr{F}_1), \overset{2}{\wedge}(\mathscr{F}_2), \overrightarrow{\det(A)})$ resp. $(\overset{2}{\wedge}(\mathscr{F}'_1), \overset{2}{\wedge}(\mathscr{F}'_2), \overrightarrow{\det(B)})$). Note that in this case $\mathscr{F}'_1 \simeq \mathscr{F}_1$ and $\mathscr{F}'_2 \simeq \mathscr{F}_2$ and $B = A^{-1}$. If $\mathscr{F}$ is not a vector bundle, then we see by Theorem 3.1 that $\mathscr{F}_p \simeq \mathscr{O}_{X,p} \oplus m_{X,p}$ and $\mathscr{F}_x \simeq \mathscr{O}_{X,x} \oplus \mathscr{O}_{X,x}$ for all $x \in X - p$. Thus $\overset{2}{\wedge}(\mathscr{F})_p/(\text{torsion}) \simeq m_{X,p}$ and $\overset{2}{\wedge}(\mathscr{F})_x \simeq \mathscr{O}_{X,x}$ for all $x \in X - p$. Also, when $\mathscr{F}$ is not a vector bundle, we see that if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$

(resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$) is the triple representing $\mathscr{F}$, then $\overset{2}{\wedge}(\mathscr{F})$/(torsion) is represented by the triple $(\overset{2}{\wedge}\mathscr{F}_1, \overset{2}{\wedge}\mathscr{F}_2, \overrightarrow{0})$ (resp. $(\overset{2}{\wedge}\mathscr{F}'_1, \overset{2}{\wedge}\mathscr{F}'_2, \overleftarrow{0})$). Note that in this case we have the following exact sequences

$$0 \to \mathscr{F}_1 \to \mathscr{F}'_1 \to k(p) \to 0,$$

$$0 \to \mathscr{F}'_2 \to \mathscr{F}_2 \to k(p) \to 0,$$

where $k(p)$ denotes the residue field at p. Thus we see that $\overset{2}{\wedge}\mathscr{F}'_1 \simeq \overset{2}{\wedge}\mathscr{F}_1 \otimes \mathscr{O}_{X_1}(p)$ and $\overset{2}{\wedge}\mathscr{F}'_2 \simeq \overset{2}{\wedge}\mathscr{F}_2 \otimes \mathscr{O}_{X_2}(-p)$. Consider the triple $(\overset{2}{\wedge}\mathscr{F}_1, \overset{2}{\wedge}\mathscr{F}_2, \overrightarrow{\lambda})$ (resp. $(\overset{2}{\wedge}\mathscr{F}'_1, \overset{2}{\wedge}\mathscr{F}'_2, \overleftarrow{\mu})$), where λ (resp. μ) is an isomorphism of one dimensional vector spaces $\overset{2}{\wedge}\mathscr{F}_1(p) \to \overset{2}{\wedge}\mathscr{F}_2(p)$ (resp. $\overset{2}{\wedge}\mathscr{F}'_1(p) \to \overset{2}{\wedge}\mathscr{F}'_2(p)$). If L_1 (resp. L_2) is the line bundle on X associated to the triple $(\overset{2}{\wedge}\mathscr{F}_1, \overset{2}{\wedge}\mathscr{F}_2, \overrightarrow{\lambda})$, resp. $((\overset{2}{\wedge}\mathscr{F}'_1, \overset{2}{\wedge}\mathscr{F}'_2, \overleftarrow{\mu}))$, then

$$L_1 \in J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\text{resp. } L_2 \in J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2)),$$

where χ_1, χ_2 are integers satisfying $a_1\chi < \chi_1 < a_1\chi + 1$, $a_2\chi + 1 < \chi_2 < a_2\chi + 2$, with $\chi_1 + \chi_2 - 2 = \chi$ and $J_{\chi_i-(1-g_i)}(X_i)$ is the Jacobian of line bundles on X_i of Euler characteristics $\chi_i - (1-g_i)$ for $i = 1, 2$. Also, note that under the natural isomorphism

$$J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\mathscr{L}_1, \mathscr{L}_2) \mapsto (\mathscr{L}_1 \otimes \mathscr{O}_{X_1}(-p), \mathscr{L}_2 \otimes \mathscr{O}_{X_2}(p)),$$

L_2 maps to L_1. Moreover,

$$\chi(\overset{2}{\wedge}\mathscr{F}/(\text{torsion})) = \chi(L_1) = \chi(L_2)$$

(see Remark 2.12).

Let $M(2,(a_1,a_2),\chi \neq 0)^0$ be the moduli space of (a_1, a_2)-semi-stable rank two vector bundles on X with Euler characteristic $\chi(\neq 0)$. Assume that $a_1\chi$ is not an integer. From the results of § 2, we see that $M(2,(a_1,a_2),\chi \neq 0)^0 = M_1^0 \cup M_2^0$, where M_1^0 (resp. M_2^0) is the moduli space of (a_1, a_2)-semi-stable vector bundles $\mathscr{F}$ on X such that

$$a_1\chi < \chi_{X_1}(\mathscr{F}|_{X_1}) < a_1\chi + 1$$

$$(\text{resp. } a_1\chi + 1 < \chi_{X_1}(\mathscr{F}|_{X_1}) < a_1\chi + 2)$$

and $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}|_{X_2}) < a_2\chi + 2$ (resp. $a_2\chi < \chi_{X_2}(\mathscr{F}|_{X_2}) < a_2\chi + 1$).

Thus we have a morphism

$$(\det)'_1 : M_1^0 \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\text{resp. } (\det)'_2 : M_2^0 \to J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2))$$

where χ_1, χ_2 are integers satisfying

$$a_1\chi < \chi_1 < a_1\chi + 1, \quad a_2\chi + 1 < \chi_2 < a_2\chi + 2,$$

with $\chi_1 + \chi_2 - 2 = \chi$. Now using the isomorphism

$$J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\mathscr{L}_1, \mathscr{L}_2) \mapsto (\mathscr{L}_1 \otimes \mathscr{O}_{X_1}(-p), \mathscr{L}_2 \otimes \mathscr{O}_{X_2}(p)),$$

we get a morphism

$$(\det) : M(2,(a_1,a_2),\chi \neq 0)^0 \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2).$$

Now the arguments of the previous paragraph show that the morphism

$$(\det): M(2,(a_1,a_2),\chi \neq 0)^0 \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

can be naturally extended to a set theoretic map from the moduli space $M(2,(a_1,a_2),$ $\chi \neq 0)$ of (a_1,a_2)- semi-stable rank two torsion free sheaves on X with Euler characteristics $\chi(\neq 0)$.

PROPOSITION 7.1

The map

$$(\det): M(2,(a_1,a_2),\chi \neq 0) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

is a morphism.

Proof. The proof follows from the following two facts:
(a) $\mathscr{F}$ is a torsion free sheaf of rank two on X with Euler characteristic χ, then

$$\chi(\overset{2}{\wedge}/(\text{torsion})) = \chi - (1-g).$$

(This can be proved considering the associated triple and the Remark 2.7).
(b) Let $\mathscr{F}_0$ be a (a_1,a_2)-semi-stable rank two torsion free non-free sheaf on X with Euler characteristic $\chi(\neq 0)$. Let $\mathscr{G}$ be a coherent sheaf on $S \times X$ which is flat over the base scheme S (finite type over the field $\mathbb{C}$ of complex numbers) and such that for all $s \in S, \mathscr{G}|_{s\times X}$ is an (a_1,a_2)-semi-stable rank two torsion free sheaf on X with Euler characteristic $\chi(\neq 0)$ and for the closed point $s_0 \in S, \mathscr{G}|_{s_0\times X} \simeq \mathscr{F}_0$. Let T be the maximal subsheaf of modules of $\overset{2}{\wedge}\mathscr{G}$ with proper support. Then $\overset{2}{\wedge}\mathscr{G}/T$ is a pure sheaf of modules on $S \times X$. The natural surjection

$$(\overset{2}{\wedge}\mathscr{G}/T)|_{s\times X} \to \overset{2}{\wedge}(\mathscr{G}|_{s\times X})$$

induces an isomorphism

$$(\overset{2}{\wedge}\mathscr{G}/T)|_{s\times X} \simeq (\overset{2}{\wedge}(\mathscr{G}|_{s\times X}))/(\text{torsion}),$$

for all $s \in S$. Thus $\overset{2}{\wedge}\mathscr{G}/T$ is a flat family of torsion free sheaves of rank one and Euler characteristic $\chi-(1-g)$ with the property

$$(\overset{2}{\wedge}\mathscr{G}/T)|_{s_0\times X} \simeq \overset{2}{\wedge}(\mathscr{G}|_{s_0\times X})/(\text{torsion}).$$

PROPOSITION 7.2

The fibres of the morphism

$$(\det): M(2,(a_1,a_2),\chi \neq 0) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

are all reduced.

Proof. Note that any $[\mathscr{F}] \in S((a_1,a_2),(2,2),\chi \neq 0)$ is represented by a triple $(\mathscr{F}_1,\mathscr{F}_2,\vec{A})$ with the property $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2$, $a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$, $\text{rk}(\vec{A}) \geqslant 1$ and $\chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2 = \chi$. Let $L_1 \in J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$ be represented by $(\mathscr{L}_1,\mathscr{L}_2,\vec{\lambda})$. Let $\mathscr{F} \in S((a_1,a_2),(2,2),\chi \neq 0)$ be such that $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1$, $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$ and 'det'$(\mathscr{F}) = L_1$. Then note that for all such $\mathscr{F}$ there exists a fixed integer n such that $\mathscr{F}(n)$ admits a trivial line subbundle. If $(\mathscr{F}_1,\mathscr{F}_2,\vec{A})$

is the triple associated to such a torsion free sheaf $\mathscr{F}$, then we see that

$$\mathscr{F}_1 \in \mathrm{Ext}^1_{\mathscr{C}_{X_1}}(\mathscr{L}_1(na), \mathscr{O}_{X_1}(-na)), \mathscr{F}_2 \in \mathrm{Ext}^1_{\mathscr{C}_{X_2}}(\mathscr{L}_2(n(b-a)), \mathscr{O}_{X_2}(-n(b-a))).$$

Thus $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is isomorphic to a triple given by a point of the rank four vector bundle $\mathrm{Hom}\,(p_1^*(F_1), p_2^*(F_2))$ on

$$\mathrm{Ext}^1_{\mathscr{C}_{X_1}}(\mathscr{L}_1(na), \mathscr{O}_{X_1}(-na)) \times \mathrm{Ext}^1_{\mathscr{C}_{X_2}}(\mathscr{L}_2(n(b-a)), \mathscr{O}_{X_2}(-n(b-a))),$$

where F_1 (resp. F_2) is the restriction to $\mathrm{Ext}^1_{\mathscr{C}_{X_1}}(\mathscr{L}_1(na), \mathscr{O}_{X_1}(-na)) \times p$ (resp. $\mathrm{Ext}^1_{\mathscr{C}_{X_2}}(\mathscr{L}_2(n(b-a)), \mathscr{O}_{X_2}(-n(b-a))) \times p$) of the universal rank two vector bundle on $\mathrm{Ext}^1_{\mathscr{C}_{X_1}}(\mathscr{L}_1(na), \mathscr{O}_{X_1}(-na)) \times X_1$ (resp. $\mathrm{Ext}^1_{\mathscr{C}_{X_2}}(\mathscr{L}_2(n(b-a))), \mathscr{O}_{X_2}(-n(b-a))) \times X_2$). Hence there is a surjective morphism from an open subvariety of the smooth variety $\mathrm{Hom}\,(p_1^*(F_1), p_2^*(F_2))$ to the variety of all $\mathscr{F} \in M(2,(a_1,a_2),\chi \neq 0)$ such that $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$ and 'det'$(\mathscr{F}) = L_1$. Thus the variety of all $\mathscr{F} \in M(2,(a_1,a_2),\chi \neq 0)$ such that $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$ and 'det'$(\mathscr{F}) = L_1$, is reduced and irreducible. Similarly, the variety of all $\mathscr{F} \in M(2,(a_1,a_2),\chi \neq 0)$ such that $a_1\chi + 1 < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2, a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 1$ and 'det'$(\mathscr{F}) = L_1$ is reduced and irreducible. This proves that the fibres of 'det' morphism are all reduced.

Remark 7.1. The 'det' morphism has good specialization properties in the folowing sense:
Let $S = \mathrm{Spec}(R)$ be a spectrum of a complete discrete valuation ring R with residue field $\mathbb{C}$ and x_0 be its closed point. Let $\mathscr{X} \to S$ be a flat family of curves with χ a two dimensional regular scheme and $\mathscr{X}|_{x_0} \simeq X$. Choose a relative ample line bundle $\mathscr{O}_\chi(1)$ on $\chi \to S$ such that its restriction to the special fibre induces the polarization of type (a_1,a_2). Then the moduli scheme

$$\mathscr{M}(2, \mathscr{O}_\mathscr{X}, \chi \neq 0) \to S$$

of rank two torsion free sheaves along the fibres with Euler characteristic χ and semi-stable with respect to $\mathscr{O}(1)$, exists as a flat scheme over S (see Theorem 4.2). By the results of §4, we see that $\mathscr{M}(2, \mathscr{O}_\mathscr{X}, \chi \neq 0)$ is an integral scheme flat over S and regular if χ is odd. Moreover,

$$\mathscr{M}(2, \mathscr{O}_\mathscr{X}, \chi \neq 0)|_{x_0} \simeq \mathscr{M}(2,(a_1,a_2),\chi \neq 0).$$

Similarly, there exists a flat family

$$J(\mathscr{X}) \to S$$

of Jacobians, the fibres of $\mathscr{X} \to S$ of Euler characteristics $\chi - (1-g)$ and

$$J(\mathscr{X}|_{x_0}) \simeq J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2).$$

Using the results proved above we see that there is a natural morphism

$$(\det)/S : \mathscr{M}(2, \mathscr{O}_{X_1}, \chi \neq 0)/S \to J(\mathscr{X})/S$$

of S schemes such that

$$(\det)/S|_{x_0} : \mathscr{M}(2, \mathscr{O}_\mathscr{X}, \chi \neq 0)/S|_{x_0} \to J(\mathscr{X})/S|_{x_0}$$

is same as the morphism

$$(\det): \mathscr{M}(2,(a_1,a_2),\chi \neq 0) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2).$$

Remark 7.2. It follows from the above remark that if we have a family such that a smooth curve Y of genus g specializes to the curve X and a line bundle $\mathscr{L}$ (with $\chi(\mathscr{L}) \neq g-1$) on Y specializes to a line bundle L, then the moduli space of rank two vector bundles on Y with fixed determinant $\mathscr{L}$ specializes to the moduli space of rank two torsion free sheaves on X with fixed determinant L. Moreover, this specialization carries the nice properties of the specializations of moduli space of vector bundles (without fixing the determinant) on Y to the moduli space of torsion free sheaves (without fixing the determinant) on X.

8. Fixed determinant moduli space over an irreducible nodal curve

Throughout this chapter, we let X denote an *irreducible* projective nodal curve with exactly one node. Let p be the node of X and $m_{X,p}$, the maximal ideal of X at p.

The notion of a triple in this paper can be suitably modified so that giving a torsion free sheaf $\mathscr{F}$ on X is equivalent a couple $(V, \vec{\phi})$, where V is a vector bundle on the normalization $\tilde{X}$ and $\vec{\phi}: V(p_1) \to V(p_2)$ is a homomorphism of the vector spaces, and $p_1, p_2 \in \tilde{X}$ are the points above the node $p \in X$ and $V(p_1)$, $V(p_2)$ are fibres of V.

Remark 8.1. Let $\mathscr{F}$ be a torsion free, but not free sheaf of rank n and degree d on X and let $(V, \vec{\phi})$ be the associated couple. Note that in this case $\det(\phi) = 0$.

(a) If $\operatorname{rank}(\vec{\phi}) = n-1$ (i.e., locally at p, $\mathscr{F}$ is of the form $(\mathscr{O}_{X,p}^{n-1} \oplus m_{X,p})$), then it is easy to see that

$$\overset{n}{\wedge}\mathscr{F}/(\text{torsion}) \simeq L_0$$

where L_0 is the torsion free (but not free) sheaf of rank one of degree d on X associated to the couple $(\overset{n}{\wedge}V, \vec{0})$.

(b) If $\operatorname{rank}(\vec{\phi}) < n-1$ (i.e., locally at p, $\mathscr{F}$ is of the form $(\mathscr{O}_{X,p}^{n-r} \oplus m_{X,p}^r)$, $(r \geqslant 2)$), then it is easy to see that for every linear map $\lambda: \overset{n}{\wedge}V(p_1) \to \overset{n}{\wedge}V(p_2)$ we have

$$\overset{n}{\wedge}\mathscr{F}/(\text{torsion}) \subset L_\lambda,$$

where L_λ is the torsion free sheaf of rank one and degree d on X associated to the couple $(\overset{n}{\wedge}V, \vec{\lambda})$.

Let Y be a non-singular projective curve which specializes to X and $n \geq 1$ be an integer. Let $M(Y,n,d)$ (resp. $M(X,n,d)$) denote the moduli space of vector bundles (resp. torsion free sheaves) of rank n and degree d on Y(resp. X) (see [8, 11]). One knows that $M(Y,n,d)$ specializes to $M(X,n,d)$ (for example this follows from the general theory of [13]). If $\mathscr{F}$ is a vector bundle of rank n and degree d on Y (resp. X), then

$$\mathscr{F} \mapsto \det(\mathscr{F}) := \overset{n}{\wedge}\mathscr{F}$$

induces a morphism $\det: M(Y,n,d) \to M(Y,1,d) =: J^d(Y)$ (resp. $\det: M(X,n,d)^0 \to M(X,1,d)^0 =: J^d(X)$, where $M(X,n,d)^0$ denotes the moduli space of vector bundles of rank n and degree d on X, which is an open subvariety of $M(X,n,d)$). For $\mathscr{L} \in J^d(Y)$, set

$$M(Y,n,\mathscr{L}) = \det^{-1}(\mathscr{L})$$

and for $L \in J^d(X)$, set

$$M(X,n,L)^0 = \det^{-1}(L)$$

DEFINITION 8.1

Let L be a torsion free sheaf of rank one and degree d on X. Define $M(X,n,L)$ as the set

$$\{[\mathscr{F}] \in M(X,n,d) | (\overset{n}{\wedge}\mathscr{F})/(\text{Torsion}) \subset L \quad \text{and} \quad m_p^n L \subset (\overset{n}{\wedge}\mathscr{F})/(\text{Torsion})\},$$

where (Torsion) denotes the maximal subsheaf of modules of $\overset{n}{\wedge}\mathscr{F}$ with proper support and m_p is the ideal sheaf of the point $p \in X$.

Remark 8.2. In fact we can define a subfunctor of the moduli functor so that this subfunctor is representable and is represented by a subscheme whose support is the set $M(X,n,L)$. Thus it is easy to see that $M(X,n,L)$ can be defined as a closed subscheme of $M(X,n,d)$.

Conjecture. (a) If L is a line bundle on X, then $M(X,n,L)$ is the closure of $M(X,n,L)^0$ in $M(X,n,d)$.
(b) Let $\mathscr{L}_0$ (resp. L_0) be a line bundle (resp. torsion free sheaf of rank one) of degree d on Y (resp. X). Assume that $\mathscr{L}_0$ specializes to L_0 as Y specializes to X. Then $M(X,n,L_0)$ is the specialization of $M(X,n,\mathscr{L}_0)$.

Below we provide some evidence to the conjecture.

Evidence. For simplicity, let n be equal to two. Let $\mathscr{F}$ be a rank two torsion free sheaf of degree d on X. If $\mathscr{F}$ is a vector bundle (resp. torsion free sheaf of type $(\mathscr{O}_{X,p} \oplus m_{X,p})$), then clearly $(\overset{n}{\wedge}\mathscr{F})$(resp. $(\overset{n}{\wedge}\mathscr{F})/(\text{torsion})$) is a line bundle (resp. torsion free sheaf of rank one) of degree d on X. If $\mathscr{F}$ is a torsion free sheaf of type $(m_{X,p} \oplus m_{X,p})$ then $(\overset{n}{\wedge}\mathscr{F})/(\text{torsion})$ is a torsion free (but not free) sheaf of rank one and degree $d-1$. Thus for any torsion free sheaf L of rank one and degree d such that $(\overset{2}{\wedge}\mathscr{F})/(\text{torsion}) \subset L$, we see $\mathscr{F} \in M(X,n,L)$: let $\mathscr{F}$ be a torsion free sheaf of type $m_{X,p} \oplus m_{X,p}$ of degree d and let L be a torsion free sheaf of rank one and degree d such that $(\overset{2}{\wedge}\mathscr{F})/(\text{torsion}) \subset L$. Let R be a d.v.r. and let $\mathbf{F}$ be a flat family of torsion free sheaves of rank two and degree d on X parametrized by $S = \text{Spec}(R)$. Assume that for the generic point of $\eta \in S$, $(\overset{2}{\wedge}\mathbf{F}|_{\eta \times X}) \simeq L \otimes Q(R)$ and for the closed point $s \in S$, $\mathbf{F}|_{s \times X} = \mathscr{F}$ where $Q(R)$ is the quotient field of R. Now if $\mathbf{L} = (\overset{2}{\wedge}\mathbf{F})/(\text{Torsion})$, where (Torsion) denotes the maximal subsheaf of modules of $\overset{2}{\wedge}\mathbf{F}$ with proper support, then $\mathbf{L}$ is a flat family of sheaves of rank one and degree d on X and there is an injective homomorphism

$$\mathbf{L} \to \pi_2^*(L),$$

where $\pi_2 : S \times X \to X$ is the second projection. Since the surjection $\mathbf{L}|_{s \times X} \to (\overset{2}{\wedge}\mathscr{F})/(\text{torsion})$ induces an isomorphism

$$(\mathbf{L}|_{s \times X})/(\text{torsion}) \simeq (\overset{2}{\wedge}\mathscr{F})/(\text{torsion}),$$

it follows that $\mathscr{F} \in M(X,n,L)$, QED.

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Generic hypersurface singularities

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Abstract. The problem considered here can be viewed as the analogue in higher dimensions of the one variable polynomial interpolation of Lagrange and Newton. Let $x_1, \ldots, x_r$ be closed points in general position in projective space Pn, then the linear subspace V of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ (the space of homogeneous polynomials of degree d on $\mathbb{P}^n$) formed by those polynomials which are singular at each x_i, is given by $r(n+1)$ linear equations in the coefficients, expressing the fact that the polynomial vanishes with its first derivatives at $x_1, \ldots, x_r$. As such, the "expected" value for the dimension of V is $\max(0, h^0(\mathcal{O}(d)) - r(n+1))$. We prove that V has the "expected" dimension for $d \geqslant 5$ (theorem A). This theorem was first proven in [A] using a very complicated induction with many initial cases. Here we give a greatly simplified proof using techniques developed by the authors while treating the corresponding problem in lower degrees.

Keywords. Waving problem; multiple points; interpolation; generic singularities.

1. Introduction

Theorem A. *Let $x_1, \ldots, x_r$ be closed points in general position in n dimensional projective space $\mathbb{P}^n$ over an algebraically closed field k. Then the vector subspace of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ of homogeneous polynomials of degree $d \geqslant 5$ on $\mathbb{P}^n$ having singularites at each x_i has the "expected" dimension* $\max(0, h^0(\mathcal{O}(d)) - r(n+1))$.

We will give a rough outline of the proof. By a standard argument we reduce to a proposition (2.6) concerning unions of double points (i.e. first infinitesimal neighbourhoods of closed points in $\mathbb{P}^n$). This is done in § 2. We attempt to prove this proposition by specializing a number of the double points to have support in a hyperplane H. Each of the specialized double points then induces a double point in H (its trace on H) and has a residual simple point. If the numbers fit (see 'adjusting' 3.2) we obtain a (lower dimensional) trace on H of the kind treated by our proposition and we can then argue by induction.

Unfortunately these numbers do not always fit! Nonetheless, we are able to overcome the problem by using the differential lemma of [AH1] (see lemma 3.3). We begin as above by specializing a number of double points to have support in H. We put in just enough double points to have at least the required number of conditions in H. Any excess of conditions in H, is then considered as coming from the trace of a single double point Z with support in H. The lemma then allows us to use the induction hypothesis (applied in H) to reduce to a residual problem (see degue in 3.3) in degree one less, involving a subscheme $d'Z$ of $Z \cap H$. This $d'Z$ depends on the union of all the residual points of the other double points we specialized to H and although it is well determined (see 3.4 (2)), it is very difficult to locate. In [AH1] we opted for a strategy of specialization, allowing us to identify $d'Z$ at the cost of a more complicated argument. Here, we are able to proceed without identifying $d'Z$ (this means that the degue of 3.3 is true for all subschemes of $Z \cap H$ having the same length as $d'Z$). We do this in 5.1, by ejecting (see 3.2) $d'Z$ out of H and specializing

further double points to have support in H. This delays the influence of $d'Z$ to a later stage in the argument, when we dispose of many points in H on whose position, $d'Z$ does not depend. By specialization, we transform the union of $d'Z$ with the right number of simple points, into a double point of $\mathbb{P}^n$. This leads to a statement (6.1) in which there is no unknown component. The proof of 6.1, uses simpler techniques of specialization and ejection.

It is interesting to note that there are numerous examples in degrees $d \leqslant 5$ where theorem A fails. In fact for $d=2$ all values of r with $2 \leqslant r \leqslant n$ give such examples (see [A], § 4), while for $d=4$, there is an example for each $n=2$, 3, 4 (see [A], § 5, and [AH2]). The only example with $d=3$ is that of a hypercubic in $\mathbb{P}^4$ with seven generic singularities (see [A], § 5, [CH] and [AH3]). Of course, unlike the one variable case, the theorem fails miserably in higher dimensions if one abandons the general position hypothesis on the x_i.

From another point of view, one can consider the general problem of determining the dimension of the space of homogeneous polynomials of degree d, having singularites of multiplicity $\geqslant m_i$ at x_i for some fixed sequence $(m_1,\ldots,m_r)$. For $n=2$, this has been persued further in [H3], where the second author has formulated a conjecture giving a clear geometric meaning to those cases where the "expected" dimension is not realized. Eventually, thorem A should be completed by a similar interpretation of the examples in the preceeding paragraph.

2. A reformulation of the theorem

Throughout the rest of the article we work over a fixed algebraically closed base field k. We write $\mathbb{P}^n$ for projective n space over this field.

In this section we formulate clearly theorem A in terms of projective cohomology. Once this has been done we show that theorem A results from proposition 1.6 below. The remaining sections are devoted to proving 1.6.

Let $x_1,\ldots,x_r$ be closed points in general position in $\mathbb{P}^n$ and let Y be the union of the first infinitesimal neighbourhoods of the x_i $(i=1,\ldots,r)$. In any degree d, the homogeneous polynomials of degree d which are singular at each x_i are canonically identified, via the usual exact sequence

$$0 \to H^0(\mathbb{P}^n,\mathscr{I}_Y(d)) \to H^0(\mathbb{P}^n,\mathcal{O}(d)) \to H^0(\mathbb{P}^n,\mathcal{O}_Y(d)) \to H^1(\mathbb{P}^n,\mathscr{I}_Y(d)) \to 0$$

with $H^0(\mathbb{P}^n,\mathscr{I}_Y(d))$, where $\mathscr{I}_Y$ is the ideal sheaf of Y as a closed subscheme of $\mathbb{P}^n$.

To say that $H^0(\mathbb{P}^n,\mathscr{I}_Y(d))$ has the expected dimension is equivalent to saying that the canonical map $H^0(\mathbb{P}^n,\mathcal{O}(d)) \to H^0(\mathbb{P}^n,\mathcal{O}_Y(d))$ has maximal rank as a map of vector spaces or, what amounts to the same thing, that one or the other of the two numbers $h^0(\mathbb{P}^n,\mathscr{I}_Y(d))$ or $h^0(\mathbb{P}^n,\mathscr{I}_Y(d))$ is zero. This motivates the following definition.

DEFINITION 2.1

Let Y be a closed subscheme of $\mathbb{P}^n$. We say that Y has maximal rank in degree d if the canonical map $H^0(\mathbb{P}^n,\mathcal{O}(d)) \to H^0(\mathbb{P}^n,\mathcal{O}_Y(d))$ has maximal rank as a map of vector spaces. We say that Y has maximal rank if it has maximal rank in all degrees $d \geqslant 0$.

We say that Y is numerically adjusted (resp. adjusted) in degree d if $\chi(\mathbb{P}^n,\mathscr{I}_Y(d))=0$ (resp. $h^i(\mathbb{P}^n,\mathscr{I}_Y(d))=0$ for $i \geqslant 0$).

Clearly if Y is adjusted in degree d then it is both numerically adjusted and of aximal rank in degree d.

The following proposition and its corollary show how these various notions are lated for zero dimensional subschemes.

ROPOSITION 1

et $Y' \subset Y \subset Y''$ be zero dimensional closed subschemes of $\mathbb{P}^n$. Fix $d \geqslant 0$ and consider ie following two maps

) $H^0(\mathbb{P}^n, \mathcal{O}(d')) \to H^0(\mathbb{P}^n, \mathcal{O}_{Y'}(d'))$; $d' \geqslant 0$,
i) $H^0(\mathbb{P}^n, \mathcal{O}(d'')) \to H^0(\mathbb{P}^n, \mathcal{O}_{Y''}(d''))$; $d'' \geqslant 0$.

hen we have (a) *If* (i) *is injective, then $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_Y(d))$ is injective for $\leqslant d'$.* (b) *If* (ii) *is surjective, then $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_Y(d))$ is surjective for $\geqslant d''$.* (c) *If* (i) *is injective for $d' = d$ and* (ii) *is surjective for $d'' = d + 1$, then Y has aximal rank.*

roof. See [H1]. □

OROLLARY 2.2

Y is a zero dimensional closed subscheme of $\mathbb{P}^n$ which is adjusted in degree d, then Y has aximal rank.

roof. Put $Y' = Y = Y''$ in the proposition. □

We will apply 2.2 and 2.3 in the following way. We first define as follows.

EFINITION 2.3

or $n, d \geqslant 1$ let $A_{n,d}$, $B_{n,d}$ be the integers defined by

$$\binom{n+d}{d} = (n+1)A_{n,d} + B_{n,d}$$

here $0 \leqslant B_{n,d} \leqslant n$. Let x_i $(i = 1, \ldots, A_{n,d})$ be generic closed points of $\mathbb{P}^n$ and let $(y_{n,d}, L_{n,d})$ e the generic couple formed by a closed point $y_{n,d}$ and a linear subspace $L_{n,d}$ of imension $B_{n,d} - 1$ incident with $y_{n,d}$. We define $Y_{n,d}$ to be the union of the first nfinitesimal neighbourhoods of $x_1, \ldots, x_{A_{n,d}}$ in $\mathbb{P}^n$ and the first infinitesimal neighbourood of $y_{n,d}$ in $L_{n,d}$.

Ve now have the following Lemma.

emma 2.4 *With the notation of* 2.4, $Y_{n,d}$ *is numerically adjusted in degree d and for any $\geqslant 0$, there is a uniquely determined $\delta \geqslant 0$ such that the generic union Y of r first nfinitesimal neighbourhoods of $\mathbb{P}^n$ is inserted in a sequence of inclusions $Y_{n,\delta} \subset Y \subset Y_{n,\delta+1}$. If $Y_{n,\delta}$ and $Y_{n,\delta+1}$ are adjusted then Y has maximal rank.*

Proof. This is a purely numerical consequence of the definitions and a simple application of 2.2 (c). □

We now show that the following proposition implies theorem A.

PROPOSITION 2

With the notation of 2.4, $Y_{n,d}$ *is adjusted in degree* d *in* $\mathbb{P}^n$ *for* $d \geq 5$.

Proof of theorem A. Let $x_1,\ldots,x_r$ be closed points in general position in projective space $\mathbb{P}^n$ and let Y be the union of the first infinitesimal neighborhoods of the x_i. Let δ be the uniquely determined integer such that $A_{n,\delta} \leqslant r < A_{n,\delta+1}$. If $d \geqslant 5$, then Y has maximal rank by 2.5 and 2.6. If $\delta < 5$, then $Y \subset Y_{n,5}$ so that Y has maximal rank in all degrees $d \geqslant 5$ by 2.2 (b) and 2.6. □

3. Preliminaries

Generally speaking the proof of 2.6 goes as follows (see [H1] for the general setting). We argue by induction on the dimension n and the degree d using the following 'lemma-definition'.

Lemma-definition 3.1. *Let* Y *be a closed subscheme of* $\mathbb{P}^n$ *numerically adjusted in degree* d *and let* H *be a hyperplane of* $\mathbb{P}^n$. *We put* $Y'' = Y \cap H$, *which we call the trace of* Y *on* H *and we let* Y' *be the closed subscheme of* $\mathbb{P}^n$ *defined by the following canonical exact sequence*

$$0 \to I_{Y'}(-1) \to I_Y \to I_{Y'',H} \to 0, \tag{1}$$

where I_Y *is the ideal of* Y *in* $\mathbb{P}^n$ and $I_{Y'',H}$, *is the ideal of* Y'' *in* H. *We say that* Y' *is the residual of* Y *with respect to* H. *In view of definition* 1.1, *it follows immediately that* Y *is adjusted in* $\mathbb{P}^n$ *in degree* d *if the following two conditions are verified:* (i) Y'' *is adjusted in* $H = \mathbb{P}^{n-1}$ *in degree* d, (ii) Y' *is adjusted in* $\mathbb{P}^n$ *in degree* $d-1$.

This allows us to resolve the proposition "Y is adjusted in $\mathbb{P}^n$ in degree d", into two similar statements, one of which is in lower degree and the other in lower dimension. Note that to exploit 3.3 in this way, it is essential to deal only with (numerically) adjusted subschemes!

3.1 *Adjusting*

Now in practice, Y is the generic member of a nice smooth irreducible flat family T. In this case we can use the upper semicontinuity of the functions $t \mapsto h^i(\mathbb{P}^n, I_{Y_t}(d))$, $t \in T$, to conclude that Y is adjusted in $\mathbb{P}^n$ in degree d if there is some point $t \in T$ where Y_t has this property (in keeping with common usage we call Y_t a specialization of Y). This is the essential element which allows us to employ 3.1. Given a familly T of closed subschemes of $\mathbb{P}^n$ numerically adjusted in degree d, we find a specialization Y_t and a hyperplane H such that (i) $Y_t'' = Y_t \cap H$ is numerically adjusted in H in degree d. By the additivity of the Euler characteristic, this implies that (ii) Y_t' is numerically adjusted in $\mathbb{P}^n$ in degree $d-1$. This necessity to adjust the Euler characteristic of the trace, is the major technical obstacle in the proof. Insufficient techniques for doing this was the source of formidible technical difficulties in the proof of theorem A given in [A]. The big advance is the differential lemma [AH1] which is given in a simplified form in 3.3.

3.2 *Ejecting*

One technique for adjusting the Euler characteristic of the trace is what we call ejecting. Fix a hyperplane H in $\mathbb{P}^n$ and a point x in some linear subspace L of H. Let X be the first infinitesimal neighbourhood of x in L and let Z be the first infinitesimal neighbourhood in $\mathbb{P}^n$ of a general point z on a line D through x, with D transverse to H. Let R be the unique, closed subscheme of $\mathbb{P}^n$ obtained as the flat limit of $Z \cup X$, as z is specialized to x along D. The process which consists in replacing $Z \cup X$ by the specialization R, will be referred to as ejecting X by Z. The sense of this terminology resides in the fact that the trace of R on H is the first infinitesimal neighbourhood of x in H, while the residual of R with respect to H is the first infinitesimal neighbourhood of x in the linear subspace spanned by $L \cup D$–loosely speaking, in the limit, X is ejected outside of H and replaced by Z.

3.3 *Extending*

Lemma 3.2. (*see* [AH2] 1.5) *Let Y, Z be a closed subscheme of $\mathbb{P}^n$. We let $Y \cap Z$ and $Y \cup Z$ denote respectively the scheme theoretic intersection and scheme theoretic union of Y and Z. If $h^i(Z, \mathscr{T}_{Y \cap Z, Z}(d)) = 0$ pour $i \geqslant 0$, then Y is adjusted in degree d in P^n if and only if $Y \cap Z$ is adjusted in $\mathbb{P}^n$ in degree d.*

Proof. An immediate consequence of the following, canonical, exact sequence of ideals

$$0 \to \mathscr{T}_{Y \cup Z} \to \mathscr{T}_Y \to \mathscr{T}_{Y \cap Z, Z} \to 0. \tag{2}$$

The process, in 2.4.1, which consists in replacing Y by $Y \cup Z$, will be referred to as extending Y by Z.

3.4 *The differentielle method*

In §4, we will use the following lemma [AH1, 1.3, 1.5].

Lemma 3.3. *We consider a closed subscheme of $\mathbb{P}^n$, numerically adjusted in degree d which is the disjoint union of two closed subschemes W, Z, with Z of finite support. We suppose that the residual Z' of Z with respect to a fixed hyperplane H is contained in the trace Z'' of Z on H. Then the union Y of W with the generic translate of Z in $\mathbb{P}^n$ is adjusted in $\mathbb{P}^n$ in degree d if the following two conditions are verified:*

(dime) *There is a subscheme $d''Z$ of Z'' containing Z' such that $W'' \cup d''Z$ is adjusted in H, in degree d.*

(degue) *There is a closed subscheme $d'Z$ contained in the intersection of the base of the linear system $|H^0(H, \mathscr{T}_{W'' \cup Z', H}(d))|$ with Z'', such that $W' \cup d'Z$ is adjusted in $\mathbb{P}^n$, in degree $d-1$.*

Remark 3.4. 1) The words dime and degue indicate that the corresponding conditions are respectively in lower dimension and lower degree. 2) When Z is the first infinitesimal neighbourhood in $\mathbb{P}^n$ of a closed point in H with

$$\chi(H, \mathscr{T}_{W'' \cup Z', H}(d)) > 0, \quad \chi(H, \mathscr{T}_{W'' \cup Z'', H}(d)) < 0$$

it is easy to see that, if $d''Z$ exists verifying dime, then the intersection $\mathscr{B}$ of the base of the linear system $|H^0(H, \mathscr{T}_{W'' \cap Z', H} \otimes \mathscr{L}_H)|$ with Z'', has degree $\deg(Z) - \deg(d''Z)$. Now this is just the condition that $W' \cup \mathscr{B}$ be numerically adjusted in $\mathbb{P}^n$ in degree $d-1$,

so that $\mathscr{B} = d'Z$ is the unique choice for $d'Z$ with $W' \cup d'Z$ numerically adjusted in $\mathbb{P}^n$ in degree $d-1$. What is more, this $d'Z$ depends only on W'' and the point Z'. That $\mathscr{B}$ have degree $\deg(Z) - \deg(d''Z)$, just says that locally at the point Z', the base of the linear system $|H^0(H, \mathscr{I}_{W'' \cup Z', H} \otimes \mathscr{L}_H)|$ is smooth of codimension $\deg(d''Z)$ in H. As such $\mathscr{B}$ is just the intersection of Z'' with some linear subspace L through Z', with L transverse to $d''Z$ and of dimension $n - \deg(d''Z)$. In particular, $\mathscr{B}$ is the first infinitesimal neighbourhood of Z' in L.

4. Proof of 2.6

4.1 *Proof of* 2.6

For $d \geqslant 5$, $Y_{n,d}$ is adjusted in $\mathbb{P}^n$ in degree d.

Proof. We proceed by induction on n, the case $n = 1$ being the classical one variable interpolation.

Let H be the generic hyperplane of $\mathbb{P}^n$ ($n \geqslant 2$). There are two cases to consider in accordance with $B_{n-1,d} = 0$ and $B_{n-1,d} > 0$.

Case $B_{n-1,d} = 0$. In this case we let $S_{n,d}$ be the specialization of $Y_{n,d}$ obtained by specializing the points x_i ($i = 1, \ldots, A_{n-1,d}$) into H. Then the trace $S''_{n,d}$ of $S_{n,d}$ on H is just $Y_{n-1,d}$. By the induction hypothesis $Y_{n-1,d}$ is adjusted in degree d, so by 3.1 it is enough to show that the residual $S'_{n,d}$ of $S_{n,d}$ with respect to H, is adjusted in degree $d-1$.

Case $B_{n-1,d} > 0$. In this case we let $S_{n,d}$ be the specialization of $Y_{n,d}$, obtained by specializing the points x_i ($i = 1, \ldots, A_{n-1,d+1}$) into H. We apply 2.5.1 with Z the first infinitesimal neighbourhood of x_1 and $d''Z$ the intersection of Z with the generic linear subspace of H passing through x_1 of dimension $B_{n-1,d-1}$. The subscheme $(S_{n,d} - Z)'' \cup d''Z$ is just $Y_{n-1,d}$ so that the dime of 2.5.1 results from the induction hypothesis. As indicated in 2.5.2 (2), there is a unique choice for $d'Z$ verifying the numerical part of the condition degue of 4.1, we must show that the union of $(S_{n,d} - Z)'$ and $d'Z$ is adjusted in degree $d-1$.

Now we define $G_{n,d}$ to be $S'_{n,d}$ if $B_{n-1,d} = 0$ (resp. $(S_{n,d} - Z)'$ if $B_{n-1,d} > 0$). This is just the generic union of $A_{n,d} - A_{n-1,d}$ (resp. $A_{n,d} - A_{n-1,d-1}$) double points in $\mathbb{P}^n$ with $A_{n-1,d}$ points in H. To complete the two cases it will be enough to prove that if Z is a generic double point of H and Z_0 is any subscheme of Z such that $T_{n,d} = G'_{n,d} Z_0$ is numerically adjusted in degree $d-1$, then $T_{n,d}$ is adjusted in degree $d-1$. This will be proven in 4. This result is more general than the degue, but, as we do not have any useful information about $d'Z$, we are obliged, in practice, to prove the more general result. □

5. The Degue

In the proof of the following proposition we simply show how the proposition can be deduced from results in lower degree. These results in lower degree are then proven in 6.

PROPOSITION 3

For $d \geqslant 5$, *let* $G_{n,d}$ *be the generic union of* $A_{n,d} - A_{n-1,d} - \varepsilon$ *double points of* $\mathbb{P}^n$ *with* $A_{n-1,d}$ *points in* H, *where* $\varepsilon = 0$ *if* $B_{n-1,d} = 0$ *and* $\varepsilon = 1$ *if* $Bn - 1, d > 0$. *Let* Z *be a generic*

double point of H and let Z_0 be any subscheme of Z such that $T_{n,d} = G_{n,d} \cup Z_0$ is numerically adjusted in degree $d-1$, then $T_{n,d}$ is adjusted in degree $d-1$.

Proof. If $n = 1$ this follows by classical one variable interpolation. We will now treat the cases $n = 2$ and $n \geqslant 2$ separately.

Case (1). $n = 2$. In this case we have $B_{2,d} = 0$ or 1, since $(d+2)(d+1)/2 \not\equiv 0$ or 1 (mod 2). If $d = 5$, then $B_{1,5} = 0 = B_{2,5}$ and $T_{2,5}$ is adjusted by 5.1. If $d = 6$ then $T_{2,6}$ is the union of five first infinitesimal neighbourhoods and one closed point in $\mathbb{P}^2$ with three closed points and one infinitesimal neighbourhood in $H = \mathbb{P}^1$. By ejecting one closed point in H with one infinitesimal neighbourhood of $\mathbb{P}^2$, the resulting specialization S has an adjusted trace on $H = \mathbb{P}^1$. The residual S' of S with respect to H, can be specialized to the generic union of four infinitesimal neighbourhoods of $\mathbb{P}^2$ with one closed point in H and one degree two closed subscheme of $\mathbb{P}^2$, transverse to H with support a closed point of H. It follows from 5.1 that S' is adjusted in $\mathbb{P}^2$ in degree $d = 4$.

Lemma 3.1. *Let s,t be the unique integers satisfying*

$$2s - t = d - (d+1)/2 + B_{1,d}/2 - B_{2,d} - B_{1,d}(3 - B_{1,d}); \quad 0 \leqslant t \leqslant 1.$$

Then for $d \geqslant 7$, we have $(s-t) \geqslant 1$ et $s \leqslant d/2 + 1/2$.

Proof of lemma. If $d = 7$ then $B_{1,d} = 0 = B_{2,d}$ and $s = 2, t = 1$. If $d = 8$, we have $B_{1,d} = 1$, $B_{2,d} = 0$ giving $s = 1$, $t = 0$. So we have the result in these cases. For $d \geqslant 9$, we consider the expression

$$2(s-t) = d/2 - 1/2 + B_{1,d}/2 - B_{1,d}(3 - B_{1,d}) - B_{2,d} - t. \tag{3}$$

Noting that $B_{1,d}$, $B_{2,d}$ and t are all less than or equal to one, we find

$$2(s-t) \geqslant 1/2 > 0$$

which gives the first part of the lemma. The second part is trivial. □

Now suppose that $d \geqslant 7$ and let s,t be as in lemma 5.1.1. Let S be the specialization of $T_{2,d}$ obtained by specializing $(s-t) \geqslant 0$ first infinitesimal neighbourhoods to have support in H, ejecting $t \leqslant 1$ of the $A_{1,d} \geqslant 1$ closed points of H with a further t infinitesimal neighbourhoods and finally specializing $y_{2,d}$ to H in the case $B_{2,d} = 1$. By (5.1.1), the trace of S on H is numerically adjusted in degree d in $H = \mathbb{P}^1$, hence it is adjusted. Now the residual S' of S with respect to H is the numerically adjusted, generic union of a number of infinitesimal neghbourhoods of $\mathbb{P}^2$ with $(s-t) \geqslant 1$ closed points in H and t finite degree two closed subschemes transverse to H with support a closed point of H. Since, by 5.1.1, $s \leqslant d/2 + 1/2$, we can conclude by 6.1, that S' is adjusted in $\mathbb{P}^2$ in degree $d-2$. This gives the result by 3.1.

Case (2). $n \geqslant 3$. The scheme $T_{n,d}$ contains the subscheme $d'Z$ which depends on the $A_{n-1,d}$ closed points in the trace $T'_{n,d}$ (see 3.4(b)). With $n \geqslant 3$ we have very little information about the linear subspace L spanned by $d'Z$ other than the fact that it is transverse to $d''Z$. Any specialization of $T_{n,d}$ which modifies the closed points in H will modify the subscheme $d'Z$. We avoid this unknown quantity by ejecting $d'Z$ by one of

the first infinitesimal neighbourhoods in $T_{n,d}$. We then specialize further first infinitesimal neighbourhoods to H, ejecting a number of closed points, in such a way that the trace of this specialization is numerically adjusted in H in degree $d-1$. In this way $d'Z$ does not appear in the trace and, when it appears in the residual, it no longer depends on the closed points in H.

As usual we begin with an accounting lemma.

Lemma 3.2. *For $n \geqslant 3, d \geqslant 5$, let s,t be the integers determined by the expression*

$$ns - t = \binom{n+d-2}{d-1} - A_{n-1,d} - B_{n,d} - n\delta_{n-1,d}; \quad 0 \leqslant t < n, \tag{4}$$

where $\delta_{n-1,d} = 0$ if $B_{n-1,d} = 0$ and $\delta_{n-1,d} = 1$ otherwise. Then we have the following inequalities:

(a) $s - t \geqslant (n-1) + \delta_{n-1,d} B_{n-1,d} - 1$,
(b) $s \geqslant n(n-1)/2 + \delta_{n-1,d} B_{n-1,d} - 1$

or one of the following

$$n = 4, d = 5 \quad \text{where } s = 5, \quad t = 0, \quad B_{4,5} = 1, \quad B_{3,5} = 0,$$

$$n = 5, d = 5 \quad \text{where } s = 8, \quad t = 0, \quad B_{5,5} = 0,$$

$$n = 6, d = 5 \quad \text{where } s = 14, \quad t = 0, \quad B_{6,5} = 0,$$

(c) $\dfrac{1}{n}\binom{n+d-2}{d-1} + (n-1)/2 \geqslant s.$

Equally,

$$A_{n-1,d} \geqslant n(n-1)/2 + d_{n,d}(n - B_{n,d}).$$

Proof of the lemma. We have $A_{n-1,d} = \dfrac{1}{n}\binom{n+d-1}{d} - B_{n-1,d}/n$, so that

$$ns - t = \frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1} - B_{n,d} - n\delta_{n-1,d} + \frac{B_{n-1,d}}{n}. \tag{5}$$

Since $t \leqslant (n-1)$, we see that (a) is implied by (b) for $n \geqslant 4$. We will prove (a) for $n = 3$. From (5) we get

$$3(s-t) = \frac{(d+1)(d-1)}{3} - B_{3,d} - 3\delta_{2,d} + \frac{B_{2,d}}{3} - 2t$$

so that (a) is equivalent, in its strict inequality form, to

$$\begin{aligned} \frac{(d+1)(d-1)}{3} &> 3(1 + \delta_{2,d} B_{2,d-1}) + B_{3,d} + 3\delta_{2,d} - \frac{B_{2,d}}{3} + 2t \\ &= 3 + B_{3,d} + (3\delta_{2,d} - 1/3) B_{2,d} + 2t. \end{aligned} \tag{6}$$

Using the inequalities $t \leqslant 2$, $B_{3,d} \leqslant 3$, $B_{2,d} \leqslant 2$ we see that it is enough to show that

$$(d+1)(d-1)/3 > 46/3.$$

This is the case for $d \geqslant 7$. For $d = 5$ we have $\delta_{2,5} = B_{2,5} = B_{3,5} = 0$ and $s = 3$, $t = 1$. For $d = 6$ we have $\delta_{2,6} = B_{2,6} = 1, B_{3,6} = 0$ and $s = 3$, $t = 0$. This gives (a) for $n = 3$ as well.

We will now prove (b). By (5) we have

$$ns = \frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1} - B_{n,d} - n\delta_{n-1,d} + \frac{B_{n-1,d}}{n} + t \tag{7}$$

so that (b) is equivalent to

$$\begin{aligned}\frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1} &> \frac{n^2(n-1)}{2} + n\delta_{n-1,d}(B_{n-1,d}-1) - n \\ &\quad + B_{n,d} - \frac{B_{n-1,d}}{n} + n\delta_{n-1,d} - t \\ &= \frac{n^2(n-1)}{2} + B_{n,d} + \left(n\delta_{n-1,d} - \frac{1}{n}\right)B_{n-1,d} - n - t.\end{aligned}$$

Using the upper bounds for $B_{n,d}$, $B_{n-1,d}$ one easily sees that it is sufficient to prove the following inequality

$$\frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1} > \frac{n^2(n-1)}{2} + \frac{(n+1)(n-1)^2}{n}.$$

Simplifying, this becomes

$$(d-1)\binom{n+d-2}{d} > \frac{1}{2}(n-1)(n^3+2n^2-2) \tag{8}$$

and one easily verifies (8) for $d = 7$, $n \geqslant 3$; $d = 6$, $n \geqslant 6$ and $d = 5$, $n \geqslant 11$. Now using the fact that the left hand side of (8) is an increasing function of d, we conclude that to prove (b) it is enough to treat the particular cases $d = 6$, $n = 3,4$; $d = 5$, $n = 3, \ldots, 10$.

For $d = 6$ we have for $n = 3$ (resp. $n = 4$) $s = 3$, $t = 0$ (resp. $s = 9$, $t = 1$) giving (b) in this case. For $d = 5$ we have the following triples (n,s,t):(3,3,1),(4,5,0),(5,8,0),(6,14,0), (7,21,4),(8,29,1),(9,39,1),(10,51,5), giving (b) in these cases as well.

One easily proves (c) using the expression (7).

Finally to prove the last part of the lemma we have

$$A_{n-1,d} = \frac{1}{n}\binom{n+d-1}{d} - B_{n-1,d}/n$$

so we must show that

$$\frac{1}{n}\binom{n+d-1}{d} > \frac{n(n-1)}{2} + \delta_{n,d}(n - B_{n,d}) + \frac{B_{n-1,d}}{n} - 1.$$

Using the upper bound for $B_{n-1,d}$ and the lower bound $1 \leqslant B_{n,d}$ for $\delta_{n,d} \neq 0$, we see that, since the left hand side is an increasing function of n, it is enough to show that

$$\binom{n+4}{5} > \frac{n^2(n-1)}{2} + n(n-1)$$

or that

$$(n+4)(n+3)(n+1) > 60(n-1).$$

One easily verifies that this holds for $n \geqslant 3$. □

Now let S be the specialization of $T_{n,d}$ obtained by specializing $(s-t)$ first infinitesimal neighbourhoods to have support in H, specializing $(y_{n,d}, L_{n,d})$ into H, then ejecting $t \leqslant n-1$ closed points of H by a further t infinitesimal neighbourhoods (one easily verifies that one has enough). In the case where $d'Z$ is non-empty we eject $d'Z$ by yet another infinitesimal neighbourhood. By (4) the trace S'' of S on H is numerically adjusted in degree $d-1$. We will apply lemma 3.1 to S.

We first show that S'' is adjusted in degree $d-1$ in $H = \mathbb{P}^{n-1}$. For this it is enough to show that S'' admits a specialization to a closed subscheme of H of the type figuring in 6.1.

If $B_{n,d} = 0$, then S'' is a generic union of closed points and first infinitesimal neighbourhoods of closed points of H. If $B_{n,d} \neq 0$, then we can specialize $\delta_{n,d}(n - B_{n,d})$ of the closed points in H to the point $y_{n,d}$, completing the first infinitesimal neighbourhood of $y_{n,d}$ in $L_{n,d}$ to a first infinitesimal neighbourhood of $y_{n,d}$ in H. This means that S''admits a specialization which is a generic union of first infinitesimal neighbourhoods of closed points in H and a number c of closed points of H where

$$c = A_{n-1,d} - \delta_{n,d}(n - B_{n,d}) - t \geqslant \frac{(n-1)(n-2)}{2}.$$

By specializing further closed points to infinitesimal neighbourhoods we obtain a specialization where the number c of closed points satisfies.

$$\frac{(n-1)(n-2)}{2} + (n-1) \geqslant c \geqslant \frac{(n-1)(n-2)}{2}$$

and we conclude that S'' admits a specialization to a closed subscheme of the type figuring in 6.1, so that S'' is adjusted in degree $d-1$.

We will now show that S' is adjusted in degree $d-2$ in $\mathbb{P}^n$. The residual S' of S with respect to H is the generic union, numerically adjusted in $\mathbb{P}^n$ in degree $d-2$, of first infinitesimal neighbourhoods of closed points in $\mathbb{P}^n$ with $t \leqslant n-1$ finite, degree two, closed subschemes with support in H, but transverse to H, $s-t$ closed points of H and res $(d'Z) \subset H$ (this is the residual of the ejected $d'Z$ which we consider as empty if $B_{n-1,d}$ is zero).

Now the subscheme res$(d'Z)$ is independent of the

$$s - t \geqslant (n-1) + \delta_{n-1,d}(B_{n-1,d} - 1)$$

closed points in H, so if res $(d'Z)$ is non-empty we can specialize $(B_{n-1,d}-1)$ of these closed points to $d'Z$ to make a first infinitesimal neighbourhood. The conditions of lemma 5.1.2, assure that this specialization of S' satisfies the hypotheses of 6.1, where the variable v is zero if $B_{n-1,d}$ is zero and one otherwise. As such S' is adjusted in $\mathbb{P}^n$ in degree $d-2$ and we have completed the proof of 5.1. □

6. Results and initial cases

PROPOSITION 4

Let $H = \mathbb{P}^{n-1}$ be a hyperplane of $\mathbb{P}^n$. In $\mathbb{P}^n$ we consider, for $d \geqslant 3$, the closed subscheme Y formed by the generic union, numerically adjusted in $\mathbb{P}^n$ in degree d, of

(a) *the first infinitesimal neighbourhoods of u closed points of* $\mathbb{P}^n$,
(b) *a closed points of* H,
(c) *b finite closed subschemes of degree two of* $\mathbb{P}^n$ *transverse to* H *with support a closed point of* H.
(d) *the first infinitesimal neighbourhood in* $\mathbb{P}^n$ *of* $v \leqslant 1$ *closed points of* H *where*

(i) $a \geqslant n-1$, *or* $d=3$ *and we have one of the following cases*
$n=4, a=5, b=0=v$; $n=5, a=8, b=0, v=1$; $n=6, a=14, b=0=v$
(ii) $\frac{1}{n}\binom{n+d}{d+1}+\frac{n-1}{2} \geqslant a+b \geqslant \frac{n(n-1)}{2}$
(iii) $0 \leqslant b < n$
(iv) *if* $n=2$ *and* $d=3$, *then* $v=0$.

Then Y *is adjusted in* $\mathbb{P}^n$ *in degree* d.

Proof. The proposition is true for $n=1$ by classical one variable interpolation. For $d=3$ the proposition results from 5 below. Henceforth we suppose that $n \geqslant 2$ and $d \geqslant 4$ and we argue by induction on n and d. Firstly we need the following accounting lemma.

Lemma 4.1. *With the same hypotheses as in* 6.1, *let* s,t *be the integers uniquely determined by the following conditions,*

$$sn-t=(n+d-1)!/(n-1)!d!-(a+b); 0 \leqslant t < n. \tag{9}$$

Then for $n \geqslant 2$, *we have* (a) $s-t+b \geqslant n-1$, (b) $s+b \geqslant (n(n-1))/2$, *or* $n=4$, $a=15$, $b=0$ *in which case* $t=0$ and $s=5$.

Proof of lemma. For (a) we have

$$s-t=\frac{1}{n}\binom{n+d-1}{d}-\frac{(a+b)}{n}-\frac{(n-1)t}{n}$$

and using (ii) of the proposition, we have

$$\begin{aligned} s-t &\geqslant \frac{1}{n}\binom{n+d-1}{d}-\frac{1}{n^2}\binom{n+d}{d+1}-\frac{n-1}{2n}-\frac{(n-1)t}{n} \\ &= \frac{(n-1)d}{n^2(d+1)}\binom{n+d-1}{d}-\frac{n-1}{2n}-\frac{(n-1)t}{n} \\ &= R(n,d,t). \end{aligned} \tag{10}$$

We want $(s-t) > (n-2)$. Since $t \leqslant (n-1)$ and $R(n,d,t)$ is a strictly increasing function of d, it is enough to show that $R(n,4,n-1) > (n-2)$, i.e.

$$\frac{4(n-1)}{5n^2}\binom{n+3}{4} > \frac{4n-7}{2}.$$

This is the case for $n=2$ or $n \geqslant 4$.

If $n=3$, then (10) becomes $s-t \geqslant (7-2t)/3 \geqslant 1$ and the last inequality is strict for $t<2$. When $t=2$, $s-t \geqslant 1$ with strict inequality if $s \geqslant 3$. Finally if $s=3$ and $t=2$,

we find by (9) that $(a+b)=8$. Using the fact that Y is adjusted in $\mathbb{P}^3$ in degree three, we find $b+8+4(u+v)=35$, so that $b\geqslant 3$ contradicting (iii) of the proposition. This gives (a).

For (b) we have

$$s=\frac{1}{n}\binom{n+d-1}{d}-\frac{a+b}{n}+\frac{t}{n} \tag{11}$$

and using (ii) of the proposition we find

$$\begin{aligned} s &\geqslant \frac{(n-1)d}{n^2(d+1)}\binom{n+d-1}{d}-\frac{n-1}{2n}+\frac{t}{n} \\ &\geqslant \frac{4(n-1)}{5n^2}\binom{n+3}{4}-\frac{n-1}{2n}+\frac{t}{n}. \end{aligned} \tag{12}$$

Since $t\geqslant 0$ and we need $s+b>\dfrac{n(n-1)}{2}-1$, it is enough to show that

$$\frac{4(n-1)}{5n^2}\binom{n+3}{4}>\frac{n(n-1)}{2}-1-\frac{n-1}{2n}.$$

This is the case for $n=2,3$ or $n\geqslant 5$.

For $n=4$, (12) becomes $s\geqslant(39+2t)/8$, so that $s>5$ for $t>0$. If $t=0$, the same expression gives $s\geqslant 5$. Finally if $s=5$ and $t=0$, then (11) gives $a+b=15$. If $b>0$, then $s+b\geqslant 6$ as required and we are left with the case $a=15$, $b=0=t$, $s=5$. This gives (b) using the fact that the right hand side of (12) is a strictly increasing function of d. □

Now to finish the proof of the proposition we use lemma 2.1. Let s,t be as in (9). We consider the specialization S of Y obtained by specializing $(s-t)-v(\geqslant 0$ by the lemma) of the u first infinitesimal neighbourhoods to have support in H and ejecting t of the $a\geqslant(n-1)$ points in H by a further t of the infinitesimal neighbourhoods. By (9), the trace S'' of S on H is numerically adjusted in H in degree d with

$$a'=a+b-t\geqslant\frac{(n-2)(n-1)}{2} \tag{13}$$

closed points contained in S''.

For $n\geqslant 2$, $S''\subset H=\mathbb{P}^{n-1}$ can be specialized to a subscheme of the same type as figures in the proposition. In fact, S'' is a generic union of closed points and first infinitesimal neighbourhoods of $\mathbb{P}^{n-1}$ and by specializing closed points to infinitesimal neighbourhoods, we can suppose that the number of closed points satisfies

$$\frac{(n-2)(n-1)}{2}+(n-1)\geqslant a'\geqslant\frac{(n-2)(n-1)}{2}. \tag{14}$$

One then verifies that

$$\binom{n+d-2}{d}+\frac{n-1}{2}\geqslant\frac{n(n-1)}{2}+\frac{n-1}{2}\geqslant\frac{(n-2)(n-1)}{2}+(n-1) \tag{15}$$

for $d\geqslant 2$. Now we conclude by the induction hypothesis that S'' is adjusted in degree d in $\mathbb{P}^{n-1}$.

The residual S' of S with respect to H is numerically adjusted in $\mathbb{P}^n$ in degree $d-1$. It the generic union of $(s-t+b)$ closed points in H, plus $t \leqslant (n-1)$ finite, degree two, losed subschemes of $\mathbb{P}^n$ with support in H, but transverse to H (these are the residuals f the ejections) and $(u-s)$ first infinitesimal neighbourhoods of closed points in $\mathbb{P}^n$. In iew of (9), (ii) and (iii) of the proposition and lemma 4.1, we have

$$(s-t+b) \geqslant (n-1)$$

$$\frac{1}{n}\binom{n+d-1}{d} + \frac{n-1}{2} \geqslant (s+b) \geqslant \frac{n-1}{2}$$

o that S' is of the type figuring in the theorem for $d \geqslant 4$. By the induction hypothesis we onclude that S' is adjusted in degree $d-1$ for $d \geqslant 4$.

The proposition now follows by 3.1. □

DEFINITION 6.1

Let $x_1, \ldots, x_{s+2}$ be closed points in $\mathbb{P}^n$ with $s \leqslant n-1$ and consider the union U of the $(s+2)(s+1))/2$ lines which join these points two by two. If H is a hyperplane in $\mathbb{P}^n$ not containing any of the points, then we say that the intersection $U \cap H$ is an -complex. The points of an s-complex span a linear subspace of dimension s and we vill admit the empty set as a (-1)-complex.

PROPOSITION 5

Let Y be the generic union, numerically adjusted in $\mathbb{P}^n$ in degree $d=3$, of

a) *the first infinitesimal neighbourhoods of u closed points of $\mathbb{P}^n$,*
b) *an $(l-2)$-complex,*
c) *a closed points of H,*
d) *b finite closed subschemes of degree two of $\mathbb{P}^n$* transverse to H with support a closed point of H,
e) *the first infinitesimal neighbourhood in $\mathbb{P}^n$ of $v \leqslant 1$ closed points of H, with $v=0$. If $n=2$, where*

(i) $a \geqslant n-1$,

(ii) $\binom{n+3}{4} + \frac{n-1}{2} \geqslant a+b \geqslant \frac{n(n-1)}{2}$ *or one of the following cases*

$n=4,\ a=5,\ b=0=v,\ l=-1,$

$n=5,\ a=8,\ b=0, v=1,\ l=-1,$

$n=6, a=14, b=0=v, l=-1,$

(iii) $0 \leqslant b < n$,

(iv) $-1 \leqslant l \leqslant (n+2)/2$.

Then Y is adjusted in $\mathbb{P}^n$ in degree $d=3$.

Proof. For $n=1$ this is just classical one variable interpolation, so from now on we suppose that $n \geqslant 2$ and we argue by induction on n using lemma 3.1. We begin with the following accounting lemma.

Lemma 5.1. *With the hypotheses of the proposition, let* $w = u + v$, *then there exists two integers* s, t *verifying the following conditions*

(α) $(w-s)n - t = \binom{n+2}{3} - (a+b) - \binom{s}{2}$,

(β) $0 \leqslant t \leqslant n$,

(γ) $1 \leqslant s \leqslant (n+1)/2$,

(δ) $(w-s) \geqslant (t+v)$.

Proof of lemma. We begin with the exceptional cases of 6.3(ii). For $n=4$, $a=5$, $b=0=v$, $l=-1$ we have $u=w=6$ and it is enough to take $s=2$ and $t=2$. For $n=5$, $a=8, b=0, v=1, l=-1$ we have $u=7$ and it is enough to take $s=3, t=1$. For $n=6$, $a=14, b=0=v, l=-1$ we have $u=w=10$ and it is enough to take $s=3, t=3$. Now, in the general case, we consider the following function of s,

$$F(s) = \frac{(n+2)(n+1)n}{6} - (w-s)n - (a+b) - \frac{s(s-1)}{2} \tag{16}$$

which is a decreasing function of s on the interval $[1,\ldots,n]$, with $0 \leqslant F(s+1) - F(s) = n - m \leqslant n-1$.

As such, there is an s verifying (α), (β) and (γ) if we know that the following two conditions are satisfied

(1) $F(1) \leqslant 0$,

(2) $F[\![(n+1)/2]\!] \geqslant -n+1$,

where $[\![\ ]\!]$ denotes the integer part. The fact that Y is adjusted in degree $d=3$ in $\mathbb{P}^n$, allows us to express w as

$$w = (n+3)(n+2)/6 - (a+b)/(n+1) - l(l-1)/2(n+1)$$

and this gives

$$F(s) = -n(n+2)/3 - (a+b)/(n+1) + nb/(n+1) + ns + nl(l-1)/2(n+1) - s(s-1)/2.$$

Now we have

$$(n+1)(F(1)-1) = -n(n+1)(n+2)/3 - (a+b) + nb + n(n+1) + nl(l-1)/2 - (n+1)$$

and using (ii) and (iv) of the proposition we find

$$\begin{aligned}(n+1)(F(1)-1) &\leq -n(n+1)(n+2)/3 - n(n-1)/2 + n(n-1) + n(n+1) \\ &\quad + n2(n+2)/8 - (n+1) \\ &= (-5n3 + 18n2 - 28n - 24)/24 \\ &< 0\end{aligned}$$

our $n \geqslant 2$. On the other hand,

$$\begin{aligned}(n+1)(F(n/2)+n) &= -n(n+1)(n+2)/3-(a+b)+bn+n^2(n+1)/2\\ &\quad +nl(l-1)/2-n(n+1)(n-2)/8+n(n+1)\\ &\geqslant -n(n+1)(n+2)/3-((n+3)(n+2)(n+1)/24\\ &\quad +(n-1))n^2(n+1)/2-n(n+1)(n-2)/8+n(n+1)\\ &= (9n^2-9n+6)/24\\ &> 0\end{aligned}$$

our $n \geqslant 1$.

Finally we must show that if s satisfies (α), (β) and (γ), then $(w-s) \geqslant (t+v)$. By (α), we ave, using (ii) and (iv) of the proposition

$$\begin{aligned}w-s &= \tfrac{1}{6}(n+2)(n+1)-\frac{1}{n}(a+b)-\frac{1}{2n}s(s-1)+\frac{t}{n}\\ &\geqslant \tfrac{1}{6}(n+2)(n+1)-\frac{1}{n}(1/24(n+3)(n+2)(n+1)+\tfrac{1}{2}(n-1))\\ &\quad -\frac{1}{8n}(n^2-1)+\frac{t}{n}\\ &= \tfrac{3}{4}(n+2)(n+1)(n-1)-\frac{n}{8}+\frac{1}{2n}+\frac{t}{n}\\ &= V(n)+\frac{t}{n}.\end{aligned}$$

Ve will show that $V(n)+t/n > t+v-1$. Since $(n-1) \geqslant t$, it will be enough to show that

$$W(n) = V(n)-((n-1)2/n)-1 > v.$$

For $n=2$, we have $v=0$ and $W(2)=1$. Now $W(n)=1/8(n^3+2n^2-10n+14)$ $-(1/2n)$, is a strictly increasing function of n for $n \geqslant 2$ so that, since $v \leqslant 1$, $W(n) > v$ for $\geqslant 2$. This completes the proof of the lemma. □

Now to complete the proof of the proposition we use lemma 2.1. Fix s, t verifying ne conditions of lemma 6.3.4. Let S be the specialization of Y obtained by specilizing $-s-t$ (which is $\geqslant 0$ by lemma 6.3.4 (d)) of the u first infinitesimal neighbourhoods to ave support in H, and ejecting t of the a closed points in H with a further t infinitesimal eighbourhoods. We will show that S is adjusted in $\mathbb{P}^n$ in degree $d=3$.

Let S_0 be obtained from S by extending the s infinitesimal neighbourhoods emaining outside of H by the union Z of the $s(s-1)/2$ lines which join them wo by two. Clearly $S \subset D$ is adjusted in each of the lines D contained in Z so, by emma 3.2, it is enough to show that S_0 is adjusted in $\mathbb{P}^n$ in degree d. We will apply emma 3.1 to S_0.

The trace S_0'' of S_0 on H, is the generic union of $u+v-s$ first infinitesimal eighbourhoods of closed points in H with

$$a' = a-t+b \geqslant (n-1)(n-2)/2$$

closed points and an $(s-2)$-complex. The scheme S_0'' is numerically adjusted in degree $d=3$ in $H=\mathbb{P}^{n-1}$ by (α) of lemma 6.3.4. By specializing closed points to first infinitesimal neighbourhoods we can suppose that

$$((n-1)(n-2)/2)+(n-1)\geqslant a'\geqslant(n-1)(n-2)/2. \tag{17}$$

Hence by (15) we conclude that S_0'' admits a specilization to a subscheme of H satisfying the conditions of the proposition in dimension $n-1$. Using the induction hypothesis we conclude that S_0'' is adjusted in H in degree $d=3$.

By lemma 3.1, the residual S_0' of S_0 with respect to H is numerically adjusted in $\mathbb{P}^n$ in degree $d=2$. It is the generic union of the first infinitesimal neighbourhoods of $1\leqslant s\leqslant(n+1)/2$ closed points of $\mathbb{P}^n$ and the $s(s-1)/2$ lines which join them two by two, with an $(l-2)$-complex, $u+v-s+b$ closed points in H and t finite, degree two, closed subschemes of $\mathbb{P}^n$ with support in H, but transverse to H. Let T be the specialization of S_0' obtained by specializing all but one of the infinitesimal neighbourhoods to have support in H and specializing all the degree two finite closed subschemes into H. By [AH2] (4.9) the trace T'' on H is adjusted in H in degree $d=2$ and it is clear that the residual T' of T with respect to H (which is the union of a first infinitesimal neighbourhood of a closed point of $\mathbb{P}^n$ with $(s-1)$ lines through the point) is adjusted in degree one in $\mathbb{P}^n$. Applying lemma 3.1 to T we conclude that S_0' is adjusted in degree $d=2$ in $\mathbb{P}^n$.

Now applying lemma 3.1 to S_0 we conclude that S_0 is adjusted in degree $d=3$ in P^n.

□

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Bredon cohomology of cyclic geometric realization of G-cyclic sets

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Abstract. We define equivariant cyclic and Hochschild cohomology modules of a cyclic object X in the category of G-sets and relate them with the Bredon cohomologies of the cyclic geometric realization $|X|^{cy}$.

Keywords. Cyclic cohomology; Hochschild cohomology; cyclic set; equivariant cohomology.

1. Introduction

Let G be a discrete group. A G-cyclic set is a cyclic object in the category $G\mathcal{S}$ of G-sets. More precisely, a G-cyclic set is a cyclic set $X = \{X_n; d_i, s_i, t_n\}$, where X_n is a G-set for each n and the maps d_i, s_i, t_n are G-equivariant. This may also be considered as a covariant functor $X: \Delta C^{op} \to G\mathcal{S}$, where ΔC is the cyclic category. Similarly, by a G-cyclic module, we shall mean a cyclic object in the category of G-modules over a commutative ring k with 1. If X is a G-cyclic set then the cyclic module $k[X]$, associated to the cyclic set X becomes in a natural way a G-cyclic module. For a cyclic set X, the geometric realization $|X|$ is an S^1-space and let $|X|^{cy} = ES^1 \times_{S^1} |X|$ be the cyclic geometric realization of X. Then a well-known result in cyclic cohomology states that

$$H^*(k[X]) \cong H^*(|X|; k),\ HC^*(k[X]) \cong H^*(|X|^{cy}; k) =: H^*_{S^1}(|X|; k).$$

We refer [3, 5, 6], for various results of this type. The aim of this paper is to prove a similar result for G-cyclic set.

Let O_G be the category of canonical orbits [1], the objects of O_G are homogeneous spaces G/H, and a morphism $\hat{g}: G/H \to G/K$ is a G-map corresponding to a subconjugacy relation $g^{-1}Hg \subseteq K$. Let k be a commutative ring with 1. An O_G-module is a contravariant functor $O_G \to k$-mod. The collection of O_G-modules along with natural transformations between them as morphisms form an abelian category Vec_G. For any k-module E, we denote by $\bar{E}$ the constant functor $O_G \to k$-mod at E, explicitly, $\bar{E}(G/H) = E$ for all G/H and $\bar{E}(\hat{g}) = id$. Then $\bar{0}$ is the zero object of Vec_G. We shall also consider O_G-groups and O_G-cyclic sets, defined similarly. If X is a G-cyclic set, then the geometric realization $|X|$ of the underlying simplicial set is an S^1-space and also a G-space, and the two actions on $|X|$ commute. As a consequence, the cyclic geometric realization $|X|^{cy} = ES^1 \times_{S^1} |X|$ is a G-space. Let $H^*_G(|X|^{cy}; \lambda)$ denote the Bredon–Illman cohomology groups of $|X|^{cy}$ with coefficients in the O_G-module λ, [1, 4]. One should expect to have an interpretation of the cohomology groups $H^*_G(|X|; \lambda)$ and $H^*_G(|X|^{cy}; \lambda)$ in terms of certain Hochschild and cyclic cohomology groups of $k[X]$. For a G-cyclic

set X, we define equivariant Hochschild and cyclic cohomology modules of $k[X]$ and prove that they are isomorphic to $H^*_G(|X|;\lambda)$ and $H^*_G(|X|^{cy};\lambda)$ respectively. We also deduce Connes' exact sequence for these equivariant Hochschild and cyclic cohomology modules.

2 Equivariant Hochschild and cyclic cohomology

Let X be a G-cyclic set, $k[X]$ the associated G-cyclic module and $\lambda: O_G \to k$-mod an O_G-module. Note that for any subgroup H of G, the H-fixed point set X^H is a cyclic set. Let $\underline{k[X]}_n: O_G \to k$-mod be the functor defined as follows. For an object G/H of O_G $\underline{k[X]}_n(G/H) = k[X^H]_n$, and for a morphism $\hat{g}: G/H \to G/K$ of O_G, $\underline{k[X]}_n(\hat{g}): k[X^K]_n \to k[X^H]_n$ is induced by $g: X^K \to X^H$, $x \mapsto gx$. Let $\partial_n: \underline{k[X]}_n \to \underline{k[X]}_{n-1}$ be the natural transformation defined by $\partial_n(G/H): k[X^H]_n \to k[X^H]_{n-1}$, $\partial_n(G/H) = \Sigma(-1)^i d_i$. Clearly $\partial_{n-1} \circ \partial_n = \bar{0}$, and thus we have a chain complex $\underline{k[X]}_*$ in Vec_G. Let $k[X]^n_G = \mathrm{Hom}(\underline{k[X]}_n, \lambda)$ where Hom is in the category Vec_G, and $\delta^n: k[X]^n_G \to k[X]^{n+1}_G$ be induced by ∂_{n+1}. Then $k[X]^*_G$ is a cochain complex in k-mod. We define equivariant Hochschild cohomology modules by $H^n_G(k[X];\lambda) =: H^n(k[X]^*_G)$.

For any cyclic set Y, let $C(k[Y])$ denote the first quadrant cyclic bicomplex (cf. [6], p. 76) associated to $k[Y]$, $C(k[Y])_{p,q} = C(k[Y_q])$, $p \geqslant 0$, and Tot $C(k[Y])$ be the corresponding total complex. Given a G-cyclic set X, define O_G-modules $\underline{\mathrm{Tot}}\ C(k[X])_n: O_G \to k$-mod by $\underline{\mathrm{Tot}}\ C(k[X])_n(G/H) = \mathrm{Tot}\ C(k[X^H])_n$ for every object G/H of O_G, and for a morphism $\hat{g}: G/H \to G/K$, $\underline{\mathrm{Tot}}\ C(k[X])_n(\hat{g})$ is induced by $g: X^K \to X^H$. Note that the map $g: X^K \to X^H$ of cyclic sets induces a map of bicomplexes $C(k[X^K]) \to C(k[X^H])$. As before, let $\partial_n: \underline{\mathrm{Tot}}\ C(k[X])_n \to \underline{\mathrm{Tot}}\ C(k[X])_{n-1}$ be the natural transformation defined by $\partial_n(G/H): \mathrm{Tot}\, C(k[X^H])_n \to \mathrm{Tot}\, C(k[X^H])_{n-1}$, where $\partial_n(G/H)$ is the boundary map of Tot $C(k[X^H])$. Set Tot $C(k[X])^n_G = \mathrm{Hom}(\underline{\mathrm{Tot}}\ C(k[X])_n; \lambda)$. Then Tot $C(k[X])^*_G$ become a cochain complex in k-mod with the coboundary induced by the natural transformation ∂_n. We define equivariant cyclic cohomology modules of $k[X]$ with coefficient λ by $HC^n_G(k[X];\lambda) =: H^n(\mathrm{Tot}\, C(k[X])^*_G)$.

Remark 2.1. For any bicomplex $C_{\cdot\cdot}$, let $C^{(2)}_{\cdot\cdot}$ denote the bicomplex consisting of the first two columns of $C_{\cdot\cdot}$. Then for a G-cyclic set X, we can define as before, a chain complex $\underline{\mathrm{Tot}}\ C(k[X])^{(2)}_*$ in Vec_G, and the corresponding cochain complex in k-mod. Now there is a quasi-isomorphism between $\underline{k[X]}_*$ and $\underline{\mathrm{Tot}}\ C(k[X])^{(2)}_*$ and thus the corresponding homology functors

$$H_n(\underline{k[X]}_*),\ H_n(\underline{\mathrm{Tot}}\ C(k[X]^{(2)}_*): O_G \to k\text{-mod}$$

are isomorphic. Also note that there are spectral sequences with E^2 terms

$$E^2_{p,q} = \begin{cases} \mathrm{Ext}^p\ (H_q(\underline{k[X]}_*), \lambda) \Rightarrow H^{p+q}_G(k[X];\lambda) \\ \mathrm{Ext}^p(H_q(\underline{\mathrm{Tot}}\ C(k[X])^{(2)}_*), \lambda) \Rightarrow H^{p+q}(\mathrm{Hom}(\underline{\mathrm{Tot}}\ C(k[X])^{(2)}_*; \lambda)). \end{cases}$$

It now follows that

$$H^n_G(k[X];\lambda) \cong H^n(\mathrm{Hom}(\underline{\mathrm{Tot}}\ C(k[X])^{(2)}_*; \lambda)).$$

Theorem 2.2. *(Connes' periodicity exact sequence). If $\lambda \in \mathrm{Vec}_G$ then for any G-cyclic set X there exists a long exact sequence*

$$\cdots \to H_G^n(k[X];\lambda) \xrightarrow{B} HC_G^{n-1}(k[X];\lambda) \xrightarrow{S} HC_G^{n+1}(k[X];\lambda)$$

$$\xrightarrow{I} H_G^{n+1}(k[X];\lambda) \xrightarrow{B} \cdots.$$

roof. For any cyclic set Y, let $C(k[Y])\,[2,0]$ be the bicomplex $(C(k[Y])\,[2,0])_{p,q}$ $= C(k[Y])_{p-2,q}$. Then for a G-cyclic set X, we have a chain complex $\underline{\mathrm{Tot}}\, C(k[X])$ $2,0]_*$ in Vec_G defined as before. Note that there is a split short exact sequence of omplexes in Vec_G

$$0 \to \underline{\mathrm{Tot}}\, C(k[X])^{(2)}_* \to \underline{\mathrm{Tot}}\, C(k[X])_* \to \underline{\mathrm{Tot}}\, C(k[X])\,[2,0]_* \to 0.$$

pplying $\mathrm{Hom}(-,\lambda)$ we get a short exact sequence of cochain complexes of k-modules. y the Remark 2.1, the associated long exact sequence yields the result. ∎

Our main result is the following theorem.

heorem 2.3. *Let X be a G-cyclic set and $k[X]$ the associated G-cyclic module and* $\in \mathrm{Vec}_G$, then

) $H_G^n(k[X];\lambda) \cong H_G^n(|X|;\lambda)$

2) $HC_G^n(k[X];\lambda) \cong H_G^n(|X|^{\mathrm{cy}};\lambda)$.

Before we prove the theorem we need to set up some more notations.

For any small category $\mathscr{C}$, let colim: Funct $(\mathscr{C}, k\text{-mod}) \to k$-mod be the covariant ight exact functor which assigns to a functor $F:\mathscr{C} \to k$-mod its colimit colim F.

Let $\mathrm{Tor}_i^{\mathscr{C}}(k,-)$: Funct $(\mathscr{C}, k\text{-mod}) \to k$-mod, $i \geqslant 0$, denote the left derived functors of olim. Then $\mathrm{Tor}_0^{\mathscr{C}}(k,-) = \mathrm{colim}$. For a G-cyclic set X we have the following con-avariant functors from O_G to k-mod.

) $\underline{\mathrm{Tor}}_n^{\Delta C^{\mathrm{op}}}(k, k[X])$,

i) $\underline{\mathrm{Tor}}_n^{\Delta C_X^{\mathrm{op}}}(k,k)$,

ii) $\underline{HC}_n(k[X])$,

v) $\underline{H}_n(\|X\|^{\mathrm{cy}};k)$,

v) $\underline{H}_n(|X|^{\mathrm{cy}};k)$.

To describe the functor (i), note that for every subgroup H of G we may regard $k[X^H]$ s an object of Funct $(\Delta C^{\mathrm{op}}, k\text{-mod})$. Then every morphism $\hat{g}:G/H \to G/K$ of O_G induces natural transformation $g:k[X^K] \to k[X^H]$. We define $\underline{\mathrm{Tor}}_n^{\Delta C^{\mathrm{op}}}(k,k[X])(G/H) = \mathrm{Tor}_n^{\Delta C^{\mathrm{op}}}$ $k, k[X^H])$ for every object G/H of O_G and for a morphism $\hat{g}:G/H \to G/K$ of O_G, $\underline{\mathrm{Tor}}_n^{\Delta C^{\mathrm{op}}}(k,k[X])\ (\hat{g})$: $\mathrm{Tor}_n^{\Delta C^{\mathrm{op}}}(k,k[X^K]) \to \mathrm{Tor}_n^{\Delta C^{\mathrm{op}}}(k,k[X^H])$ is the homomorphism $\mathrm{Tor}_n^{\Delta C^{\mathrm{op}}}(k,g)$, where $g:k[X^K] \to k[X^H]$ is the natural transformation induced by $\hat{g}$.

To describe the functor (ii) we proceed as follows. For any subgroup H of G regard $X^H:\Delta C^{\mathrm{op}} \to \mathscr{S}$ as a functor and let $g:X^K \to X^H$ be the natural transformation induced by $\hat{g}:G/H \to G/K$. Let $\Delta C^{\mathrm{op}}_{X^H}$ denote the Bousfield–Kan category corresponding to the unctor X^H. The natural transformation $g:X^K \to X^H$ induces in an obvious way functor $\Delta C^{\mathrm{op}}_{X^K} \to \Delta C^{\mathrm{op}}_{X^H}$ which we denote again by g and this in turn induces a functor $\underline{g}$: Funct $(\Delta C^{\mathrm{op}}_{X^H}, k\text{-mod}) \to$ Funct $(\Delta C^{\mathrm{op}}_{X^K}, k\text{-mod})$. It is easy to see that $\underline{g}$ is an exact functor. Let colim^H: Funct$(\Delta C^{\mathrm{op}}_{X^H}, k\text{-mod}) \to k$-mod denote the colimit functor 'colim' as

mentioned before to distinguish it from the colimit functor colim^K:Funct($\Delta C^{\text{op}}_{X^K}$, k-mod)$\to k$-mod. Then $\text{Tor}_i^{\Delta C^{\text{op}}_{X^H}}(k,-)$ are the left derived functors of colim^H. Let k^H denote the constant functor k^H:$\Delta C^{\text{op}}_{X^H}\to k$-mod at k. Define $\underline{\text{Tor}}_n^{\Delta C^{\text{op}}_X}(k,k)(G/H)=\text{Tor}_n^{\Delta C^{\text{op}}_{X^H}}(k,k^H)$ for objects G/H of O_G. For a morphism $\hat{g}:G/H\to G/K$ we must define a module homomorphism $\underline{\text{Tor}}_n^{\Delta C^{\text{op}}_X}(k,k)(\hat{g}):\text{Tor}_n^{\Delta C^{\text{op}}_{X^K}}(k,k^K)\to\text{Tor}_n^{\Delta C^{\text{op}}_{X^H}}(k,k^H)$. Note that by the universal property defining 'colim', the functor $\underline{g}$ induces a natural transformation $\tilde{g}$: $\text{colim}^K\circ\underline{g}\to\text{colim}^H$. Let $P.\to k^H\to 0$ be a projective resolution of k^H in Funct ($\Delta C^{\text{op}}_{X^H}$, k-mod). Since $\underline{g}$ is exact, $\underline{g}(P.)\to\underline{g}(k^H)=k^K\to 0$ is a resolution of k^K in Funct ($\Delta C^{\text{op}}_{X^H}$, k-mod). Let $Q.\to k^K\to 0$ be any projective resolution of k^K. Then there exists a map of complexes $Q.\to\underline{g}(P.)$ (which is a quasi-isomorphism). Therefore we get a map of complexes of k-modules $\text{colim}^K Q.\to\text{colim}^K\underline{g}(P.)$. By composing with $\tilde{g}$ we get a map of complexes of k-modules $\text{colim}^K Q.\to\text{colim}^H P.$, and hence an induced map in homology $H_n(\text{colim}^K Q.)\to H_n(\text{colim}^H P.)$. Note that $H_n(\text{colim}^K Q.)=\text{Tor}_n^{\Delta C^{\text{op}}_{X^K}}(k,k^K)$ and $H_n(\text{colim}^H P.)=\text{Tor}_n^{\Delta C^{\text{op}}_{X^H}}(k,k^H)$. We define $\underline{\text{Tor}}_n^{\Delta C^{\text{op}}_X}(k,k)(\hat{g})$ to be this induced map in homology. This makes $\underline{\text{Tor}}_n^{\Delta C^{\text{op}}_X}(k,k)$ a contravariant functor from O_G to k-mod.

The functor (iii) is defined as follows. For objects G/H of O_G, $\underline{HC}_n(k[X])(G/H)=HC_n(k[X^H])$ and for a morphism $\hat{g}$, $\underline{HC}_n(k[X])(\hat{g}):HC_n(k[X^K])\to HC_n(k[X^H])$ is the homomorphism of cyclic homologies induced by $g:X^K\to X^H$.

To describe the functor (iv), let ΔC^{op}_X be the Bousfield–Kan category of the underlying cyclic set of the G-cyclic set X. The nerve $B.\Delta C^{\text{op}}_X$ of the small category ΔC^{op}_X inherits an obvious G-action as follows. Let u denote a typical element $([n_0],x_0)\xrightarrow{f_1}([n_1],x_1)\xrightarrow{f_2}\cdots\xrightarrow{f_j}([n_j],x_j)$ of $B_j\Delta C^{\text{op}}_X$ and $g\in G$, then $gu=([n_0],gx_0)\xrightarrow{f_1}([n_1],gx_1)\to\cdots\xrightarrow{f_j}([n_j],gx_j)$. This makes sense as X is a G-cyclic set. This makes $B.\Delta C^{\text{op}}_X$ a G-simplicial set and consequently the classifying space $B\Delta C^{\text{op}}_X=|B.\Delta C^{\text{op}}_X|$ becomes a G-space. Let $\|X\|^{\text{cy}}=B\Delta C^{\text{op}}_X$. Moreover we have $(\|X\|^{\text{cy}})^H=\|X^H\|^{\text{cy}}$ for any subgroup H of G. We may now define the functor (iv) as in (iii) by singular homology. Explicitly, $\underline{H}_n(\|X\|^{\text{cy}};k)(G/H)=H_n(\|X^H\|^{\text{cy}};k)$ for objects G/H and for a morphism $\hat{g}$, $H_n(\|X\|^{\text{cy}};k)(\hat{g})$ is the homomorphism induced by $g:X^K\to X^H$. The functor (v) is defined similarly using the G-space $|X|^{\text{cy}}$.

Lemma 2.4. *With the above notations, there is a sequence of isomorphisms between the functors*

$$\underline{HC}_n(k[X])\overset{\eta_1}{\cong}\underline{\text{Tor}}_n^{\Delta C^{\text{op}}}(k,k[X])\overset{\eta_2}{\cong}\underline{\text{Tor}}_n^{\Delta C^{\text{op}}_X}(k,k)\overset{\eta_3}{\cong}\underline{H}_n(\|X\|^{\text{cy}};k)$$

$$\overset{\eta_4}{\cong}\underline{H}_n(|X|^{\text{cy}};k).$$

Proof. Recall that for any cyclic module E there is an isomorphism $\text{Tor}_n^{\Delta C^{\text{op}}}(k,E)\cong HC_n(E)$ (cf. Theorem 6.2.8, [6]). Define $\eta_1(G/H)$ to be the isomorphism $\text{Tor}_n^{\Delta C^{\text{op}}}(k,k[X^H])\cong HC_n(k[X^H])$. That η_1 is a natural equivalence follows from the fact that the above isomorphism is natural with respect to any morphism of cyclic modules $E\to F$. Define $\eta_2(G/H):\text{Tor}_n^{\Delta C^{\text{op}}}(k,k[X^H])\cong\text{Tor}_n^{\Delta C^{\text{op}}_{X^H}}(k,k^H)$ to be the isomorphism given by the Shapiro's lemma (cf. c.12, p. 414, [6]). One checks that this isomorphism is natural with respect to natural transformation $g:k[X^K]\to k[X^H]$ induced by $\hat{g}:G/H\to G/K$. The natural equivalence η_3 is given by the natural isomorphism $\eta_3(G/H):\text{Tor}_n^{\Delta C^{\text{op}}_{X^H}}(k,k^H)\cong H_n(\|X^H\|^{\text{cy}};k)$ (cf. [7]). To describe η_4, recall (cf. proposition 7.2.6, p. 233, [6]) that for any cyclic set Y, there is a canonical homotopy

quivalence $\|Y\|^{cy} \simeq |Y|^{cy}$. Now if X is a G-cyclic set, then it is straight forward to check at the homotopy equivalence constructed in [6] is actually a G-homotopy equivalence, whose restriction to H-fixed point set is the homotopy equivalence $X^H\|^{cy} \simeq |X^H|^{cy}$. Define $\eta_4(G/H)$ to be the isomorphism in homology induced by the omotopy equivalence $\|X^H\|^{cy} \simeq |X^H|^{cy}$. This completes the proof of the lemma. ∎

roof of Theorem 2.3. We shall prove (2), the proof of (1) is similar. Recall that there is spectral sequence with E^2-term $E^2_{p,q} = \mathrm{Ext}^p\ (\underline{HC}_q(k[X]), \lambda) \Rightarrow HC_G^{p+q}(k[X]; \lambda)$. 'e also have a spectral sequence with E^2-term $E^2_{p,q} = \mathrm{Ext}^p(\underline{H}_q(|X|^{cy}; k), \lambda) \Rightarrow H_G^{p+q}$ $X|^{cy}; \lambda)$. Now the natural equivalence $\underline{HC}_q(k[X]) \cong \underline{H}_q(|X|^{cy}; k)$ given by Lemma 2.4 xtends to an isomorphism between the above spectral sequences and thus we have $C_G^{p+q}(k[X]; \lambda) \cong H_G^{p+q}(|X|^{cy}; \lambda)$. ∎

Examples

xample 3.1. Let Z be an S^1-space as well as a G-space and the two actions commute. or instance, we can take Z to be the free loop space of a G-space X. Set $S_n(Z) = \mathrm{Map}_{S^1}$ $^1 \times \Delta_n, Z)$, the S^1-equivariant maps from $S^1 \times \Delta_n$ to Z, where S^1 acts on $S^1 \times \Delta_n$ by ultiplication on the first factor. The G-action on Z makes $S_n(Z)$ a G-set so that $S.(Z)$ ecomes a G-cyclic set.

To give our next example, we need the following result which is a construction due to lmendorf [2] adapted in the context of cyclic sets. Let $O_G\mathscr{S}^c$ and $G\mathscr{S}^c$ denote espectively the category of O_G-cyclic sets and G-cyclic sets. Let Φ: $G\mathscr{S}^c \to O_G\mathscr{S}^c$ be the nctor given by $\Phi X(G/H) = X^H$ and $\Phi X(\hat{g})$ is the cyclic map g: $X^K \to X^H$.

heorem 3.2. *(Elmendorf). There is a functor $\mathscr{E}$:$O_G\mathscr{S}^c \to G\mathscr{S}^c$, and a natural transformation η: $\Phi\mathscr{E} \to id$ such that for each O_G-cyclic set T and each subgroup H, $\eta(T)(G/H)$: $\mathscr{E}T)^H \to T(G/H)$ is a homotopy equivalence.*

roof. Given an O_G-cyclic set T, we form the bar complex $B_*(T, O_G)$, the n-mplexes of which are $(n+2)$-tuples $(u; f_1, f_2, \ldots, f_n; x)$ where $G/H_n \xrightarrow{f_n} G/H_{n-1}$ $\xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} G/H_1 \xrightarrow{f_1} G/H_0$ are composable morphisms in O_G, $u \in T(G/H_0)_n$ and $x \in G/H_n$. he face and degeneracy maps are given by

$$d_i(u; f_1, \ldots, f_n; x) = \begin{cases} (d_0 T(f_1)(u); f_2, \ldots, f_n; x) & i = 0 \\ (d_i u; f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n; x) & 0 < i < n \\ (d_n u; f_1, \ldots, f_{n-1}; f_n(x)) & i = n \end{cases}$$

$$s_i(u; f_1, \ldots, f_n; x) = (s_i u; f_1, \ldots, f_i, id, f_{i+1}, \ldots, f_n; x),$$

here the face and the degeneracy maps on the right are those of the appropriate omplexes. There is an obvious action of G on the last coordinate which makes $B_*(T, O_G)$ a G-simplicial set. Define $\mathscr{E}T = B_*(T, O_G)$. For an n-simplex $(u; f_1, \ldots, f_n; x)$ he action of $\mathbb{Z}_{n+1}$ is given by

$$t_n(u; f_1, \ldots, f_n; x) = (t_n u; f_1, \ldots, f_n; x),$$

here the action of t_n on the right hand side is the action of $\mathbb{Z}_{n+1}$ on $T(G/H_0)_n$. This nakes $\mathscr{E}T$ a G-cyclic set. For a subgroup H, the n-simplexes of the cyclic set $(\mathscr{E}T)^H$ are f the form $(u; f_1, \ldots, f_n; f)$ where f: $G/H \to G/H_n$ is a morphism in O_G. There are cyclic

maps η_H^T: $(\mathscr{E}T)^H \to T(G/H)$ and ξ_H^T: $T(G/H) \to (\mathscr{E}T)^H$ given by $\eta_H^T(u; f_1, \ldots, f_n; f) = T(f_1 \circ f_2 \circ \cdots \circ f_n \circ f)(u)$ and $\xi_H^T(u) = (u; id, \ldots, id; id)$ where id is the identity map $G/H \to G/H$. Then $\eta_H^T \circ \xi_H^T = id$ and $\xi_H^T \circ \eta_H^T \simeq id$. The homotopy is given by the functions $h_i: (\mathscr{E}T)_n^H \to (\mathscr{E}T)_{n+1}^H$, $0 \leqslant i \leqslant n$, where $h_i\ (u; f_1, \ldots, f_n; f) = (s_i u, f_1, \ldots, f_i, f_{i+1} \circ \cdots \circ f_n \circ f, id, \ldots, id; id)$. This completes the proof. ∎

Example 3.3. Let λ be an abelian O_G-group, that is, $\lambda(G/H)$ is abelian for all subgroups H. Let $B.\lambda(G/H)$ be the nerve of $\lambda(G/H)$, where $B_n\lambda(G/H) = \lambda(G/H)^n$. The assignment $G/H \to B.\lambda(G/H)$ yields an O_G-simplicial set $B.\lambda$. Let $z_G \in \lambda(G/G)$. For any subgroup H of G, let $z_H \in \lambda(G/H)$ be the element $z_H = \lambda(G/H \to G/G)(z_G)$, where $G/H \to G/G$ is induced by the inclusion $H \subseteq G$. Then for any morphism $\hat{g}$: $G/H \to G/K$, $\lambda(\hat{g})(z_K) = z_H$. Now define an action of the cyclic operator t_n on $B_n\lambda(G/H)$ by $t_n(a_1, \ldots, a_n) = (z_H(a_1 a_2, \ldots, a_n)^{-1}, a_1, \ldots, a_{n-1})$ (cf. [6], 7.3.3). This makes $B.\lambda$ an O_G-cyclic set. Therefore by Theorem 3.2, $\mathscr{E}B.\lambda$ is a G-cyclic set. The construction of $\mathscr{E}B.\lambda$ from λ is of course functorial.

Let $\mathscr{E}B\lambda = |\mathscr{E}B.\lambda|$. Then $\mathscr{E}B\lambda$ is a G-space and an S^1-space and the two actions commute. The S^1-action on $\mathscr{E}B\lambda$ may be described as follows. Let $\bar{\mathbb{Z}}$ be the constant O_G-group and $z = \{z_H\}$ be a natural family of elements from λ chosen as above. Then there is an action of $\bar{\mathbb{Z}}$ on λ given by the natural transformation γ_z: $\bar{\mathbb{Z}} \times \lambda \to \lambda$, where $\gamma_z(G/H)\ (n, a) = z_H^n a$, $a \in \lambda(G/H)$. Then it follows from Proposition 7.3.4, [6] and Theorem 3.2, that the S^1-action is given by the composition $S^1 \times \mathscr{E}B\lambda \simeq_G \mathscr{E}B(\bar{\mathbb{Z}} \times \lambda) \xrightarrow{\mathscr{E}B(\gamma_z)} \mathscr{E}B\lambda$. Let $X(\lambda, z) = ES^1 \times_{S^1} \mathscr{E}B\lambda$. Thus $X(\lambda, z) = |\mathscr{E}B.\lambda|^{cy}$. Suppose now that z_H is of infinite order for every subgroup H. This happens, for instance, if z_G is of infinite order and the subgroup $\{z_G\}$ of $\lambda(G/G)$ does not intersect the kernel of the homomorphism $\lambda(G/H \to G/G)$ for any subgroup H, except at identity. Then it follows that $X(\lambda, z)$ is an equivariant Eilenberg–MacLane space $K(\lambda/\{z\}, 1)$, where $\lambda/\{z\}$ is the O_G-group $\lambda/\{z\}\ (G/H) = \lambda(G/H)/\{z_H\}$. If some z_H is of finite order then $X(\lambda, z)^H$ has only two non-trivial homotopy groups, $\pi_1(X(\lambda, z)^H) = \lambda(G/H)/\{z_H\}$, $\pi_2(X(\lambda, z)^H) = \mathbb{Z}$, (cf. 7.3.5, [6], and § 2, [2]). From Theorem 2.3, we get as follows.

COROLLORY 3.4

Let λ be an abelian O_G-group, $z = \{z_H\}$ be a natural family of elements as described above. Let $\lambda' \in \mathrm{Vec}_G$. Then

$$HC_G^*(k[\mathscr{E}B.\lambda]; \lambda') \cong H_G^*(X(\lambda, z); \lambda').$$

Moreover, if each z_H is of infinite order, then

$$HC_G^*(k[\mathscr{E}B.\lambda]; \lambda') \cong H_G^*(K(\lambda/\{z\}, 1); \lambda') \cong [K(\lambda/\{z\}, 1), K(\lambda', *)]_G.$$ ∎

The last identification follows from the fact that equivariant Eilenberg–MacLane spaces classify Bredon cohomology. In particular, we have

$$HC_G^1(k[\mathscr{E}B.\lambda]; \lambda') \cong \mathrm{Hom}(\lambda/\{z\}, \lambda').$$

As an example consider $\lambda = \bar{\mathbb{Z}}_p$, where $p > 1$ a prime, and let z_G be any element other than the identity. Then it is easy to see that $X(\lambda, z) = K(\bar{\mathbb{Z}}, 2)$. Therefore it follows from

Corollary 3.4 that

$$HC_G^1(k[\mathscr{E}B.\lambda];\lambda') = 0,$$

$$HC_G^2(k[\mathscr{E}B.\lambda];\lambda') \cong \mathrm{Hom}(\overline{\mathbb{Z}}, \lambda').$$

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A note on equivariant Euler characteristic

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Abstract. We give a new equivariant cohomological characterization of the equivariant Euler characteristic of a G-simplicial set as defined by Brown. This implies in particular that the equivariant Euler characteristic is a G-homotopy invariant.

Keywords. G-simplicial set; equivariant cohomology; Euler characteristic.

. Introduction

erre [5] and Brown [2] defined the Euler characteristic of a discrete group G of nite homological type. Recall that the cohomological dimension $\mathrm{cd}\, G$ is inf n so hat the $\mathbb{Z}G$-module $\mathbb{Z}$ with trivial G-action admits projective resolutions of ength $\leqslant n$, and that G has finite virtual cohomological dimension $\mathrm{vcd}\, G$ if there xists a finite index subgroup with finite cohomological dimension. A group G is f finite homological type if

1) $\mathrm{vcd}\, G < \infty$, and
2) every torsion-free subgroup of finite index has finitely generated rational homology.

If the rational homology of G is finitely generated, then the 'naive' Euler characteristic of G is the integer

$$\tilde{\chi}(G) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(G; \mathbb{Q}),$$

nd if G is of finite homological type, then its Euler characteristic is the rational umber

$$\chi(G) = \tilde{\chi}(H)/[G:H],$$

where H is a torsion-free subgroup of finite index $[G:H]$. It is a result of Brown [2] that $\chi(G)$ is independent of the choice of H.

Next, if K is a G-simplicial set, where G is discrete, such that the simplicial set K/G has initely many non-degenerate cells, and each isotropy subgroup G_x of a simplex x of K is of finite homological type, then the equivariant Euler characteristic of K is

$$\chi_G(K) = \sum (-1)^{\dim x} \chi(G_x),$$

where the sum is over the representatives of non-degenerate simplexes of K/G.

Now suppose that O_G denotes the category of canonical orbits of G whose objects are eft coset spaces G/H and whose morphisms are G-maps $\hat{g}: G/H \to G/H'$ coming from a subconjugacy relation $g^{-1} Hg \leqslant H'$. Let $\lambda_{\mathbb{Q}}$ denote the contravariant functor from O_G o the category of vector spaces over $\mathbb{Q}$ such that $\lambda_{\mathbb{Q}}(G/H) = \mathrm{Hom}(\mathbb{Q}(G/H), \mathbb{Q})$ where $\mathbb{Q}(G/H)$ is the vector space over $\mathbb{Q}$ with basis G/H, and $\lambda_{\mathbb{Q}}(\hat{g}) = \mathrm{Hom}(\mathbb{Q}(\hat{g}), id)$.

In § 2, we define equivariant cohomology $H^*_G(K;\lambda_{\mathbb{Q}})$ of K which is G-homotopy invariant. In fact, this cohomology is the simplicial analogue of the Bredon cohomology [1]. The purpose of the present paper is to prove the following theorems.

Theorem 1. *If the action of a discrete group G on a G-simplicial set K is such that (i) K/G has only finitely many non-degenerate cells, and (ii) every isotropy subgroup of every simplex of K has finite index in G, and if $\lambda_{\mathbb{Q}}$ is as defined above, then the cohomology groups $H^*_G(K;\lambda_{\mathbb{Q}})$ are finitely generated.*

Theorem 2. *If G acts freely on a simplicial set K satisfying the conditions (i) and (ii) of Theorem 1 then $\chi_G(K) = \chi(G)\sum(-1)^i \dim_{\mathbb{Q}} H^i_G(K;\lambda_{\mathbb{Q}})$, where the summation is from $i=0$ to $i = \dim K/G$.*

Theorem 3. *If G is of finite cohomological dimension and finite homological type, and K is a G-simplicial set satisfying the conditions of the above theorem, then*

$$\chi_G(K) = \chi(G)\sum(-1)^i \dim_{\mathbb{Q}} H^i_G(K;\lambda_{\mathbb{Q}}),$$

where the summation is from $i=0$ to $i = \dim K/G$.

Thus $\chi_G(K)$ is a G-homotopy invariant. In particular, if G is free of rank n, then it is of finite homological type because its virtual cohomological dimension is n. Moreover its rational homology is finitely generated because there exists a $K(G,1)$ with one 0-cell and n 1-cells. Therefore $\chi(G) = 1-n$, and

$$\chi_G(K) = (1-n)\sum(-1)^i \dim_{\mathbb{Q}} H^i_G(K;\lambda_{\mathbb{Q}}),$$

if K is as in Theorem 1.

The plan of the paper is as follows. In § 2, we discuss some basic results with only sketches of proofs, the details of which may be worked out without difficulty. The proofs of the theorems appear in § 3.

2. Equivariant cohomology of a G-simplicial set

A G-simplicial set is a simplicial set K together with an action of G by simplicial maps which commute with the face and degeneracy maps d_i and s_i.

We define the equivariant cohomology $H^*_G(K;\lambda)$ of K with a cofficient system λ (which is a contravariant functor from O_G to the category R-mod of R-modules, R being a commutative ring with (1) as follows. Let $C^n(K;\lambda)$ be the R-module of functions c defined on n-simplexes x of K such that $c(x)\in\lambda(G/G_x)$, where G_x is the isotropy subgroup of x in G. Then define coboundary $\delta: C^n(K;\lambda)\to C^{n+1}(K;\lambda)$ by

$$\delta(c)(x) = \sum_{i=0}^{n+1}(-1)^i \lambda(d_i x \to x)c(d_i x),$$

where $\lambda(d_i x \to x)$ is the homomorphism $\lambda(G/G_{d_i x})\to\lambda(G/G_x)$ induced from the G-map $G/G_x \to G/G_{d_i x}$ given by the inclusion $G_x \subseteq G_{d_i x}$.

We define an action of G on $C^n(K;\lambda)$ by $(gc)(x) = \lambda(\hat{g})(c(g^{-1}x))$ where $\lambda(\hat{g})$ is the isomorphism $\lambda(G/G_{g^{-1}x})\to\lambda(G/G_x)$ induced by the conjugacy relation

$g^{-1}G_x g = G_{g^{-1}x}$. Let $C^n_G(K;\lambda)$ be the submodule of G-invariant cochains $(C^n(K;\lambda))^G$. Clearly this makes $C^*_G(K;\lambda)$ a cochain complex, and so we may define

$$H^n_G(K;\lambda) = H_n(C^*_G(K;\lambda)).$$

Note that if the action of G on K is free, then we have

$$H^*_G(K;\lambda) \cong H^*(K/G;\lambda(G/\{e\}))$$

for every coefficient system λ.

Clearly a G-simplicial map $f: K \to L$ induces a cochain map $f^\#: C^*_G(L;\lambda) \to C^*_G(K;\lambda)$ defined by $f^\#(c)(x) = \lambda(fx \to x)c(fx)$, where $\lambda(fx \to x): \lambda(G/G_{fx}) \to \lambda(G/G_x)$ is the homomorphism induced by the inclusion $G_x \subseteq G_{fx}$. Then $f^\#$ induces homomorphism $f^*: H^*_G(L;\lambda) \to H^*_G(K;\lambda)$ satisfying the usual functorial properties.

Lemma 4. *If $f, g: K \to L$ are G-homotopic G-simplicial maps, then*

$$f^* = g^*: H^*_G(L;\lambda) \to H^*_G(K;\lambda).$$

Sketch of Proof. The cochain maps $f^\#, g^\#: C^*_G(L;\lambda) \to C^*_G(K;\lambda)$ are cochain homotopic by $h: C^n_G(L;\lambda) \to C^{n-1}_G(K;\lambda)$ given by

$$h(c)(x) = \sum_{j=0}^{n-1} (-1)^j \lambda(h_j x \to x) c(h_j x),$$

where $h_j: K_n \to L_{n+1}$ are G-functions constituting a G-homotopy from f to g. ∎

Alternatively, the cochain complex $C^*_G(K;\lambda)$ may be defined as follows. Consider for each $n = \geqslant 0$ a coefficient system $\underline{C}_n(K): O_G \to R$-mod by setting $\underline{C}_n(K)(G/H) = C_n(K^H; R)$ which is the free R-module generated by the n-simplexes of K^H, and, for a G-map $\hat{g}: G/H \to G/H'$, $g^{-1}Hg \subseteq H'$, setting $\underline{C}_n(K)(\hat{g}) = g_*$ which is the chain map induced by the left translation $g: K^{H'} \to K^H$. This gives a chain complex $\underline{C}_*(K)$ in the abelian category of coefficient systems, and if λ is a coefficient system, then $\mathrm{Hom}(\underline{C}_*(K), \lambda)$, which is the R-module of natural transformations $\underline{C}_*(K) \to \lambda$, becomes a cochain complex.

Lemma 5. *There is an isomorphism of cochain complexes*

$$\alpha: C^*_G(K;\lambda) \to \mathrm{Hom}(\underline{C}_*(K), \lambda).$$

Sketch of Proof. Define α by $\alpha(c)(G/H)(x) = \lambda(G_x \to H)(c(x))$, where $x \in K^H_n$ and $\lambda(G_x \to H): \lambda(G/G_x) \to \lambda(G/H)$ is the homomorphism induced by the inclusion $H \subseteq G_x$. Next, define the inverse α' of α by $\alpha'(T)(x) = T(G/G_x)(x)$. ∎

Note that $\underline{C}_*(K)$ is projective in the abelian category of coefficient systems which has sufficiently many injectives, and if λ^* is an injective resolution of λ, then we have a double complex $\mathrm{Hom}(\underline{C}_*(K), \lambda^*)$. The homological algebra applied to this double complex yields a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(\underline{H}_q(K), \lambda) \Rightarrow H^{p+q}_G(K;\lambda),$$

where $\underline{H}_q(K): O_G \to R$-mod is the cofficient system given by

$$\underline{H}_q(K)(G/H) = H_q(K^H; R) \quad \text{and} \quad \underline{H}_q(K)(\hat{g}) = H_q(g).$$

Lemma 6. *If* $f: K \to L$ *is a* G*-simplicial map such that each* $f^H = f|K^H: K^H \to L^H$, $H \subseteq G$, *induces isomorphism in the classical homology with* R *coefficients, then*

$$f^*: H^*_G(L; \lambda) \to H^*_G(K; \lambda)$$

is an isomorphism for every coefficient system λ.

Sketch of Proof. We have an isomorphism $f_*: \underline{H}_q(K) \to \underline{H}_q(L)$ given by $f_*(G/H) = f^H_*$. This extends to an isomorphism f^* between the spectral sequences. ■

For a G-simplicial set K, let RK denote the G-simplicial R-module with the set of n-simplexes $(RK)_n = RK_n$ which is the free R-module with basis K_n, and the face and degeneracy maps as the linear extensions of the corresponding maps of K. The G-action on RK is also defined similarly.

Lemma 7. *There is an isomorphism* $H^*_G(K; \lambda) \cong H^*_G(RK; \lambda)$.

Sketch of Proof. We have a cochain isomorphism $\theta: \mathrm{Hom}(\underline{C}_*(K), \lambda) \to \mathrm{Hom}(\underline{C}_*(RK), \lambda)$ given by $\theta(T)(G/H)(\sum n_i x_i) = T(G/H)(\sum n_i x_i)$. ■

Let NRK denote the G-pre-simplicial module (degeneracy not considered) where the set of n-simplexes is $\{x \in RK_n : d_i x = 0, 0 \leqslant i < n\}$, and the nth face operator is d_n.

Lemma 8. *There is an isomorphism* $H^*_G(K; \lambda) \cong H^*_G(NRK; \lambda)$.

Sketch of Proof. Consider the inclusion map i: $\underline{C}_*(NRK) \to \underline{C}_*(RK)$. By May [4, (22.3)], $i(G/H): C_*(NRK^H; R) \to C_*(RK; R)$ induces isomorphism on homology for each $H \subseteq G$. The proof then follows from Lemmas 6 and 7. ■

It may be noted in passing that if X is a G-space and SX the associated singular G-simplicial set, then the cohomology $H^*_G(SX; \lambda)$ is isomorphic to the equivariant singular cohomology of X with coefficient system λ (see Illman [3]), for every λ.

3. Proofs of theorems

Proof of Theorem 1. In view of Lemma 8, it is sufficient to prove that the vector space $C^n_G(NRK; \lambda_{\mathbb{Q}})$ is finitely generated. Let $x_1, \ldots, x_k$ denote the representatives of the orbit classes of the non-degenerate n-simplexes which lie in NRK. Suppose that for $1 \leqslant l \leqslant k$, the isotropy group G_{x_l} has index m_l in G. Fix a coset representation

$$G/G_{x_l} = \{a_{l_1} G_{x_l}, \ldots, a_{l_{m_l}} G_{x_l}\}, \quad 1 \leqslant l \leqslant k, \quad a_{l_i} \in G.$$

Then define cochains c_{ij} by

$$c_{ij}(x_l) = \begin{cases} 0 & j \neq l \\ (a_{l_i} G_{x_l})^* & j = l, \ 1 \leqslant i \leqslant m_l \\ 0 & j = l, i > m_l \end{cases}$$

where $(a_{l_i} G_{x_l})^*$ are basis dual to $a_{l_i} G_{x_l}$. There is a unique way to define c_{ij} on the orbit of x_l so that $c_{ij} \in C^n_G(NRK; \lambda_{\mathbb{Q}})$. It is also clear that the set $\{c_{ij}\}$ is a linearly independent set, and that any invariant cochain can be written in terms of the c_{ij}'s. This proves the theorem.

Proof of Theorem 2. The group G is necessarily finite. Therefore $\chi_G(K)$ is defined, and, by Theorem 1, the groups $H_G^*(K;\lambda_{\mathbb{Q}})$ are finitely generated. Also, as the action is free, we have

$$\chi_G(K) = \sum_i (-1)^i N_i = \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(K/G;\mathbb{Q}),$$

where N_i denotes the number of non-degenerate i-simplexes of K modulo the action. Consequently, $\chi_G(K) = \chi(K/G)$, the Euler characteristic of K/G. On the other hand the nature of the action implies $H_G^*(K;\lambda_{\mathbb{Q}}) \cong H^*(K/G;\mathbb{Q}(G))$ and, as

$$\dim_{\mathbb{Q}} H^*(K/G;\mathbb{Q}(G)) = |G| \dim_{\mathbb{Q}} H^*(K/G;\mathbb{Q}),$$

the theorem follows.

Proof of Theorem 3. Since G has finite cohomological dimension, it is torsion free. Also, since G is of finite homological type, the isotropy subgroups G_x also have finite homological type, by a result of Brown [2, IX (6.3)]. Therefore $\chi(G_x)$ is defined, and

$$\begin{aligned}\chi_G(K) &= \sum (-1)^{\dim x} \chi(G_x) = \sum (-1)^{\dim x} \chi(G)\,[G : G_x] \\ &= \chi(G) \sum_{i=0}^{\dim(K/G)} (-1)^i \dim_{\mathbb{Q}} C_G^i(K;\lambda_{\mathbb{Q}}), \\ &= \chi(G) \sum_{i=0}^{\dim(K/G)} (-1)^i \dim_{\mathbb{Q}} H_G^i(K;\lambda_{\mathbb{Q}}).\end{aligned}$$

The last step follows since we are dealing with vector spaces. This completes the proof.

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Degree of approximation of functions in the Hölder metric by Borel's means

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Abstract. After establishing the Fourier character of the series the authors have studied the degree of approximation of functions associated with the same series in the Hölder metric using Borel's mean.

Keywords. Fourier character of the series; Banach space; Hölder metric; Borel's mean.

1. Definitions

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[-\pi, \pi]$ under sup-norm.

For $0 < \alpha \leqslant 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leqslant K |x - y|^\alpha\}. \tag{1.1}$$

The space H_α is a Banach space [6] with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x, y} \{\Delta^\alpha f(x, y)\}, \tag{1.2}$$

where

$$\|f\|_c = \sup_{-\pi \leqslant x \leqslant \pi} |f(x)|$$

and

$$\Delta^\alpha f(x, y) = |f(x) - f(y)| \, |x - y|^{-\alpha}, \quad (x \neq y). \tag{1.3}$$

We shall use the convention that $\Delta^0 f(x, y) = 0$. The metric induced by the norm (1.2) on H_α is called Hölder metric.

Let f be a periodic function of period 2π and integrable in the Lebesgue sense over $[-\pi, \pi]$.

Let the Fourier series associated with f at x be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x). \tag{1.4}$$

Let us write

$$\Phi_x(t) = \tfrac{1}{2}\{f(x + t) + f(x - t) - 2f(x)\}, \tag{1.5}$$

$$S_n(x) = \sum_{k=0}^{n} A_k(x), \tag{1.6}$$

$$\sigma_n(x)=\frac{1}{n}\sum_{k=0}^{n} S_k(x) \quad (n>0). \tag{1.7}$$

Given any sequence $\{t_n(x)\}$, its Borel's exponential mean $B_p(t,x)$ is defined by ([2], p. 182)

$$B_p(t,x)=e^{-p}\sum_{n=0}^{\infty} t_n(x)\frac{p^n}{n!} \quad (p>0). \tag{1.8}$$

2. Introduction

The convergence and $(C,1)$ summability problem of the series

$$\sum_{n=1}^{\infty}\frac{S_n(x)-f(x)}{n} \tag{2.1}$$

have been studied by Hardy and Littlewood [3] (also see [7], p. 125). In a recent work Das, Ojha and Ray [1] have investigated the degree of approximation of (2.1) in the Hölder metric by Borel's mean.

In a recent paper Mohanty and Ray [4] have studied the convergence and absolute convergence of the series

$$\sum_{n=1}^{\infty}\frac{\sigma_n(x)-f(x)}{n}. \tag{2.2}$$

As regards convergence they have shown that if

$$\int_0^t \Phi_x(u)\,du = o(t),$$

then the series (2.2) converges if and only if

$$\int_{+0}^{\pi}\frac{\Phi_x(u)}{u}\,du$$

exists.

3. Main results

Premchandra [5] has studied the degree of approximation problems for Fourier series by Borel's means. The purpose of the present paper is to investigate the degree of approximation problem of the series (2.2) in Hölder metric by Borel's mean. Before we state the main theorem we first proceed to exhibit the Fourier character of the series (2.2).

Fourier character of the series (2.2)

We write

$$\theta_x(u)=\int_u^{\pi}\frac{\Phi_x(v)}{(2\sin v/2)^2}\,dv, \tag{3.1}$$

$$\Theta_x(t)=-\int_0^t \theta_x(u)\,du. \tag{3.2}$$

For every $f \in \mathrm{Lip}\,\alpha, 0 < \alpha \leqslant 1, \theta \in L(0,\pi)$ and $\Theta_x(t)$ is even, let

$$\Theta_x(t) \sim \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt, \tag{3.3}$$

where

$$\begin{aligned} c_0 = c_0(x) &= \frac{2}{\pi}\int_0^{\pi} \Theta_x(t)\,\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi}\left(-\int_0^t \theta_x(u)\,\mathrm{d}u\right)\mathrm{d}t \\ &= -\frac{2}{\pi}\int_0^{\pi}\mathrm{d}t\left(\int_0^t \theta_x(u)\,\mathrm{d}u\right) = -\frac{2}{\pi}\int_0^{\pi}\theta_x(u)\,\mathrm{d}u\int_u^{\pi}\mathrm{d}t \\ &= -\frac{2}{\pi}\int_0^{\pi}(\pi-u)\,\theta_x(u)\,\mathrm{d}u = -\frac{2}{\pi}\int_0^{\pi}(\pi-u)\,\mathrm{d}u\int_u^{\pi}\frac{\Phi_x(v)}{(2\sin v/2)^2}\,\mathrm{d}v \\ &= -\frac{2}{\pi}\int_0^{\pi}\frac{\Phi_x(v)}{(2\sin v/2)^2}\,\mathrm{d}v\int_0^{v}(\pi-u)\,\mathrm{d}u = -\frac{2}{\pi}\int_0^{\pi}\frac{(\pi v - v^2/2)}{(2\sin v/2)^2}\Phi_x(v)\,\mathrm{d}v \end{aligned} \tag{3.4}$$

and for $n \geqslant 1$

$$\begin{aligned} c_n = c_n(x) &= \frac{2}{\pi}\int_0^{\pi}\Theta_x(t)\cos nt\,\mathrm{d}t \\ &= \frac{2}{\pi}\left[\Theta_x(t)\frac{\sin nt}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\theta_x(t)\frac{\sin nt}{n}\,\mathrm{d}t \\ &= \frac{2}{\pi n}\int_0^{\pi}\theta_x(t)\sin nt\,\mathrm{d}t \\ &= \frac{2}{\pi n}\left[\theta_x(t)\left(\frac{1-\cos nt}{n}\right)\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\frac{\Phi_x(t)}{(2\sin t/2)^2}\left(\frac{1-\cos nt}{n}\right)\mathrm{d}t \\ &= \frac{2}{\pi n^2}\int_0^{\pi}\Phi_x(t)\frac{2\sin^2 nt/2}{(2\sin t/2)^2}\,\mathrm{d}t \\ &= \frac{1}{\pi n^2}\int_0^{\pi}\Phi_x(t)\left(\frac{\sin nt/2}{\sin t/2}\right)^2\mathrm{d}t = \frac{\sigma_n(x)-f(x)}{n}. \end{aligned} \tag{3.5}$$

Theorem. *Let* $0 \leqslant \beta < \alpha \leqslant 1$ *and* $f \in H_\alpha$. *Then*

$$\| B_p(T) + \tfrac{1}{2}c_0 \|_\beta = O(1)\begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha-\beta \neq 1 \\ \dfrac{\log p}{p}, & \alpha-\beta = 1 \ \text{(i.e. } \beta = 0, \alpha = 1) \end{cases}, \tag{3.6}$$

where

$$B_p(T) = B_p(T,x) = \mathrm{e}^{-p}\sum_{k=0}^{\infty}\frac{p^k}{k!}T_k(x)$$

$$T_0(x) = 0,\ T_n(x) = \sum_{k=1}^{n}\frac{\sigma_k(x)-f(x)}{k}, n \geqslant 1$$

and c_0 *is defined by* (3.4)

Additional notations

Throughout the present paper we suppose that $0 < \delta < \pi/4$ and

$$G(t) = \Theta_x(t) - \Theta_y(t), \tag{3.7}$$

$$Q_p = Q_p(t) = \sin(\tfrac{1}{2}t + p \sin t), \tag{3.8}$$

$$H_p(t) = e^{-p} \sum_{n=0}^{\infty} \frac{p^n}{n!} \sin(n + \tfrac{1}{2})t, \tag{3.9}$$

$$l_p(x) = B_p(T, x) + \tfrac{1}{2}c_0, \tag{3.10}$$

$$\eta = \eta(p) = \pi/p, \tag{3.11}$$

$$P = P(p) = \frac{\pi \log p}{\sqrt{p}}. \tag{3.12}$$

By formal computation we see that

$$\begin{aligned} H_p(t) &= \operatorname{Im}\left\{ e^{-p} \sum_{n=0}^{\infty} \frac{p^n}{n!} e^{i(n+\frac{1}{2})t} \right\} \\ &= e^{-p(1-\cos t)} \sin(\tfrac{1}{2}t + p \sin t) \\ &= e^{-p(1-\cos t)} Q_p(t). \end{aligned} \tag{3.13}$$

4. Lemmas

We use the following lemmas:

Lemma 1. *If* $f \in H_\alpha$, $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < \alpha < 1$, *then*

$$\Phi_x(v) - \Phi_y(v) = O(|v|^\alpha), \tag{4.1}$$

$$\Phi_x(v) - \Phi_y(v) = O(|x - y|^\alpha), \tag{4.2}$$

$$\Phi_x(v) - \Phi_y(v) = O(1)\,|x - y|^\beta\,|v|^{\alpha-\beta}. \tag{4.3}$$

Proof. Estimates (4.1) and (4.2) follow immediately from the definition of $\Phi_x(v)$ and H_α. Using (4.2) and (4.1), we get

$$\begin{aligned} \Phi_x(v) - \Phi_y(v) &= [\Phi_x(v) - \Phi_y(v)]^{\beta/\alpha} [\Phi_x(v) - \Phi_y(v)]^{1-\beta/\alpha} \\ &= O(1)\,|x - y|^\beta\,|v|^{\alpha-\beta}. \end{aligned}$$

Lemma 2. *Let* $0 < \delta < \pi/4$. *Then for* $0 \leqslant t \leqslant \delta$

$$\sin(p \sin t) - \sin pt = O(p t^3). \tag{4.4}$$

Proof. We have

$$\sin(p\sin t) - \sin pt = 2\cos\left(\frac{p\sin t + pt}{2}\right)\sin\left(\frac{p\sin t - pt}{2}\right)$$
$$= O(1)\left|\sin\frac{1}{2}p(\sin t - t)\right| = O(pt^3).$$

Lemma 3. *Let* $0 < \delta < \pi/4$. *Then for* $c = 2/\pi^2$ *and* $\eta = \pi/p$

(i) $e^{-p(1-\cos t)} = O(e^{-pct^2})$ (4.5)

(ii) $e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))} = O(pe^{-pct^2})$. (4.6)

Proof. (i) Since

$$1 - \cos t \geqslant \frac{2}{\pi^2}t^2 = ct^2 \text{ for all } 0 \leqslant t \leqslant \pi,$$

the estimate (4.5) follows at once. (ii) We have

$$\exp\{-p(1-\cos t)\} - \exp\{-p(1-\cos(t+\eta)\}$$
$$= \exp\{-p(1-\cos t)\}[1 - \exp\{p(\cos(t+\eta) - \cos t)\}]$$
$$= O(1)p\exp\{-p(1-\cos t)\}|\cos(t+\eta) - \cos t|$$
$$= O(pe^{-pct^2}) \quad \text{by (4.5).}$$

Lemma 4. *If* $f \in H_\alpha$, $0 \leqslant \beta < \alpha \leqslant 1$ *and* $0 < t \leqslant \pi$, *then*

$$\Theta_x(t) = O(1)\begin{cases} t^\alpha, 0 < \alpha < 1 \\ t|\log t|, \alpha = 1 \end{cases}, \tag{4.7}$$

$$G(t) = O(1)\begin{cases} |x-y|^\beta t^{\alpha-\beta}, \alpha - \beta \neq 1 \\ t|\log t|, \alpha - \beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}, \tag{4.8}$$

$$G(t+\eta) - G(t) = O(1)\begin{cases} |x-y|^\beta \eta\, t^{\alpha-\beta-1}, \alpha - \beta \neq 1 \\ \eta|\log t|, \alpha - \beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}. \tag{4.9}$$

Proof of (4.7). Proof follows at once from the definition of $\Theta_x(t)$.

Proof of (4.8). Since $f \in H_\alpha \Rightarrow \Phi_x(v) = O(|v|^\alpha)$ from (3.1) and (3.2), we get

$$\Theta_x(t) = -\int_0^t du\left(\int_u^\pi \frac{\Phi_x(v)}{(2\sin v/2)^2}dv\right)$$
$$= O(1)|x-y|^\beta \int_0^t du \int_u^\pi v^{\alpha-\beta-2}\,dv$$
$$= O(1)\begin{cases} |x-y|^\beta \int_0^t u^{\alpha-\beta-1}\,du, & \alpha - \beta \neq 1 \\ \int_0^t |\log u|\,du, & \alpha - \beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}$$

$$= O(1)\begin{cases} |x-y|^{\beta}\, t^{\alpha-\beta}, & \alpha-\beta \neq 1 \\ t\,|\log t|, & \alpha-\beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}.$$

Proof of (4.9). Using Lemma 1, we get

$$G'(t) = -\int_t^{\pi} \frac{\Phi_x(v) - \Phi_y(v)}{(2\sin v/2)^2}\,dv$$

$$= O(1)|x-y|^{\beta} \int_t^{\pi} v^{\alpha-\beta-2}\,dv$$

$$= O(1)\begin{cases} |x-y|^{\beta}\, t^{\alpha-\beta-1}, & \alpha-\beta \neq 1 \\ |\log t|, & \alpha-\beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}.$$

By mean value theorem for some t_1 with $t < t_1 < t + \eta$

$$G(t+\eta) - G(t) = \eta\, G'(t_1)$$

$$= O(1)\eta \begin{cases} |x-y|^{\beta}\, t^{\alpha-\beta-1}, & \alpha-\beta \neq 1 \\ |\log t|, & \alpha-\beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}.$$

5. Proof of the theorem

Using (3.5) and writing

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin t/2}$$

we have

$$T_n(x) = \sum_{k=1}^{n} \frac{\sigma_k(x) - f(x)}{k} = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} c_k - \tfrac{1}{2}c_0$$

$$= \frac{2}{\pi}\int_0^{\pi} \Theta_x(t)\, D_n(t)\,dt - \tfrac{1}{2}c_0$$

and hence

$$B_p(T,x) = \mathrm{e}^{-p} \sum_{n=0}^{\infty} \frac{p^n}{n!}\, T_n(x)$$

$$= \mathrm{e}^{-p} \sum_{n=0}^{\infty} \left\{\frac{2}{\pi}\int_0^{\pi} \Theta_x(t)\, D_n(t)\,dt - \tfrac{1}{2}c_0\right\}\frac{p^n}{n!}$$

$$= \frac{2}{\pi}\mathrm{e}^{-p}\int_0^{\pi} \Theta_x(t)\left(\sum_{n=0}^{\infty}\frac{p^n}{n!} D_n(t)\right)dt - \tfrac{1}{2}c_0$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t)}{2\sin t/2}\left\{\mathrm{e}^{-p}\sum_{n=0}^{\infty}\frac{p^n}{n!}\sin\left(n+\frac{1}{2}\right)t\right\}dt - \tfrac{1}{2}c_0$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t)}{2\sin t/2}\, H_p(t)\,dt - \tfrac{1}{2}c_0$$

ɔm which it follows that

$$I_p(x) = B_p(T,x) + \frac{1}{2}c_0 = \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t)}{2\sin t/2} H_p(t)\,dt. \tag{5.1}$$

ow by (5.1), (3.7) and (3.13) we get

$$\begin{aligned} I_p(x) - I_p(y) &= \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t) - \Theta_y(t)}{2\sin t/2} H_p(t)\,dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{G(t)}{2\sin t/2} e^{-p(1-\cos t)} \sin\left(\frac{1}{2}t + p\sin t\right) dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{G(t)}{2\sin t/2} e^{-p(1-\cos t)} Q_p(t)\,dt \\ &= \frac{2}{\pi}\left[\int_0^{\eta} + \int_{\eta}^{\delta} + \int_{\delta}^{\pi}\right] \frac{G(t)\,e^{-p(1-\cos t)}}{2\sin t/2} Q_p(t)\,dt \\ &= I_1 + I_2 + I_3, \quad \text{say}. \end{aligned} \tag{5.2}$$

or the proof of our theorem we are required to estimate

$\sup_{x,y} \Delta^{\beta} l_p(x,y) = \sup_{x,y} \frac{|l_p(x) - l_p(y)|}{|x-y|^{\beta}}, (x \neq y)$ and

) $\|l_p\|_c$.

what follows, we first consider $\sup_{x,y} \Delta^{\beta} l_p(x,y)$ in the case when

$$\alpha - \beta \neq 1 \text{ (i.e. } 0 < \beta < \alpha \leqslant 1 \text{ or } \beta = 0, \quad 0 < \alpha < 1).$$

y Lemma 4

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^{\eta} \frac{G(t)\,e^{-p(1-\cos t)}}{2\sin t/2} Q_p(t)\,dt \\ &= O(1)|x-y|^{\beta}\int_0^{\eta} t^{\alpha-\beta}(1+p)\,dt \\ &= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \end{aligned} \tag{5.3}$$

y Lemmas 3 and 4

$$\begin{aligned} I_3 &= \frac{2}{\pi}\int_{\delta}^{\pi} \frac{G(t)}{2\sin t/2} e^{-p(1-\cos t)} Q_p(t)\,dt \\ &= O(1)|x-y|^{\beta}\int_{\delta}^{\pi} t^{\alpha-\beta-1}\, e^{-pct^2}\,dt, (\alpha - \beta \neq 1) \\ &= O(1)\,|x-y|^{\beta}\, e^{-pc\delta^2}\int_{\delta}^{\pi} t^{\alpha-\beta-1}\,dt \\ &= O(1)\frac{|x-y|^{\beta}}{p^{\Delta}}, \text{positive } \Delta \text{ however large.} \end{aligned} \tag{5.4}$$

Now we write

$$\pi I_2 = \int_\eta^\delta \frac{G(t)}{\sin t/2} e^{-p(1-\cos t)} \sin\left(\frac{1}{2}t + p \sin t\right) dt$$

$$= \int_\eta^\delta G(t) e^{-p(1-\cos t)} \cos(p \sin t)\, dt$$

$$+ \int_\eta^\delta G(t) \cot(t/2)\, e^{-p(1-\cos t)} \sin(p \sin t)\, dt$$

$$= I_{2,1} + I_{2,2}, \quad \text{say.} \tag{5.5}$$

Using Lemmas 3 and 4 we have for $\alpha - \beta \neq 1$

$$I_{2,1} = \int_\eta^\delta G(t) e^{-p(1-\cos t)} \cos(p \sin t)\, dt$$

$$= O(1)|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p} \int_\eta^\delta t^{\alpha-\beta-1} \frac{d}{dt}(-e^{-pct^2})\, dt$$

$$= O(1)|x-y|^\beta p^{-1} |[t^{\alpha-\beta-1} e^{-pct^2}]_\delta^\eta|$$

$$+ O(1)|x-y|^\beta p^{-1} \int_\eta^\delta t^{\alpha-\beta-2} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p^{\alpha-\beta}} + O(1)\frac{|x-y|^\beta}{p} \int_\eta^\delta t^{\alpha-\beta-2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p^{\alpha-\beta}} \qquad (\alpha - \beta \neq 1). \tag{5.6}$$

We write

$$I_{2,2} = 2\int_\eta^\delta \frac{G(t)}{2\tan t/2} e^{-p(1-\cos t)} \sin(p \sin t)\, dt$$

$$= 2\int_\eta^\delta \frac{G(t)}{t} e^{-p(1-\cos t)} \sin(p \sin t)\, dt$$

$$+ 2\int_\eta^\delta G(t)\left\{\frac{1}{2\tan(t/2)} - \frac{1}{t}\right\} e^{-p(1-\cos t)} \sin(p \sin t)\, dt$$

$$= J_1 + J_2, \quad \text{say.} \tag{5.7}$$

By Lemmas 3 and 4 we have for $\alpha - \beta \neq 1$

$$J_2 = O(1)|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta+1} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p} \int_\eta^\delta t^{\alpha-\beta} \frac{d}{dt}(-e^{-pct^2})\, dt$$

$$= O(1)\frac{|x-y|^\beta}{p}\left\{|[t^{\alpha-\beta}\,e^{-pct^2}]_\delta^\eta| + (\alpha-\beta)\int_\eta^\delta t^{\alpha-\beta-1}\,e^{-pct^2}\,dt\right\}$$

$$= O(1)\,\frac{|x-y|^\beta}{p^{1+\alpha-\beta}} + O(1)\,\frac{|x-y|^\beta}{p}\int_\eta^\delta t^{\alpha-\beta-1}\,dt$$

$$= O(1)\frac{|x-y|^\beta}{p}. \tag{5.8}$$

Now we write

$$J_1 = 2\int_\eta^\delta \frac{G(t)}{t}\,e^{-p(1-\cos t)}\sin(p\sin t)\,dt$$

$$= 2\int_\eta^\delta \frac{G(t)}{t}\,e^{-p(1-\cos t)}\sin pt\,dt$$

$$+ 2\int_\eta^\delta \frac{G(t)}{t}\,e^{-p(1-\cos t)}\{\sin(p\sin t) - \sin pt\}dt$$

$$= J_{1,1} + J_{1,2}, \qquad \text{say}. \tag{5.9}$$

By Lemmas 2, 3 and 4 we have for $\alpha - \beta \neq 1$

$$J_{1,2} = 2\int_\eta^\delta \frac{G(t)}{t}\,e^{-p(1-\cos t)}\{\sin(p\sin t) - \sin pt\}dt$$

$$= O(1)\,|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta-1}\,pt^3\,e^{-pct^2}\,dt$$

$$= O(1)\,|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta+1}\frac{d}{dt}(-e^{-pct^2})\,dt$$

$$= O(1)\,|x-y|^\beta\,|[t^{\alpha-\beta+1}\,e^{-pct^2}]_\delta^\eta|$$

$$+ O(1)\,|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta}\,e^{-pct^2}\,dt$$

$$= O(1)\,\frac{|x-y|^\beta}{p^{\alpha-\beta}}, \tag{5.10}$$

using (5.6) for the estimation of the last integral.

Now we write

$$J_{1,1} = 2\int_\eta^\delta \frac{G(t)}{t}\,e^{-p(1-\cos t)}\sin pt\,dt$$

$$= \left[\int_\eta^\delta + \int_{2\eta}^{\delta+\eta} + \int_\eta^{2\eta} - \int_\delta^{\delta+\eta}\right]\frac{G(t)}{t}\,e^{-p(1-\cos t)}\sin pt\,dt$$

$$= \int_\eta^\delta \left[\frac{G(t)}{t}\,e^{-p(1-\cos t)} - \frac{G(t+\eta)}{t+\eta}\,e^{-p(1-\cos(t+\eta))}\right]\sin pt\,dt$$

$$+ \left[\int_\eta^{2\eta} - \int_\delta^{\delta+\eta}\right]\frac{G(t)}{t}\,e^{-p(1-\cos t)}\sin pt\,dt$$

$$= \int_{\eta}^{\delta} \left\{ \frac{G(t) - G(t+\eta)}{t} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$+ \int_{\eta}^{\delta} G(t+\eta) \left\{ \frac{1}{t} - \frac{1}{t+\eta} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$+ \int_{\eta}^{\delta} \frac{G(t+\eta)}{t+\eta} \{ e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))} \} \sin pt \, dt$$

$$+ \left(\int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta} \right) \frac{G(t)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= R_1 + R_2 + R_3 + R_4 - R_5, \quad \text{say.} \tag{5.11}$$

We write

$$R_1 = \left(\int_{\eta}^{P} + \int_{P}^{\delta} \right) \left\{ \frac{G(t) - G(t+\eta)}{t} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= R_{1,1} + R_{1,2}, \quad \text{say.} \tag{5.12}$$

Using (4.9) of Lemma 4, we get

$$R_{1,1} = \int_{\eta}^{P} \frac{G(t) - G(t+\eta)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1) |x-y|^{\beta} \eta \int_{\eta}^{P} t^{\alpha-\beta-2} \, dt$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}} (\alpha - \beta \neq 1). \tag{5.13}$$

Again using (4.9) and Lemma 3

$$R_{1,2} = \int_{P}^{\delta} \frac{G(t) - G(t+\eta)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1) |x-y|^{\beta} \eta \int_{P}^{\delta} t^{\alpha-\beta-2} e^{-pct^2} dt$$

$$= O(1) \frac{|x-y|^{\beta}}{p} e^{-pc\delta^2} \int_{P}^{\delta} t^{\alpha-\beta-2} \, dt$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\Delta}}, \text{ positive } \Delta \text{ however large.} \tag{5.14}$$

From (5.12), (5.13) and (5.14), we have

$$R_1 = O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \tag{5.15}$$

Using Lemmas 3 and 4 and adopting the arguments similar to those used in estimating $I_{2,1}$, we get

$$R_3 = \int_{\eta}^{\delta} \frac{G(t+\eta)}{t+\eta} \{e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))}\} \sin pt \, dt$$

$$= O(1)|x-y|^{\beta} \int_{\eta}^{\delta} t^{\alpha-\beta} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1). \tag{5.16}$$

By Lemma 4, we have

$$R_4 = \int_{\eta}^{2\eta} \frac{G(t)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1)|x-y|^{\beta} \int_{\eta}^{2\eta} t^{\alpha-\beta-1} dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1). \tag{5.17}$$

Again by Lemmas 3 and 4

$$R_5 = \int_{\delta}^{\delta+\eta} \frac{G(t)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1)|x-y|^{\beta} \int_{\delta}^{\delta+\eta} t^{\alpha-\beta-1} e^{-pct^2} dt, \quad (\alpha-\beta \neq 1)$$

$$= O(1)|x-y|^{\beta} e^{-pc\delta^2} \int_{\delta}^{\delta+\eta} t^{\alpha-\beta-1} dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{\Delta}}, \quad \text{positive } \Delta \text{ however large.} \tag{5.18}$$

Now we write

$$2R_2 = 2\int_{\eta}^{\delta} G(t+\eta)\left\{\frac{1}{t} - \frac{1}{t+\eta}\right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= 2\eta \int_{\eta}^{\delta} \frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= \eta\left[\int_{\eta}^{\delta} + \int_{2\eta}^{\delta+\eta} + \int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta}\right] \frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= \eta \int_{\eta}^{\delta} \left\{\frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} - \frac{G(t+\eta)}{(t+\eta)(t+2\eta)} e^{-p(1-\cos(t+\eta))}\right\} \sin pt \, dt$$

$$+ \eta\left[\int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta}\right] \frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= \eta \int_{\eta}^{\delta} \frac{G(t+\eta) - G(t+2\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$+ \eta \int_{\eta}^{\delta} \frac{G(t+2\eta)}{t+\eta} \left\{ \frac{1}{t} - \frac{1}{t+2\eta} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$+ \eta \int_{\eta}^{\delta} \frac{G(t+2\eta)}{(t+\eta)(t+2\eta)} \{ e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))} \} \sin pt \, dt$$

$$+ \eta \left\{ \int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta} \right\} \frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= L_1 + L_2 + L_3 + L_4 - L_5, \text{ say.} \tag{5.19}$$

By (4.9) of Lemma 4, we get

$$L_1 = \eta \int_{\eta}^{\delta} \frac{G(t+\eta) - G(t+2\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1) |x-y|^{\beta} \eta^2 \int_{\eta}^{\delta} t^{\alpha-\beta-3} \, dt$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \tag{5.20}$$

By Lemma 4 $(\alpha - \beta \neq 1)$, we have

$$L_2 = \eta \int_{\eta}^{\delta} \frac{G(t+\eta)}{t+\eta} \left\{ \frac{1}{t} - \frac{1}{t+2\eta} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1) |x-y|^{\beta} \eta^2 \int_{\eta}^{\delta} t^{\alpha-\beta-3} \, dt$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}. \tag{5.21}$$

Usinig Lemmas 3, 4 and integrating by parts, we have $(\alpha - \beta \neq 1)$

$$L_3 = \eta \int_{\eta}^{\delta} \frac{G(t+2\eta)}{(t+\eta)(t+2\eta)} \{ e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))} \} \sin pt \, dt$$

$$= O(1) |x-y|^{\beta} \eta \int_{\eta}^{\delta} t^{\alpha-\beta-2} (t e^{-pct^2}) \, dt$$

$$= O(1) |x-y|^{\beta} \eta^2 \int_{\eta}^{\delta} t^{\alpha-\beta-2} \frac{d}{dt} (-e^{-pct^2}) \, dt$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \tag{5.22}$$

By Lemma 4, $(\alpha - \beta \neq 1)$ we have

$$L_4 = \eta \int_{\eta}^{2\eta} \frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1)|x-y|^{\beta}\eta \int_{\eta}^{2\eta} t^{\alpha-\beta-2}\,dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, (\alpha-\beta \neq 1). \tag{5.23}$$

Lastly by Lemma 4 $(\alpha-\beta \neq 1)$, we get

$$L_5 = \eta \int_{\delta}^{\delta+\eta} \frac{G(t+\eta)}{t(t+\eta)} e^{-p(1-\cos t)} \sin pt\,dt$$

$$= O(1)|x-y|^{\beta}\eta \int_{\delta}^{\delta+\eta} t^{\alpha-\beta-2}\,dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p}, \quad (\alpha-\beta \neq 1). \tag{5.24}$$

Collecting the results from (5.19) to (5.24), we get

$$R_2 = O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1)\cdot \tag{5.25}$$

From (5.11), (5.15), (5.16), (5.17), (5.18) and (5.25) we get

$$J_{1,1} = O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1)\cdot \tag{5.26}$$

Collecting the results from (5.5) to (5.10) and using (5.26) we get

$$I_2 = O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1)\cdot \tag{5.27}$$

Using the estimates of $I_k (k = 1, 2, 3)$ in (5.2) (note that estimate of I_2 dominates over that of I_1 and I_3), we obtain for $\alpha-\beta \neq 1$

$$\sup_{\substack{x,y \\ x\neq y}} (\Delta^{\beta} l_p(x,y)) = \sup_{x,y} \frac{|l_p(x)-l_p(y)|}{|x-y|^{\beta}} = O(1)\frac{1}{p^{\alpha-\beta}}\cdot \tag{5.28}$$

In the case where $\alpha-\beta = 1$, (i.e., $\beta = 0$, $\alpha = 1$) proceeding in the lines similar to those used above in the case $\alpha-\beta \neq 1$, we can prove that

$$\sup_{\substack{x,y \\ x\neq y}} (\Delta^{\beta} l_p(x,y)) = O(1)\frac{\log p}{p}, \ (\alpha-\beta = 1). \tag{5.29}$$

As $f \in H_{\alpha} \Rightarrow \Phi_x(t) = O(|t|^{\alpha})$ and so proceeding as above (in fact cases $\beta = 0$ of (5.28) and (5.29)), we get

$$\|l_p(x)\|_c = O(1)\begin{cases} \dfrac{1}{p^{\alpha}}, & 0<\alpha<1 \\ \dfrac{\log p}{p}, & \alpha = 1 \end{cases} \tag{5.30}$$

From (5.28), (5.29) and (5.30), we have

$$\|l_p(x)\|_\beta = \|l_p(x)\|_c + \sup_{\substack{x,y \\ x \neq y}} (\Delta^\beta l_p(x,y))$$

$$= O(1) \begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \dfrac{\log p}{p}, & \alpha - \beta \equiv 1, \quad (\text{i.e.}, \beta = 0, \alpha = 1) \end{cases}.$$

This completes the proof of the theorem.

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On a general theorem concerning some absolute summability methods

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Abstract. In this note a general theorem covering several absolute summability methods e.g, $|\bar{N}, p_n|_k, |R, p_n|_k$, is proved.

Keywords. Summability methods

1. Introduction

Let Σa_n be a given infinite series with the sequence of partial sums (s_n). By u_n we denote the nth $(C, 1)$ mean of the sequence (s_n). The series Σa_n is said to be summable $|C, 1|_k$, $k \geqslant 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \quad (P_{-i} = p_{-i} = 0, \quad i \geqslant 1).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]). The series Σa_n is said to be summable $|R, p_n|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty.$$

The series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, (see [1]), if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then both of $|R, p_n|_k$ and $|\bar{N}, p_n|_k$ is the same as $|C, 1|_k$ (resp. $|R, p_n|$, $|\bar{N}, p_n|$) summability.

We assume that (f_n), (g_n) and (h_n) are positive sequences.

2. Main theorem

Theorem. *Let* $Y_n = (1/F_{n-1}H_n)\,\Sigma_{v=1}^{n} F_{v-1} x_v \xi_v$ *such that* $F_n = \Sigma_{v=1}^{n} f_v \to \infty$,

$X_n = (1/G_n)\,\Sigma_{v=1}^{n} g_v x_v$. *Suppose that*

$$F_n \Delta g_n^{-1} = O(f_n g_n^{-1}),$$

$$\sum_{n=v+1}^{\infty} \frac{1}{F_{n-1} H_n^k} = O\left(\frac{1}{f_v H_v^k}\right),$$

and

$$\sum_{n=1}^{\infty} \left| \Delta \left| \frac{1}{F_{n-1}} \sum_{v=1}^{n-1} \frac{F_v G_v}{g_{v+1} H_v} \Delta \varepsilon_v \right|^k \right| < \infty.$$

Then the necessary and sufficient conditions that $\Sigma |Y_n|^k < \infty$ whenever $\Sigma |X_n|^k < \infty$ are

(i) $\varepsilon_n = O\left\{\dfrac{g_n H_n}{G_n}\right\}$,

(ii) $\Delta \varepsilon_n = O\left\{\dfrac{g_{n+1} H_n f_n}{G_n F_n}\right\}$.

Remark. It may be mentioned that on putting $f_n = q_n$, $g_n = P_{n-1}$, we obtain

1. $|\bar{N}, q_n|_k$, $|R, q_n|_k$ summability of $\Sigma a_n \varepsilon_n$ by putting $H_n = (Q_n/q_n)^{1/k}$, $H_n = n^{1/k-1}(Q_n/q_n)$ respectively, and
2. $|\bar{N}, p_n|_k$, $|R, p_n|_k$ summability of Σa_n by putting $G_n = (P_n/p_n)^{1/k} P_{n-1}$, $G_n = n^{1/k-1}(P_n P_{n-1}/p_n)$ respectively.

3. Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1. (*Bor* [2]). *Let* $k \geqslant 1$ *and let* $A = (a_{nv})$ *be an infinite matrix. In order that* $A\varepsilon(l^k; l^k)$ *it is necessary that*

$$a_{nv} = O(1) \quad (\textit{all } n, v).$$

Lemma 2. *Let* $\Sigma_{v=1}^{n} x_v = X_n$, $x_v \geqslant 0$, $\Sigma_{v=1}^{n} x_v \varepsilon_v = O(X_n)$, $\Sigma |\Delta \varepsilon_n| < \infty$. *Then* $\varepsilon_n = O(1)$.

Proof. Abel's transformation gives

$$\sum_{v=1}^{n-1} X_v(\Delta \varepsilon_v) + X_n \varepsilon_n = O(X_n)$$

$$X_n |\varepsilon_n| = O(X_n) + O(X_n) \sum_{v=1}^{n-1} |\Delta \varepsilon_n|$$

$$= O(X_n) + O(X_n)$$

$$= O(X_n),$$

Then

$$\varepsilon_n = O(1).$$

. Proof of the theorem

Sufficiency. We have via Abel's transformation

$$\begin{aligned} Y_n &= \frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v \Delta(F_{v-1} g_v^{-1}\varepsilon_v) + \frac{G_n X_n}{g_n H_n}\varepsilon_n \\ &= \frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1}\{-G_v X_v f_v g_v^{-1}\varepsilon_v + G_v X_v\, F_v \Delta g_v^{-1}\varepsilon_v \\ &\quad + G_v X_v\, F_v g_{v+1}^{-1}\Delta\varepsilon_v\} + \frac{G_n X_n}{g_n H_n}\varepsilon_n \qquad (2) \\ &= Y_{n,1} + Y_{n,2} + Y_{n,3} + Y_{n,4}, \text{ say.} \end{aligned}$$

By Minkowski's inequality,

$$\sum_{n=1}^{m} |Y_n|^k = O(1)\sum_{n=1}^{m}\sum_{r=1}^{4} |Y_{n,r}|^k.$$

Applying Hölder's inequality

$$\begin{aligned} \sum_{n=2}^{m+1} |Y_{n,1}|^k &= \sum_{n=2}^{m+1}\left|\frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v f_v g_v^{-1}\varepsilon_v\right|^k \\ &\leqslant \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{v=1}^{n-1} G_v^k X_v^k f_v g_v^{-k}|\varepsilon_v|^k\left\{\frac{1}{F_{n-1}}\sum_{v=1}^{n-1} f_v\right\}^{k-1} \\ &\leqslant O(1)\sum_{v=1}^{m} G_v^k X_v^k f_v g_v^{-k}|\varepsilon_v|^k \sum_{n=v+1}^{m+1}\frac{1}{F_{n-1}H_n^k} \\ &\leqslant O(1)\sum_{v=1}^{m}\frac{G_v^k X_v^k}{H_v^k g_v^k}|\varepsilon_v|^k, \end{aligned}$$

$$\begin{aligned} \sum_{n=2}^{m+1} |Y_{n,2}|^k &= \sum_{n=2}^{m+1}\left|\frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v\left(\frac{F_v}{f_v}\right) f_v \Delta g_v^{-1}\varepsilon_v\right|^k \\ &\leqslant \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{v=1}^{n-1} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v|\Delta g_v^{-1}|^k|\varepsilon_v|^k\left\{\frac{1}{F_{n-1}}\sum_{v=1}^{n-1} f_v\right\}^{k-1} \\ &\leqslant O(1)\sum_{v=1}^{m} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v|\Delta g_v^{-1}|^k|\varepsilon_v|^k \sum_{n=v+1}^{m+1}\frac{1}{F_{n-1}H_n^k} \\ &\leqslant O(1)\sum_{v=1}^{m}\frac{G_v^k X_v^k}{H_v^k g_v^k}|\varepsilon_v|^k, \end{aligned}$$

$$\begin{aligned} \sum_{n=2}^{m+1} |Y_{n,3}|^k &= \sum_{n=2}^{m+1}\left|\frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v^k X_v^k\left(\frac{F_v}{f_v}\right) f_v g_{v+1}^{-1}\Delta\varepsilon_v\right|^k \\ &\leqslant \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{v=1}^{n-1} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v g_{v+1}^{-k}|\Delta\varepsilon_v|^k\left\{\frac{1}{F_{n-1}}\sum_{v=1}^{n-1} f_v\right\}^{k-1} \\ &\leqslant O(1)\sum_{v=1}^{m} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v g_{v+1}^{-k}|\Delta\varepsilon_v|^k \sum_{n=v+1}^{m+1}\frac{1}{F_{n-1}H_n^k} \end{aligned}$$

$$\leqslant O(1)\sum_{v=1}^{m}\frac{G_v^k X_v^k F_v^k}{H_v^k g_{v+1}^k f_v^k}|\Delta\varepsilon_v|^k,$$

$$\sum_{n=1}^{m}|Y_{n,4}|^k=\sum_{n=1}^{m}\left|\frac{G_n X_n}{g_n H_n}\varepsilon_n\right|^k$$

$$\leqslant O(1)\sum_{n=1}^{m}\frac{G_n^k X_n^k}{g_n^k H_n^k}|\varepsilon_n|^k.$$

The result follows by the hypothesis.

Necessity of (i). By (1), we are able to write the matrix transforming (X_n) into (Y_n). Since $\Sigma|X_n|^k<\infty\Rightarrow\Sigma|Y_n|^k<\infty$, the matrix $\varepsilon(l^k;l^k)$. By lemma 1, a necessary condition for this implication is that the elements (in particular the diagonal elements) of this matrix should be bounded. Hence (i).

Necessity of (ii). From (2), we have

$$|Y_{n,3}|\leqslant|Y_{n,1}|+|Y_{n,2}|+|Y_{n,4}|+|Y_n|.$$

By Minkowski's inequality,

$$\sum_{n=1}^{m}|Y_{n,3}|^k\leqslant O(1)\sum_{n=1}^{m}\{|Y_{n,1}|^k+|Y_{n,2}|^k+|Y_{n,4}|^k+|Y_n|^k\}.$$

Suppose that $\Sigma|X_n|^k<\infty\Rightarrow\Sigma|Y_n|^k<\infty$. Then we have via the proof of sufficiency using (i),

$$\sum_{n=1}^{m}|Y_{n,3}|^k\leqslant O(1)\sum_{n=1}^{m}|X_n|^k,$$

$$\sum_{n=1}^{m}\frac{1}{H_n^k}\left|\frac{1}{F_{n-1}}\sum_{v=1}^{n-1}G_v X_v F_v g_{v+1}^{-1}\Delta\varepsilon_v\right|^k=O(1)\sum_{n=1}^{m}|X_n|^k.$$

Now, put $X_n=1/H_n$, we obtain

$$\sum_{n=1}^{m}\frac{1}{H_n^k}\left|\frac{1}{F_{n-1}}\sum_{v=1}^{n-1}f_v\left\{\frac{F_v G_v}{f_v g_{v+1}H_v}\Delta\varepsilon_v\right\}\right|^k=O(1)\sum_{n=1}^{m}1/H_n^k.$$

This, by Lemma 2, implies

$$\frac{1}{F_{n-1}}\sum_{v=1}^{n-1}f_v\left|\frac{F_v G_v}{f_v g_{v+1}H_v}\Delta\varepsilon_v\right|^k=O(1).$$

But

$$\sum_{v=1}^{n-1}f_v=F_{n-1}\to\infty,$$

Then, we should have

$$\frac{F_v G_v}{f_v g_{v+1}H_v}\Delta\varepsilon_v=O(1).$$

This completes the proof of the theorem.

5. Applications

With the aid of the remark, the following results are consequences of the theorem.

Theorem 2. *Let $p_nQ_n = O(P_nq_n)$. Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$ is summable $|N, q_n|_k$, whenever Σa_n is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, are*

$$\varepsilon_n = O\left\{\left(\frac{p_nQ_n}{P_nq_n}\right)^{1/k}\right\}, \quad \Delta\varepsilon_n = O\left\{\left(\frac{p_n}{P_{n-1}}\right)\left(\frac{P_nq_n}{p_nQ_n}\right)^{1-1/k}\right\},$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_{v-1}Q_v}{P_v}\right)\left(\frac{P_vq_v}{p_vQ_v}\right)^{1/k}\Delta\varepsilon_v\right|^k\right| < \infty.$$

Theorem 3. *Let $p_nQ_n = O(P_nq_n)$ and*

$$\sum_{n=v}^{m}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}} = O\left(\frac{v^{k-1}q_v^{k-1}}{Q_v^k}\right). \tag{3}$$

Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$ is summable $|R, q_n|_k$ whenever Σa_n is summable $|R, p_n|_k$, $k \geqslant 1$, are

$$\varepsilon_n = O\left(\frac{p_nQ_n}{P_nq_n}\right), \quad \Delta\varepsilon_n = O\left(\frac{p_n}{P_{n-1}}\right),$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}q_v}{p_v}\Delta\varepsilon_v\right|^k\right| < \infty.$$

Theorem 4. *Let $p_nQ_n = O(P_nq_n)$. Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$ is summable $|\bar{N}, q_n|_k$ whenever Σa_n is summable $|R, p_n|_k$, $k \geqslant 1$, are*

$$\varepsilon_n = O\left\{\left(\frac{np_n}{P_n}\right)\left(\frac{Q_n}{nq_n}\right)^{1/k}\right\}, \quad \Delta\varepsilon_n = O\left\{\left(\frac{p_n}{P_{n-1}}\right)\left(\frac{nq_n}{Q_n}\right)^{1-1/k}\right\},$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}Q_v}{vp_v}\left(\frac{vq_v}{Q_v}\right)^{1/k}\Delta\varepsilon_n\right|^k\right| < \infty.$$

Theorem 5. *Let $p_nQ_n = O(P_nq_n)$ and (3) is satisfied. Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$ is summable $|R, q_n|_k$ whenever Σa_n is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, are*

$$\varepsilon_n = O\left\{\left(\frac{Q_n}{nq_n}\right)\left(\frac{np_n}{P_n}\right)^{1/k}\right\}, \quad \Delta\varepsilon_n = O\left\{\left(\frac{p_n}{P_{n-1}}\right)\left(\frac{P_n}{np_n}\right)^{1-1/k}\right\},$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{vq_vP_{v-1}}{P_v}\right)\left(\frac{P_v}{vp_v}\right)^{1/k}\right|^k\right| < \infty.$$

Theorem 6. (Bor and Thorpe [1]). *Let* $p_nQ_n = O(P_nq_n)$ *and* $P_nq_n = O(p_nQ_n)$. *Then* Σa_n *is summable* $|\bar{N}, q_n|_k$ *iff it is summable* $|\bar{N}, p_n|_k$, $k \geqslant 1$.
The proof follows from theorem 2 by putting $\varepsilon_n = 1$.

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Inequalities for the derivative of a polynomial

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < k$, where $k > 0$. For $k \leqslant 1$, it is known that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, where $Q(z) = z^n \overline{P(1/\bar{z})}$. In this paper we obtain certain refinements of this result. We also present a refinement of a generalization of the theorem of Turán.

Keywords. Maximum modules; inequalities; polynomials.

1. Introduction and statements of results

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ on the unit disk $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1}$$

This inequality is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial (for reference, see [9]). In (1) equality holds if and only if $P(z)$ has all its zeros at the origin, and so it is natural to seek improvement in (1) under appropriate assumptions on the zeros of $P(z)$. It was conjectured by Erdös and later verified by Lax [7] (see also [2]) that if $P(z)$ does not vanish in $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

On the other hand, Turán [10] showed that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

Thus in (2) as well as in (3) equality holds for those polynomials of degree n which have all their zeros on $|z| = 1$.

As an extension of (2) and (3), Malik [8] proved that if $P(z) \neq 0$ in $|z| < k$ where $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{4}$$

whereas if $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{5}$$

As an analogous result to (4) in case $k \leqslant 1$, it was shown by Govil [4] that if $P(z) \neq 0$ in $|z| < k, k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{6}$$

provided $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on the circle $|z| = 1$. Here $Q(z) = z^n \overline{P(1/\bar{z})}$.

In this paper we shall first present a refinement of (6) and obtain a bound that depends on the location of all the zeros of $P(z)$. We prove the following theorem.

Theorem 1. *Let*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n which does not vanish in $|z| < k$ where $k \leqslant 1$, and let

$$Q(z) = z^n \overline{P(1/\bar{z})}.$$

If $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on the circle $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1}{1+k^n} \left\{ n - k^n \sum_{j=1}^{n} \frac{|z_j| - k}{|z_j| + k} \right\} \max_{|z|=1} |P(z)|. \tag{7}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + k^n$, where $k \leqslant 1$.

Next we present the following refinement of Malik's result (5).

Theorem 2. *Let*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n. If $|z_j| \leqslant k_j \leqslant 1, 1 \leqslant j \leqslant n$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \left\{ 1 + \frac{1}{1 + \left(\frac{2}{n}\right) \sum_{j=1}^{n} \frac{k_j}{1 - k_j}} \right\} \max_{|z|=1} |P(z)|. \tag{8}$$

The result is best possible and equality holds for the polynomial $P(z) = (z + k)^n$ where $k \leqslant 1$.

It can be easily seen that Theorem 2 includes as special cases Turán's result (3) and Malik's result (5).

Theorem 3. *Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < k$ where $k \leqslant 1$ and $Q(z) = z^n \overline{P(1/\bar{z})}$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right. \tag{9}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + k^n$, where $k \leqslant 1$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1. *If $P(z)$ is a polynomial of degree n, then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leqslant n \max_{|z|=1} |P(z)|,$$

where

$$Q(z) = z^n \overline{P(1/\bar{z})}.$$

This is a special case of a result due to Govil and Rahman [6] (see also [2]).

We also need the following result which is due to Aziz [1, Theorem 1].

Lemma 2. *If all the zeros of the polynomial*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

of degree n lie in $|z| \leqslant k$ where $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{2}{1+k^n} \sum_{j=1}^{n} \frac{k}{k+|z_j|} \max_{|z|=1} |P(z)|.$$

The next result which is due to Gardner and Govil [3] is needed for the proof of Theorem 2.

Lemma 3. *Let*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n. If $|z_j| \geqslant k_j \geqslant 1, 1 \leqslant j \leqslant n$, then for $|z| = 1$

$$|Q'(z)/P'(z)| \geqslant 1 + \left(\frac{n}{\sum_{j=1}^{n} \frac{1}{k_j - 1}} \right).$$

For the proof of Theorem 3, we need Lemma 4, due to Govil [5].

Lemma 4. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \geqslant 1$, then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}.$$

3. Proofs of the theorems

Proof of Theorem 1. Since all the zeros of $P(z)$ lie in $|z| \geqslant k, k \leqslant 1$, it follows that all the zeros of

$$Q(z) = z^n \overline{P(1/\bar{z})}$$

lie in $|z| \leqslant 1/k, 1/k \geqslant 1$ and $|P(z)| = |Q(z)|$ for $|z| = 1$.

Applying Lemma 2 to the polynomial $Q(z)$, we get

$$\max_{|z|=1} |Q'(z)| \geqslant \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \max_{|z|=1} |Q(z)|$$

$$= \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \max_{|z|=1} |P(z)|. \tag{10}$$

Since by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, if

$$\max_{|z|=1} |P'(z)| = |P'(e^{i\alpha})|, \quad 0 \leqslant \alpha < 2\pi, \tag{11}$$

then

$$\max_{|z|=1} |Q'(z)| = |Q'(e^{i\alpha})|. \tag{12}$$

Now by Lemma 1, we have

$$|P'(e^{i\alpha})| + |Q'(e^{i\alpha})| \leqslant n \max_{|z|=1} |P(z)|.$$

This gives with the help of (10), (11) and (12)

$$n \max_{|z|=1} |P(z)| \geqslant \max_{|z|=1} |P'(z)| + \max_{|z|=1} |Q'(z)|$$

$$\geqslant \max_{|z|=1} |P'(z)| + \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \max_{|z|=1} |P(z)|,$$

which implies

$$\max_{|z|=1} |P'(z)| \leqslant \left\{ n - \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \right\} \max_{|z|=1} |P(z)|,$$

and after simplification, we get

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1}{1+k^n} \left\{ n - k^n \sum_{j=1}^{n} \frac{|z_j|-k}{|z_j|+k} \right\} \max_{|z|=1} |P(z)|.$$

This proves the desired result.

Proof of Theorem 2. Since

$$P(z) = \prod_{j=1}^{n} (z - z_j),$$

it follows that

$$Q(z) = z^n \overline{P(1/\bar{z})} = \prod_{j=1}^{n} (1 - z\bar{z}_j)$$

and

$$|P(z)| = |Q(z)| \quad \text{for} \quad |z| = 1.$$

Now by hypothesis $|z_j| \leqslant k_j \leqslant 1$, $j = 1, 2, \ldots, n$. Therefore, $|1/z_j| \geqslant |1/k_j| \geqslant 1$, $j = 1, 2, \ldots, n$ and hence by Lemma 3, for $|z| = 1$

$$|P'(z)/Q'(z)| \geqslant 1 + \frac{n}{\sum_{j=1}^{n} \frac{k_j}{1 - k_j}}$$

$$= \frac{\sum_{j=1}^{n} \left(\frac{k_j}{1 - k_j} + 1 \right)}{\sum_{j=1}^{n} \frac{k_j}{1 - k_j}}$$

$$= \frac{\sum_{j=1}^{n} \frac{1}{1 - k_j}}{\sum_{j=1}^{n} \frac{k_j}{1 - k_j}}$$

which gives, for $|z| = 1$

$$|Q'(z)/P'(z)| \leqslant \frac{\sum_{j=1}^{n} \frac{k_j}{1 - k_j}}{\sum_{j=1}^{n} \frac{1}{1 - k_j}}$$

$$= \frac{\sum_{j=1}^{n} \left(\frac{1}{1 - k_j} - 1 \right)}{\sum_{j=1}^{n} \frac{1}{1 - k_j}}$$

$$= 1 - \frac{n}{\sum_{j=1}^{n} \frac{1}{1 - k_j}}.$$

Hence

$$t|P'(z)| \geqslant |Q'(z)| \quad \text{for } |z| = 1, \tag{13}$$

where

$$t = 1 - \frac{n}{\sum_{j=1}^{n} \frac{1}{1 - k_j}}.$$

Also, we have

$$Q'(z) = n z^{n-1} \overline{P(1/\bar{z})} - z^{n-2} \overline{P'(1/\bar{z})}.$$

Therefore, for $z = e^{i\theta}, 0 \leqslant \theta < 2\pi$

$$Q'(e^{i\theta}) = n e^{i(n-1)\theta} \overline{P(e^{i\theta})} - e^{i(n-2)\theta} \overline{P'(e^{i\theta})}, \tag{14}$$

which gives for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \geqslant n|P(z)| - |P'(z)|$$

or

$$|P'(z)| + |Q'(z)| \geqslant n|P(z)| \quad \text{for} \quad |z| = 1. \tag{15}$$

Using (13) in (15), we obtain for $|z| = 1$

$$\begin{aligned}(1+t)|P'(z)| &= |P'(z)| + t|P'(z)| \\ &\geqslant |P'(z)| + |Q'(z)| \geqslant n|P(z)|.\end{aligned}$$

Equivalently,

$$|P'(z)| \geqslant \frac{n}{1+t}|P(z)| \quad \text{for} \quad |z| = 1, \tag{16}$$

where t is defined by (13).

Since

$$\begin{aligned}\frac{1}{1+t} &= \frac{1}{2 - \left\{ n \Big/ \sum_{j=1}^{n} \frac{1}{1-k_j} \right\}} = \sum_{j=1}^{n} \frac{1}{1-k_j} \Big/ \sum_{j=1}^{n} \left[\frac{2}{1-k_j} - 1 \right] \\ &= \frac{1}{2} \sum_{j=1}^{n} \frac{1+k_j+1-k_j}{1-k_j} \Big/ \sum_{j=1}^{n} \frac{1+k_j}{1-k_j} \\ &= \frac{1}{2}\left[1 + n \Big/ \sum_{j=1}^{n} \frac{1+k_j}{1-k_j} \right] = \frac{1}{2}\left[1 + \left(n \Big/ \frac{\Sigma 1 - k_j + 2k_j}{1-k_j} \right] \right. \\ &= \frac{1}{2}\left[1 + n \Big/ \left(n + 2 \sum_{j=1}^{n} \frac{k_j}{1-k_j} \right) \right] \\ &= \frac{1}{2}\left[1 + \frac{1}{1 + \frac{2}{n} \sum_{j=1}^{n} \frac{k_j}{1-k_j}} \right],\end{aligned}$$

from (16) we get

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2}\left[1 + \frac{1}{1 + \frac{2}{n} \sum_{j=1}^{n} \frac{k_j}{1-k_j}} \right] \max_{|z|=1} |P(z)|,$$

and this completes the proof of Theorem 2.

roof of Theorem 3. Since all the zeros of $P(z)$ lie in $|z| \geqslant k$ where $k \leqslant 1$, all the zeros $Q(z) = z^n\overline{P(1/\bar{z})}$ lie in $|z| \leqslant (1/k)$ where $(1/k) \geqslant 1$ and $|Q(z)| = |P(z)|$ for $|z| = 1$. Apping Lemma 4 to the polynomial $Q(z)$, we get

$$\max_{|z|=1} |Q'(z)| \geqslant \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|Q(z)| + \min_{|z|=(1/k)} |Q(z)|\right\}$$

$$= \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=(1/k)} |Q(z)|\right\}. \tag{17}$$

low

$$\min_{|z|=(1/k)} |Q(z)| = \min_{|z|=1} |Q(z/k)|$$

$$= \min_{|z|=1} \left|\frac{z^n}{k^n}\overline{P(k/\bar{z})}\right|$$

$$= \frac{1}{k^n}\min_{|z|=k} |P(z)|.$$

Using this in (17), we obtain

$$\max_{|z|=1} |Q'(z)| \geqslant \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \frac{1}{k^n}\min_{|z|=k} |P(z)|\right\}. \tag{18}$$

ince by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, we choose α such that

$$\max_{|z|=1} |P'(z)| = |P'(e^{i\alpha})| \quad \text{where} \quad 0 \leqslant \alpha < 2\pi,$$

hen

$$\max_{|z|=1} |Q'(z)| = |Q'(e^{i\alpha})|.$$

Hence from (18), we have

$$|Q'(e^{i\alpha})| \geqslant \frac{nk^n}{1+k^n}\left\{\max_{|z|=1} |P(z)| + \frac{1}{k^n}\min_{|z|=1} |P(z)|\right\}.$$

This in conjunction with Lemma 1 gives

$$n\max_{|z|=1}|P(z)| \geqslant |P'(e^{i\alpha})| + |Q'(e^{i\alpha})|$$

$$\geqslant |P'(e^{i\alpha})| + \frac{nk^n}{1+k^n}\left\{\max_{|z|=1} |P(z)| + \frac{1}{k^n}\min_{|z|=1} |P(z)|\right\}.$$

This implies

$$|P'(e^{i\alpha})| \leqslant \frac{n}{1+k^n}\left\{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\right\},$$

which is equivalent to the desired result.

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On self-reciprocal polynomials

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Abstract. In this paper we establish a sharp result concerning integral mean estimates for self-reciprocal polynomials.

Keywords. Integral mean estimates; self-reciprocal polynomials; inequalities.

Let $P(z)$ be a polynomial of degree at most n. Then $P(z)$ is said to be self-reciprocal if it satisfies the condition

$$z^n P(1/z) = P(z) \quad \text{for all} \quad z \in C. \tag{1}$$

Polynomials $P(z)$ satisfying (1) were studied in the past by other authors (see for example [1, 2, 4–7]). Govil, Jain and Labelle [6] have proved that if $P(z)$ is a polynomial satisfying (1) and if in addition $P(z)$ has all its zeros either in the left half plane or in the right half plane, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2}} \max_{|z|=1} |P(z)| \tag{2}$$

and

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

It was shown by Dewan and Govil [3] and independently by the first author [1, Theorem 5] that inequality (3) holds for all polynomials $P(z)$ satisfying (1). Here we present the following interesting result which is L^2 analogue of (2) and (3) for all polynomials satisfying (1).

Theorem. *If $P(z)$ is a self-reciprocal polynomial of degree at most n,* then

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P'(e^{i\theta})|^2 d\theta\right\}^{1/2} \leqslant \frac{n}{\sqrt{2}}\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta\right\}^{1/2} \tag{4}$$

and

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P'(e^{i\theta})|^2 d\theta\right\}^{1/2} \geqslant \frac{n}{2}\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta\right\}^{1/2}. \tag{5}$$

Both the estimates are sharp. Equality in (4) *holds for $P(z) = c(z^n + 1)$ for all $n \geqslant 1$ and in* (5) *for $P(z) = cz^{n/2}$ where n is an even positive integer.*

Proof of the Theorem. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree at most n and let

$$Q(z) = z^n P(1/z) = a_n + a_{n-1}z + \cdots + a_1 z^{n-1} + a_0 z^n.$$

Then we have

$$P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + 2a_2 z + a_1$$

and

$$Q'(z) = na_0 z^{n-1} + (n-1)a_1 z^{n-2} + \cdots + 2a_{n-2}z + a_{n-1}.$$

Now

$$\frac{1}{2\pi}\int_0^{2\pi} \{|P'(e^{i\theta})|^2 + |Q'(e^{i\theta})|^2\}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |P'(e^{i\theta})|^2\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} |Q'(e^{i\theta})|^2\,d\theta$$

$$= \sum_{j=0}^{n} j^2|a_j|^2 + \sum_{j=0}^{n} (n-j)^2|a_j|^2 = \sum_{j=0}^{n} (j^2 + (n-j)^2)|a_j|^2$$

$$\leqslant \sum_{j=0}^{n} n^2|a_j|^2 = \frac{n^2}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2\,d\theta. \tag{6}$$

Now suppose $P(z)$ is a self-reciprocal polynomial of degree at most n. Then we have $z^n P(1/z) = P(z)$ for all $z \in C$, or equivalently $Q(z) = P(z)$ for all $z \in C$. This implies $Q'(z) = P'(z)$ for all $z \in C$ and hence in particular

$$|Q'(e^{i\theta})|^2 = |P'(e^{i\theta})|^2 \quad \text{for} \quad 0 \leqslant \theta < 2\pi.$$

Using this in (6), we get

$$\frac{1}{2\pi}\int_0^{2\pi} |P'(e^{i\theta})|^2 \leqslant \frac{n^2}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2\,d\theta,$$

from which (4) follows immediately.

To prove (5), we take $A = zP'(z)$ and $B = nP(z) - zP'(z)$ in the identity

$$2|A|^2 + 2|B|^2 = |A+B|^2 + |A-B|^2,$$

we get

$$2|zP'(z)|^2 + 2|nP(z) - zP'(z)|^2 = n^2|P(z)|^2 + |2zP'(z) - nP(z)|^2.$$

This in particular for $z = e^{i\theta}$, $0 \leqslant \theta < 2\pi$, implies that

$$|P'(e^{i\theta})|^2 + |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|^2 \geqslant \frac{n^2}{2}|P(e^{i\theta})|^2,$$

which gives

$$\int_0^{2\pi} |P'(e^{i\theta})|^2\,d\theta + \int_0^{2\pi} |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|^2\,d\theta$$

$$\geqslant \frac{n^2}{2}\int_0^{2\pi} |P(e^{i\theta})|^2\,d\theta. \tag{7}$$

Now by hypothesis, $P(z) = z^n P(1/z)$ for all $z \in C$ and therefore, we have

$$zP'(z) = nz^n P(1/z) - z^{n-1} P'(1/z).$$

This in particular for $z = e^{-i\theta}$, $0 \leqslant \theta < 2\pi$, gives

$$\begin{aligned} |P'(e^{-i\theta})| &= |e^{-i\theta} P'(e^{-i\theta})| = |n\, e^{-in\theta} P(e^{i\theta}) - e^{-i(n-1)\theta} P'(e^{i\theta})| \\ &= |nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta})|. \end{aligned} \tag{8}$$

Using (8) in (7), we obtain

$$\int_0^{2\pi} |P'(e^{i\theta})|^2 \, d\theta + \int_0^{2\pi} |P'(e^{-i\theta})|^2 \, d\theta \geqslant \frac{n^2}{2} \int_0^{2\pi} |P(e^{i\theta})|^2 \, d\theta. \tag{9}$$

Since it can be easily verified that

$$\int_0^{2\pi} |P'(e^{-i\theta})|^2 d\theta = \int_0^{2\pi} |P'(e^{i\theta})|^2 \, d\theta,$$

from (9) it follows that

$$\int_0^{2\pi} |P'(e^{i\theta})|^2 \, d\theta \geqslant \frac{n^2}{4} \int_0^{2\pi} |P(e^{i\theta})|^2 \, d\theta$$

which is equivalent to (5) and this completes the proof of the theorem.

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Random commutation

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Abstract. We investigate the commutation between a continuous linear random operator and a continuous linear deterministic operator on a Banach space. From this we obtain probabilistic versions of theorems by Fuglede and Putnam, both of them dealing with the commutation between continuous linear operators with continuous normal operators on a Hilbert space.

Keywords. Random operator; normal operator; compact normal operator.

1. Introduction

Operator commutators are a classical topic appearing in several branches of functional analysis and operator theory [1,6].

In this paper we investigate the commutation of a continuous linear random operator Φ and a continuous linear deterministic operator F both of them acting on a Banach space X. This commutation may be understood in a very broad sense; namely, for each element x in X and each continuous linear functional f on X, the event $[f(\Phi Fx) = f(F\Phi x)]$ can happen, that is the set $\{\omega : f((\Phi Fx)(\omega)) = f(F(\Phi x)(\omega))\}$ has a positive probability which depends upon the element x and the functional f. In such a case, there exists a measurable set Δ with a positive probability in such a way that, for every $x \in X$, the random variables ΦFx and $F\Phi x$ coincide almost surely.

Fuglede solved in [3] an outstanding problem in functional analysis, proposed by von Neumann, namely that every continuous linear operator T commuting with a continuous normal operator F on a Hilbert space, also commutes with the adjoint operator of F. Also the study of the commutator $[T, F]$ when $[[T, F], F]$ equals zero, and the operators T and F act on a Banach space, has received considerable attention. A fundamental result in this area was the theorem of Kleinecke [4] and Sirokov [9] who proved independently that in such a case $[T, F]$ is generalized nilpotent operator. A special case of this with the additional assumption that F is a continuous normal operator on a Hilbert space, was solved by Putnam [5] by stating that, in such a case the operator $[T, F]$ equals zero. We show several probabilistic versions of those theorems by Fuglede and Putnam.

Finally we show that if there is a sufficiently large probabilistic commutation between a random operator Φ and a compact normal operator on a Hilbert space, then $\mathbb{P}[\Phi \equiv \varphi(K)] > 0$ for a suitable random function φ acting on the spectrum of K.

2. On the continuity of linear random operators

Throughout the paper, $(\Omega, \Sigma, \mathbb{P})$ denotes a complete probability space. Every measurable subset Δ of Ω is considered as a new probability space with the inherited structure from Ω,

whose induced probability on Δ is the conditional probability $\mathbb{P}_\Delta$. As it is usual the *expectation* of a given real random variable ξ, denoted by $\mathbb{E}(\xi)$, is defined to be the number $\int_\Omega \xi d\mathbb{P}$.

A mapping ξ from Ω into a given Banach space Y, over either the real or the complex field, denoted by $\mathbb{K}$, is said to be a *Bochner random variable* on Y if it is the almost sure limit of a sequence of *simple random variables* on Y. We denote, by $\mathscr{L}_0(\mathbb{P}, Y)$ the linear space of all Y-valued Bochner random variables, which with the almost sure identification becomes a metrizable complete linear topological space $L_0(\mathbb{P}, Y)$ for the convergence in probability. This topology can be derived from the paranorm given by $\|\xi\|_0 = \mathbb{E}(\|\xi\|/(1+\|\xi\|))$. Relevant subspaces of $\mathscr{L}_0(\mathbb{P}, Y)$ are the spaces $\mathscr{L}_r(\mathbb{P}, Y) = \{\xi \in \mathscr{L}_0(\mathbb{P}, Y): \mathbb{E}\|\xi\|^r < \infty\}$ of all Y-valued Bochner random variables having rth moment. Besides the inherited topology from $\mathscr{L}_0(\mathbb{P}, Y)$, the space $\mathscr{L}_r(\mathbb{P}, Y)$ has its appropriate topology, namely that associated to convergence in r-mean, which can be derived from the paranorm given by $\|\xi\|_r = \mathbb{E}\|\xi\|^r$, when $0 < r < 1$, while if $1 \leqslant r$ it can be derived from the seminorm given by $\|\xi\|_r = (\mathbb{E}\|\xi\|^r)^{1/r}$. Given $\xi \in \mathscr{L}_0(\mathbb{P}, Y)$, $[\xi]$ denotes the equivalence class of ξ for the usual almost sure identfication. The space $L_r(\mathbb{P}, Y) = \{[\xi]: \xi \in \mathscr{L}_r(\mathbb{P}, Y)\}$ becomes a metrizable complete linear space.

Given Banach spaces X and Ydenote by $BL(X, Y)$ the linear space of all continuous linear operators from X into Y endowed with the usual operator norm given by $\|F\| = \sup_{\|x\|=1} \|Fx\|$. To shorten notation we write $BL(X)$ instead of $BL(X, X)$ and X' instead of $BL(X, \mathbb{K})$.

A map $\Phi: X \times \Omega \to Y$ is said to be a random operator from X to Y if, for each $x \in X$, the map $\omega \mapsto \Phi(x, \omega)$, noted Φx, lies in $\mathscr{L}_0(\mathbb{P}, Y)$. For a full discussion of random operators the reader is referred to [2] and [10]. Such an operator is said to have rth *moment* if the maps $\omega \mapsto \Phi(z, \omega)$ lie in $\mathscr{L}_r(\mathbb{P}, Y)$ and is said to be *linear* if $\mathbb{P}[\Phi(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \Phi x_1 + \alpha_2 \Phi x_2] = 1$, for all $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{K}$. The *continuity in probability* of Φ at $x_0 \in X$ means that $\lim_{x \to x_0} \mathbb{P}[\|\Phi x - \Phi x_0\| > \varepsilon] = 0$, $\forall \varepsilon > 0$. If Φ has rth moment, then there is a notion of continuity more convenient than the continuity in probability introduced above; namely the *continuity in r-mean*, which means that $\lim_{x \to x_0} \mathbb{E}\|\Phi x - \Phi x_0\|^r = 0$. We define the *conditional operator*, Φ_Δ, of Φ as the restriction of Φ to $X \times \Delta$.

Theorem 2.1. *Let X and Y be Banach spaces and Φ be a linear random operator from X to Y. Then the following assertions are equivalent:*

1. *Φ is continuous in probability at every point in X.*
2. $\lim_{x \to 0} \mathbb{P}[\|\Phi x\| > \varepsilon] = 0$ *for every* $\varepsilon > 0$.
3. $\lim_{(x,f) \to (0,0)} \mathbb{P}[|f(\Phi x)| > \varepsilon] = 0$ *for every* $\varepsilon > 0$.

Furthermore, if in addition Φ has rth moment, then the preceding conditions are equivalent to the following ones:

4. *Φ is continuous in r-mean at every point in X.*
5. $\lim_{x \to 0} \mathbb{E}\|\Phi x\|^r = 0$.
6. $\lim_{(x,f) \to (0,0)} \mathbb{E}|f(\Phi x)|^r = 0$.
7. $\exists M > 0: \mathbb{E}\|\Phi x\|^r \leqslant M\|x\|^r \quad \forall x \in X$.

Proof. It is clear that 1 implies 2, 4 implies 5, and 6 implies 3.
Since $|f(\Phi x)| \leqslant \|f\| \|\Phi x\|$, we have $\lim_{(x,f)\to(0,0)} |f(\Phi x)| = 0$ whenever 2 is fulfilled.

Assume that Φ satisfies condition 3. To deduce assertion 1, in view of the closed graph theorem, it suffices to show that the graph of Φ is closed in $X \times \mathscr{L}_0(\mathbb{P}, Y)$. To this end, consider a sequence $\{x_n\}$ converging to 0 in X with $\{\Phi x_n\}$ converging in probability to ξ. For every $f \in Y'$ we have $f(\xi) = f(\lim_{n\to\infty} \Phi x_n) = \lim_{n\to\infty} f(\Phi x_n) = 0$. This gives $\mathbb{P}[\xi \in \ker f] = 1$. Since $\{\ker f : f \in Y'\}$ is a family of closed subsets in Y, we can apply [11; Lemma 3.1] in order to obtain

$$\mathbb{P}\left[\xi \in \bigcap_{f\in Y'} \ker f\right] = \inf\{\mathbb{P}[\xi \in \ker f_1 \cap \cdots \cap \ker f_k] : k \in \mathbb{N},\ f_1, \ldots, f_k \in Y'\} = 1.$$

By Hahn–Banach theorem, $\cap_{f\in Y'} \ker f = 0$, and thus we get $\mathbb{P}[\xi = 0] = 1$. This establishes 1. To prove 4 we only need to show that Φ has closed graph in $X \times \mathscr{L}_r(\mathbb{P}, Y)$. Let $\{x_n\}$ be a sequence in X converging to zero with $\{\Phi x_n\}$ converging in r-mean to ξ. For every $f \in Y'$, we have $\mathbb{E}|f(\xi)|^r = \lim_{n\to\infty} \mathbb{E}|f(\Phi x_n)|^r = 0$. Therefore $\mathbb{P}[\xi \in \ker f] = 1$. Arguing as above we get $\mathbb{P}[\xi = 0] = 1$ and this gives 4.

Assume that Φ satisfies 5. Since $\mathbb{E}|f(\Phi x)|^r \leqslant \|f\|^r \mathbb{E}\|\Phi x\|^r$, it may be concluded that $\lim_{(x,f)\to(0,0)} \mathbb{E}|f(Tx)|^r = 0$.

It is clear that 5 follows from 7. Conversely, assume that 5 holds and we claim that there are $N, \delta > 0$ such that $\mathbb{E}\|\Phi x\|^r \leqslant N$ if $\|x\| = \delta$. Otherwise for every $n \in \mathbb{N}$ we choose $x_n \in X$ with $\|x_n\| = 1/n$ and $\mathbb{E}\|\Phi x_n\|^r > n$ which contradicts assertion 5. If $x \in X \setminus \{0\}$, then $\mathbb{E}\|\Phi(\delta \|x\|)^{-1} x\|^r \leqslant N$ and therefore $\mathbb{E}\|\Phi x\|^r \leqslant \delta^r N \|x\|^r$, as desired.

We leave it to the reader to state the notion of continuous multilinear random operator.

3. Commutation probabilities

To investigate the commutation properties between a random operator and a deterministic one we first require an improvement of [11, Theorem 4.1]. To do this we establish a technical result in the following.

Lemma 3.1. *Let X and Y be Banach spaces and Φ a continuous linear random operator from X to Y. Then the set $C_\delta = \{x \in X : \mathbb{P}[\Phi x = 0] \geqslant \delta\}$ is closed, for every $0 \leqslant \delta \leqslant 1$.*

Proof. Let $\{x_n\}$ be a sequence in C_δ converging to an element x in X. The sequence $\{\Phi x_n\}$ converges in probability to Φx and consequently $\lim_{n\to\infty} \mathbb{E}(\|\Phi x_n\|/(1 + \|\Phi x_n\|)) = \mathbb{E}(\|\Phi x\|/(1 + \|\Phi x\|))$. Let $\Omega_n = \{\omega \in \Omega : \Phi x_n \neq 0\}$. For every $n \in \mathbb{N}$, we have $\mathbb{E}(\|\Phi x_n\|/(1 + \|\Phi x_n\|)) = \int_{\Omega n} \|\Phi x_n\|/(1 + \|\Phi x_n\|) d\mathbb{P} \leqslant \mathbb{P}[\Omega_n] \leqslant 1 - \delta$. Therefore, $\mathbb{E}(\|\Phi x\|/(1 + \|\Phi x\|)) \leqslant 1 - \delta$. Given $k \in \mathbb{N}$, it is clear that $kx_n \in C_\delta$ for every $n \in \mathbb{N}$ and $\{kx_n\}$ converges in probability to kx. Accordingly $\mathbb{E}(k\|\Phi x\|/(1 + k\|\Phi x\|)) \leqslant 1 - \delta$. Letting $k \to \infty$, we deduce that $\mathbb{P}[\Phi x \neq 0] \leqslant 1 - \delta$ and $x \in C_\delta$. □

Theorem 3.2. *Let $X_1, \ldots, X_N$, Y be Banach spaces and Φ a continuous multilinear random operator from $X_1 \times \cdots \times X_N$ to Y. If, for all $x \in X_1 \times \cdots \times X_N$ and $f \in Y'$*

$$\mathbb{P}[f(\Phi x) = 0] > 0,$$

then the set

$$\{\mathbb{P}[f(\Phi x)=0]: x\in X_1\times\cdots\times X_N, f\in Y'\}$$

has a nonzero minimum, say δ, *and there exists a measurable set* Δ *with* $\mathbb{P}[\Delta]=\delta$ *in such a way that, for every* $x\in X_1\times\cdots\times X_N$,

$$\Phi x=0 \text{ almost surely on } \Delta.$$

Proof. For simplicity we assume $N=1$. For each $n\in\mathbb{N}$, let C_n be the closed subset of $X_1\times Y'$ given by $C_n=\{(x,\rho)\in X_1\times Y': \mathbb{P}[f(\Phi x)=0]\geqslant 1/n\}$. Then $X_1\times Y'=\cup_{n=1}^{\infty}C_n$. From Baire theorem C_n has an interior point, say (x',f'), for a suitable natural number n. Note that, if $(x,f)\in X_1\times Y'$ and $\lambda,\mu\in\mathbb{K}\setminus\{0\}$ with $|\lambda|$ and $|\mu|$ small enough, then

$$\mathbb{P}[(f'-\mu f)\Phi(x'-\lambda x)=0]\geqslant\frac{1}{n}.$$

Therefore

$$\frac{1}{n}\leqslant\mathbb{P}[f'-\mu f)\Phi(x'-\lambda x)=0]=\mathbb{P}[f'\Phi(x'-\lambda x)-\mu f\Phi(x'-\lambda x)=0]$$

$$=\mathbb{P}[f'\Phi(x'-\lambda x)=f\Phi(x'-\lambda x)=0]$$

$$+\mathbb{P}[f'\Phi(x'-\lambda x)=\mu f\Phi(x'-\lambda x),\ f'\Phi(x'-\lambda x)\neq 0],$$

and letting $\mu\to 0$ we have

$$\frac{1}{n}\leqslant\mathbb{P}[f'\Phi(x'-\lambda x)=f\Phi(x'-\lambda x)=0]$$

$$\leqslant\mathbb{P}[f'\Phi x'=f'\Phi x=0,\quad f\Phi x'=f\Phi x=0]$$

$$+\mathbb{P}[f'\Phi x'=\lambda f'\Phi x, f'\Phi x\neq 0]+\mathbb{P}[f\Phi x'=\lambda f\Phi x, f\Phi x\neq 0].$$

Further, letting $\lambda\to 0$ we get

$$\frac{1}{n}\leqslant\mathbb{P}[(f'\Phi x'=f'\Phi x=0, f\Phi x'=f\Phi x=0]\leqslant\mathbb{P}[f\Phi x=0].$$

This shows that $\inf\{\mathbb{P}[f\Phi x=0]: x\in X_1,\ f\in Y'\}>0$. Therefore we can apply [11, Theorem 4.1] to show that there exists a measurable set Δ with

$$\mathbb{P}[\Delta]=\min\{\mathbb{P}[f\Phi x=0]: x\in X_1, f\in Y'\}$$

$$=\max\{\mathbb{P}[\Delta]: f\Phi x=0 \text{ almost surely on } \Delta\ \forall x\in X_1,\ f\in Y'\}.$$

For each $x\in X_1\times\cdots\times X_N$ we have

$$\mathbb{P}_\Delta[\Phi_\Delta x=0]=\mathbb{P}_\Delta\left[\Phi_\Delta x\in\bigcap_{f\in Y'}\ker f\right].$$

Since $\{\ker f: f\in Y'\}$ is a family of closed subsets of Y we can apply [11, Lemma 3.1] to deduce that $\mathbb{P}_\Delta[\Phi_\Delta x=0]$ equals $\inf\{\mathbb{P}_\Delta[f_1(\Phi_\Delta x)=\cdots=f_k(\Phi_\Delta x)=0]: k\in\mathbb{N}, f_1,\ldots,f_k\in Y'\}=1$. Consequently $\Phi x=0$ almost surely on Δ. □

The preceding theorem is used in the following to compute the equivalence of two continuous linear random operators.

COROLLARY 3.3

Let Φ and Ψ be continuous linear random operators from a Banach space X to a Banach space Y. If $\mathbb{P}[f(\Phi x)=f(\Psi x)]>0$ for all $x\in X$ and $f\in Y'$, then the set $\{\mathbb{P}|f(\Phi x)=f(\Psi x)]:x\in X, f\in Y'\}$ has a nonzero minimum, say δ, and there exists a measurable set Δ with $\mathbb{P}[\Delta]=\delta$ in such a way that, for every $x\in X$, $\Phi x=\Psi x$ almost surely on Δ.

Two random operators Φ and Ψ from a Banach space X to a Banach space Y are said to be *equivalent*, written $\Phi\equiv\Psi$, if $\mathbb{P}[f(\Phi x)=f(\Psi x)]=1\ \forall x\in X, \forall f\in Y'$. The preceding result shows that $\Phi\equiv\Psi$ if and only if, $\mathbb{P}[\Phi x=\Psi x]=1\ \forall x\in X$. The quantity $\min\{\mathbb{P}[f(\Phi x)=f(\Psi x)]:x\in X, f\in Y'\}$ is considered as the probability of Φ and Ψ being equivalent and it is denoted as $\mathbb{P}[\Phi\equiv\Psi]$.

If F and G are continuous linear deterministic operators on Banach spaces X and Y, respectively, and Φ is a continuous linear random operator from X to Y, then the operators $(x,\omega)\mapsto\Phi(Fx,\omega)$ and $(x,\omega)\mapsto G(\Phi(x,\omega))$ from $X\times\Omega$ into Y are continuous linear random operators from X to Y and we denote them by ΦF and $G\Phi$, respectively. Given a continuous linear random operator Φ and a continuous linear deterministic operator F, both of them acting on a Banach space X, we define the *commutator* of Φ and F as the continuous linear random operator $[\Phi,F]=\Phi F-F\Phi$. If $\mathbb{P}[f([\Phi,F]x)=0]>0$ for all $x\in X$ and $f\in X'$, then it is reasonable to consider the quantity $\mathbb{P}[\Phi F\equiv F\Phi]$ as a measure of the commutation likelihood between both of the operators. To determine the existence of a measurable set on which Φ behaves as an operator commuting with F, it suffices to check the quantities $\mathbb{P}[f([\Phi,F]x)=0]$. If all of them are nonzero, then it is likely that Φ commutes with F and we measure the likelihood that can be expected by computing the minimum of all of them. In such a case, by Corollary 3.3, there exists a measurable set Δ with $\mathbb{P}(\Delta)=\mathbb{P}[\Phi F\equiv F\Phi]$ and satisfying for every $x\in X$ the equality $\Phi Fx=F\Phi x$ almost surely on Δ, that is $\Phi_\Delta F\equiv F\Phi_\Delta$.

In the next we apply Theorem 3.2 to study a large commutation.

COROLLARY 3.4

Let Φ be a continuous linear random operator and $\mathscr{F}$ be a norm-closed subspace of continuous linear deterministic operators, all of them acting on a Banach space X. If, for all $x\in X$, $f\in X'$, and $F\in\mathscr{F}$, $\mathbb{P}[f(\Phi Fx)=f(F\Phi x)]>0$ then the set $\{\mathbb{P}[f(\Phi Fx)=f(F\Phi x)]:x\in X, f\in X', F\in\mathscr{F}\}$ has a nonzero minimum, say δ, and there exists a measurable set Δ with $\mathbb{P}[\Delta]=\delta$ in such a way that, for all $x\in X$ and $F\in\mathscr{F}$, $\Phi Fx=F\Phi x$ almost surely on Δ.

Proof. Consider the continuous bilinear random operator $(x,F)\mapsto[\Phi,F]x$ from $X\times\mathscr{F}$ to X and apply Theorem 3.2. □

4. Commutation with normal operators

Commutation theory is concerned mostly with operators acting on a Hilbert space. This has its origin in the commutation relations occurring in quantum mechanics. For a deeper discussion of this topic we refer the reader to [6].

Fuglede theorem (see [8, Corollary 2.2.6]) states that, if a continuous linear operator T on a Hilbert space H commutes with a continuous normal operator F on H, then T also commutes with the adjoint operator of F, from now on denoted by F^*. Unfortunately we don't know whether Fuglede theorem remains true if we replace T by a continuous linear random operator Φ. However we state some relevant results about this. To do this we first require a number of technical results.

Lemma 4.1. *Let Φ be a continuous linear random operator from a finite-dimensional Banach space X to a Banach space Y. Then Φ is equivalent to a Bochner random variable on $BL(X, Y)$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis in X and, for each $i = 1, \ldots, n$, let $\{y_{i,k}\}$ be a sequence of simple random variables on Y converging almost surely to Φx_i. For each $k \in \mathbb{N}$, we define a simple random variable Ψ_k on $BL(X, Y)$ by

$$\Psi_k\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i y_{i,k} \quad \forall\, \lambda_1, \ldots, \lambda_n \in \mathbb{K}.$$

For all $p, q \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, we have

$$\left\|(\Psi_p - \Psi_q)\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\| = \left\|\sum_{i=1}^{n} \lambda_i (y_{i,p} - y_{i,q})\right\| \leqslant \left(\sum_{i=1}^{n} |\lambda_i|\right) \sum_{j=1}^{n} \|y_{j,p} - y_{j,q}\|.$$

Since the map $\lambda_1 x_1 + \cdots + \lambda_n x_n \mapsto |\lambda_1| + \cdots + |\lambda_n|$ defines a norm on X necessarily equivalent to the original one $\|\cdot\|$, there exists a positive constant M satisfying $|\lambda_1| + \cdots + |\lambda_n| \leqslant M \|\lambda_1 x_1 + \cdots + \lambda_n x_n\|$, $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Therefore

$$\left\|(\Psi_p - \Psi_q)\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\| \leqslant M \left(\sum_{j=1}^{n} \|y_{j,p} - y_{j,q}\|\right) \left\|\sum_{i=1}^{n} \lambda_i x_i\right\|,$$

and $\|\Psi_p - \Psi_q\| \leqslant M \sum_{j=1}^{n} \|y_{j,p} - y_{j,q}\|$, which shows that the sequence $\{\Psi_k\}$ converges almost surely to a random variable Ψ on $BL(X, Y)$. Moreover, for every $x \in X$ the sequence $\{\Psi_k x\}$ converges almost surely to Φx and therefore $\Phi \equiv \Psi$. □

We can now formulate a Fuglede type theorem dealing with the random commutation with a compact normal operator.

Theorem 4.2 *Let Φ be a continuous linear random operator and K be a compact normal deterministic operator, both of them acting on a complex Hilbert space H. Then*

$$\mathbb{P}[\Phi K \equiv K\Phi] = \mathbb{P}[\Phi K^* \equiv K^*\Phi].$$

Proof. Let $\{\lambda_n\}$ denote the sequence of spectral values of K with $\lambda_0 = 0$ and, for each $n \in \mathbb{N} \cup \{0\}$, π_n denote the projection from H onto $\ker(\lambda_n I - K)$. By [1, Theorem 3.2], $K = \sum_{n=0}^{\infty} \lambda_n \pi_n$ and $\sum_{n=0}^{\infty} \pi_n = I$ pointwise.

Fix a measurable set Δ with $\mathbb{P}[\Delta] = \mathbb{P}[\Phi K \equiv K\Phi]$ and $\Phi_\Delta K \equiv K\Phi_\Delta$. Given $n \in \mathbb{N}$, we have $\Phi_\Delta K \pi_n \equiv K \Phi_\Delta \pi_n$ and therefore $(\Phi_\Delta \pi_n) K \equiv K(\Phi_\Delta \pi_n)$. Since π_n has finite rank [1, Theorem 2.4], $\Phi_\Delta \pi_n$ is equivalent to a random variable on $BL(H)$. From this it may be concluded that $(\Phi_\Delta \pi_n) K^* \equiv K^* (\Phi_\Delta \pi_n)$, and consequently $[\Phi_\Delta, K^*]\pi_n \equiv 0$. Further we note that $K\pi_0 = \pi_0 K = 0$ and therefore $K(\Phi_\Delta \pi_0) \equiv \Phi_\Delta K \pi_0 \equiv 0$. Since $\Phi_\Delta \pi_0$ maps every

element almost surely into $\ker K$ and $\ker K = \ker K^*$, we have $K^*\Phi_\Delta\pi_0 \equiv 0$. Accordingly $[\Phi_\Delta, K^*]\pi_0 \equiv 0$. Since $[\Phi_\Delta, K^*]\pi_n \equiv 0\ \forall n \in \mathbb{N}\cup\{0\}$ and $\sum_{n=0}^{\infty}\pi_n = I$, it follows that $[\Phi_\Delta, K^*] \equiv 0$. Therefore $\mathbb{P}[\Phi K^* \equiv K^*\Phi] \geqslant \mathbb{P}[\Phi K \equiv K\Phi]$. Replacing K by K^* we have $\mathbb{P}[\Phi K \equiv K\Phi] \geqslant \mathbb{P}[\Phi K^* \equiv K^*\Phi]$, as required.

In order to avoid the compactness condition, we require the random operator to have 1st moment.

Theorem 4.3. *Let Φ be a continuous linear random operator having* 1st *moment and F be a continuous normal deterministic operator, both of them acting on a complex Hilbert space H. Then*

$$\mathbb{P}[\Phi F \equiv F\Phi] = \mathbb{P}[\Phi F^* \equiv F^*\Phi].$$

Proof. Our proof follows on the same lines as given by Rosenblum (see [8, Theorem 2.2.5]). Let Δ be a measurable set with $\mathbb{P}[\Delta] = \mathbb{P}[\Phi F \equiv F\Phi]$ and $\Phi_\Delta F \equiv F\Phi_\Delta$. For all $n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$, we have $\Phi_\Delta(\bar\lambda F)^n \equiv (\bar\lambda F)^n\Phi_\Delta$. Since Φ_Δ is continuous in mean, for every $x\in H$, we have the almost surely equalities

$$\Phi_\Delta \exp(\bar\lambda F)x = \Phi_\Delta \sum_{n=0}^{\infty}\frac{1}{n!}(\bar\lambda F)^n x = \sum_{n=0}^{\infty}\Phi_\Delta\frac{1}{n!}(\bar\lambda F)^n x$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(\bar\lambda F)^n\Phi_\Delta x = \exp(\bar\lambda F)\Phi_\Delta x.$$

Hence $\Phi_\Delta \exp(\bar\lambda F) \equiv \exp(\bar\lambda F)\Phi_\Delta$, which implies $\Phi_\Delta \equiv \exp(\bar\lambda F)\Phi_\Delta \exp(-\bar\lambda F)$ and $\exp(-\lambda F^*)\Phi_\Delta \exp(\lambda F^*) \equiv \exp(\bar\lambda F - \lambda F^*)\Phi_\Delta \exp(-\bar\lambda F + \lambda F^*)$. Since $\exp(\bar\lambda F - \lambda F^*)$ and $\exp(-\lambda F + \lambda F^*)$are unitary operators we have $\| \exp.(\bar\lambda F - \lambda F^*)\| = \| \exp(-\bar\lambda F + \lambda F^*)\| = 1$. Given $x, y \in H$, we define a complex $L_1(\mathbb{P}_\Delta, \mathbb{C})$-valued function φ holomorphic on the whole complex plane by

$$\varphi(\lambda) = [(\Phi_\Delta \exp(\lambda F^*)x | \exp(-\bar\lambda F)y)] = [(\exp(-\lambda F^*)\Phi_\Delta \exp(\lambda F^*)x | y)]$$

$$= [(\exp(\bar\lambda F - \lambda F^*)\Phi_\Delta \exp(-\bar\lambda F + \lambda F^*)x | y)]$$

$$= [(\Phi_\Delta \exp(-\bar\lambda F + \lambda F^*)x | \exp(-\bar\lambda F + \lambda F^*)y)].$$

From Theorem 2.1 we have $\mathbb{E}\|\Phi_\Delta x\| \leqslant M\|x\| \forall x\in X$ for a suitable $M > 0$. Therefore $\mathbb{E}|\varphi(\lambda)| \leqslant M\|\exp(\bar\lambda F - \lambda F^*)x\|\,\|\exp(-\bar\lambda F + \lambda F^*)y\| \leqslant M\|x\|\,\|y\|$, for a suitable $M > 0$. Liouville theorem [7, Theorem 3.32] assures that the vector-valued function φ is constant. Hence $\varphi(\lambda) = \varphi(0) = [(\Phi_\Delta x | y)]$ (which mean that, for every $\lambda\in\mathbb{C}$, $\varphi(\lambda) = (\Phi x | y)$ almost surely on Δ) and $0 = \varphi'(\lambda) = [(\Phi_\Delta \exp(\lambda F^*)F^*\ x | \exp(-\bar\lambda F)y)] + [(\Phi_\Delta \exp\lambda F^*)x | \exp(-\bar\lambda F)(-F)y)]$. Accordingly $0 = \varphi'(0) = [(\Phi F^* x | y) - (\Phi x | Fy)]$ almost surely, for all $x, y\in H$. This shows that $F^*\Phi \equiv \Phi F^*$. Therefore $\mathbb{P}[\Phi F^* \equiv F^*\Phi] \geqslant \mathbb{P}[\Phi F \equiv F\Phi]$, and replacing F by F^* we have $\mathbb{P}[\Phi F \equiv F\Phi] \geqslant \mathbb{P}[\Phi F^* \equiv F^*\Phi]$, which ends the proof. □

For a continuous linear operator T and a continuous normal operator F on a Hilbert space H, for which the commutator $[T, F]$ commutes with F, Putnam theorem [8, Corollary 2.2.9] shows that $[T, F] = 0$. It seems to be unknown whether Putnam theorem remains true for a continuous linear random operator T. We establish several probabilistic versions of that theorem.

In order to translate Putnam theorem to random operators we first restrict our attention to compact deterministic operators.

Theorem 4.4. *Let Φ be a continuous linear random operator and K be a compact normal deterministic operator, both of them acting on a complex Hilbert space H. Then*

$$\mathbb{P}[[[\Phi, K], K] \equiv 0] = \mathbb{P}[[\Phi, K] \equiv 0].$$

Proof. As in the proof of Theorem 4.2 we put $K = \sum_{n=0}^{\infty} \lambda_n \pi_n$.

It is clear that $\mathbb{P}[[[\Phi, K], K] \equiv 0] \geqslant \mathbb{P}[[\Phi, K] \equiv 0]$.

Let Δ be a measurable set with $[[\Phi_\Delta, K], K] \equiv 0$ and $\mathbb{P}[\Delta] = \mathbb{P}[[[\Phi, K], K] \equiv 0]$. For every $n \in \mathbb{N}$, we have $[[\Phi_\Delta \pi_n, K], K] \equiv [[\Phi_\Delta, K], K]\pi_n \equiv 0$. Since $\Phi_\Delta \pi_n$ is equivalent to a random variable on $BL(H)$, it may be concluded that $[\Phi_\Delta \pi_n, K] \equiv 0$ and therefore $[\Phi_\Delta, K]\pi_n \equiv 0$. Since $[\Phi_\Delta, K]K \equiv K[\Phi_\Delta, K]$, we have $0 \equiv [\Phi_\Delta, K]K\pi_0 \equiv K[\Phi_\Delta, K]\pi_0$. Consequently $[\Phi_\Delta, K]\pi_0$ maps every element almost surely into $\ker K$, which gives $\pi_n[\Phi_\Delta, K]\pi_0 \equiv 0 \; \forall n \in \mathbb{N}$. Further $\pi_0[\Phi_\Delta, K]\pi_0 \equiv 0$ and therefore $[\Phi_\Delta, K]\pi_0 \equiv 0$, since $\sum_{n=0}^{\infty} \pi_n = I$. Accordingly $[\Phi_\Delta, K] \equiv 0$ and so $\mathbb{P}[[\Phi, K] \equiv 0] \geqslant \mathbb{P}[[[\Phi, K], K] \equiv 0]$, which concludes the proof. □

Now compactness condition may be omitted by requiring the expectation property on the random operator.

Lemma 4.5. *Let Φ be a continuous linear random operator having 1st moment and F be a continuous normal deterministic operator, both of them acting on a complex Hilbert space H. Assume that $F = \int z dE(z)$ for a suitable spectral measure E on the spectrum* sp (F) *of F. Then $\Phi F \equiv F\Phi$ and $\Phi F^* \equiv F^*\Phi$ if and only if, $\Phi E(\Lambda) \equiv E(\Lambda)\Phi$ for every measurable set Λ in* sp (F).

Proof. If $\Phi F \equiv F\Phi$ and $\Phi F^* \equiv F^*\Phi$, then $\Phi p(F, F^*) \equiv \Phi p(F, F^*)$ for every complex polynomial p in F and F^*. Given a measurable subset. Λ of sp(F), the projection $E(\Lambda)$ can be obtained as the strong limit in $BL(H)$ of a suitable sequence $\{p_n(F, F^*)\}$ of complex polynomials $p_n(F, F^*)$ in F and F^* with $p_n(F, F^*)^* = p_n(F, F^*)$. Therefore, for every $x \in H$, we have

$$\Phi E(\Lambda)x = \|\cdot\|_1 - \lim_{n\to\infty} \Phi p_n(F,F^*)x = \|\cdot\|_1 - \lim_{n\to\infty} p_n(F,F^*)\Phi x = E(\Lambda)\Phi x$$

(where $\|\cdot\|_1 - \lim$ denotes the limit in mean) and thus $\Phi E(\Lambda) \equiv E(\Lambda)\Phi$.

We now assume that, $\Phi E(\Lambda) \equiv E(\Lambda)\Phi$ for every measurable subset Λ of sp(F). Let ε be a positive number, $\Lambda_1, \ldots, \Lambda_n$ measurable subsets of sp(F), and $z_1 \in \Lambda_1, \ldots, z_n \in \Lambda_n$ such that $\|F - \sum_{k=1}^{n} z_k E(\Lambda_k)\| < \varepsilon$. Then

$$\begin{aligned}\|[\Phi, F]x\| &= \left\| \Phi\left(F - \sum_{k=1}^{n} z_k E(\Lambda_k)\right)x - \left(F - \sum_{k=1}^{n} z_k E(\Lambda_k)\right)\Phi x \right\| \\ &\leqslant \left\| \Phi\left(F - \sum_{k=1}^{n} z_k E(\Lambda_k)\right)x \right\| + \left\| \left(F - \sum_{k=1}^{n} z_k E(\Lambda_k)\right)\Phi x \right\|,\end{aligned}$$

almost surely, for every $x \in H$. Since $[\Phi, F]$ is a continuous linear 1st order random operator, Theorem 2.1 shows that there exists $M > 0$ such that $\mathbb{E}\|[\Phi, F]x\| \leqslant M\|x\| \; \forall x \in X$.

Accordingly

$$\mathbb{E}\|[\Phi,F]x\| \leqslant 2M\left\|F-\sum_{k=1}^{n} z_k E(\Lambda_k)\right\|\|x\| \leqslant 2M\varepsilon\|x\|,$$

for a suitable $M>0$. Letting $\varepsilon\to 0$ we have $\mathbb{E}\|[\Phi,F]x\|=0$ and therefore $[\Phi,F]\equiv 0$.

In the previous argument, just replace $z_1,\ldots,z_n$ by $\bar{z}_1,\ldots,\bar{z}_n$, and it follows that $[T,F^*]\equiv 0$. □

Theorem 4.6. *Let Φ be a continuous linear random operator having 1st moment and F be a continuous normal deterministic operator, both of them acting on a complex Hilbert space H. Then*

$$\mathbb{P}[[[\Phi,F],F]\equiv 0]=\mathbb{P}[[\Phi,F]\equiv 0].$$

Proof. It is clear that $\mathbb{P}[[[\Phi,F],F]\equiv 0]\geqslant\mathbb{P}[[\Phi,F]\equiv 0]$.
Let Δ be a measurable set with $\mathbb{P}[\Delta]=\mathbb{P}[[[\Phi,F],F]\equiv 0]$ and $[[\Phi_\Delta,F],F]\equiv 0$. Since $[\Phi_\Delta,F]\,F\equiv F[\Phi_\Delta,F]$, Theorem 4.3 shows that $[\Phi_\Delta,F]F^*\equiv F^*[\Phi_\Delta,F]$. Let E be a spectral measure on sp (F) such that $F=\int z\mathrm{d}E(z)$. By applying the preceding lemma, for every measurable subset Λ in sp(F), we have $[\Phi_\Delta,F]\,E(\Lambda)\equiv E(\Lambda)[\Phi_\Delta,F]$. Since

$$\begin{aligned}[\Phi_\Delta,E(\Lambda)]F &\equiv \Phi_\Delta E(\Lambda)F-E(\Lambda)\Phi_\Delta F\equiv \Phi_\Delta FE(\Lambda)-E(\Lambda)-\Phi_\Delta F\\ &\equiv[\Phi_\Delta,F]E(\Lambda)+F\Phi_\Delta E(\Lambda)-E(\Lambda)\Phi_\Delta F\\ &\equiv E(\Lambda)[\Phi_\Delta,F]+F\Phi_\Delta E(\Lambda)-E(\Lambda)\Phi_\Delta F\\ &\equiv F\Phi_\Delta E(\Lambda)-E(\Lambda)F\Phi_\Delta\equiv F[\Phi_\Delta,E(\Lambda)].\end{aligned}$$

Theorem 4.3 shows that $[\Phi_\Delta,E(\Lambda)]F^*\equiv F^*[\Phi_\Delta,E(\Lambda)]$ and the preceding lemma gives $[\Phi_\Delta,E(\Lambda)]E(\Lambda)\equiv E(\Lambda)[\Phi_\Delta,E(\Lambda)]$. Accordingly $\Phi_\Delta E(\Lambda)-E(\Lambda)\Phi_\Delta E(\Lambda)\equiv E(\Lambda)\Phi_\Delta E(\Lambda)-E(\Lambda)\Phi_\Delta$. By multiplying on the left and on the right by $E(\Lambda)$ we obtain $\Phi_\Delta E(\Lambda)\equiv E(\Lambda)\Phi_\Delta$. By Lemma 4.5 $[\Phi_\Delta,F]\equiv 0$ and therefore $\mathbb{P}[[\Phi,F]\equiv 0]\geqslant\mathbb{P}[[[\Phi,F],F]\equiv 0]$. □

5. Commutation with compact normal operators

Throughout this section, K stands for a compact normal deterministic operator on a complex Hilbert space H. It decomposes into a pointwise sum $K=\sum_{n=0}^{\infty}\lambda_n\pi_n$ where $\{\lambda_n\}$ is the sequence of its spectral values with $\lambda_0=0$ and, for each n, π_n is the projection onto ker $(\lambda_n I-K)$.

It is well known that any continuous linear deterministic operator F that commutes with K can be decomposed into a pointwise sum $F=\sum_{n=0}^{\infty}\varphi(\lambda_n)\pi_n$ for a suitable complex valued bounded function acting on sp (K). Actually for every such function the above series converges pointwise to a continuous linear deterministic operator, usually denoted by $\varphi(K)$, satisfying that commutation property.

In this section we study whether the above assertions remain true in the random setting, replacing F by a random operator and φ by a random function.

Theorem 5.1. *Let Φ be a continuous linear random operator on H. If*

$$\mathbb{P}[([\Phi,F]x|y)=0]>0$$

for every continuous linear deterministic operator F on H commuting with K and all x, $y \in H$, then there exists a sequence bounded in probability $\{\xi_n\}$ of complex-valued random variables such that, for every $x \in H$, the random series $\Sigma \xi_n \pi_n x$ converges in probability and

$$\mathbb{P}\left[\Phi x = \sum_{n=0}^{\infty} \xi_n \pi_n x\right] > 0.$$

Furthermore, if Φ has rth moment, then the random variables ξ_n have rth moment, the sequence $\{\xi_n\}$ is bounded in r-mean and the series $\Sigma \xi_n \pi_n x$ converges in r-mean.

Proof. By considering the closed subspace $\mathscr{F} = \{F \in BL(H) : FK = KF\}$ of $BL(H)$ we can choose a measurable set Δ with $\mathbb{P}[\Delta] > 0$ and $[\Phi_\Delta, F] \equiv 0, \forall F \in \mathscr{F}$. For every λ_n we fix $u_n \in \ker(\lambda_n I - K)$ with $\| u_n \| = 1$ and we define the random variable ξ_n to be $(\Phi u_n | u_n)$ on Δ and 0 otherwise. Since Φ is continuous $\{\xi_n\}$ is bounded in probability. For every $n \in \mathbb{N}$ and every $u \in \ker(\lambda_n I - K)$, we consider the continuous linear operator F on H given by $Fx = (x|u_n)u$. Then F commutes with K and so $\Phi_\Delta F \equiv F \Phi_\Delta$. For every $y \in H$, $(\Phi_\Delta u | y) = (\Phi_\Delta F u_n | y) = (F \Phi_\Delta u_n | y) = (\Phi_\Delta u_n | F^* y) = \xi_{n|\Delta}(u|y)$. Therefore Φ_Δ is equivalent to $\xi_{n|\Delta} \pi_n$ on $\ker(\lambda_n I - K)$. For every $x \in H$, $\Sigma_{n=0}^{\infty} \pi_n x$ converges in probability (in r-mean if Φ has rth moment) to x and consequently $\Phi_\Delta x = \Sigma_{n=0}^{\infty} \Phi_\Delta \pi_n x$. From this we deduce $\Phi_\Delta x = \Sigma_{n=0}^{\infty} \xi_{n|\Delta} \pi_n x$. Therefore the random series $\Sigma \xi_{n|\Delta} \pi_n x$, and consequently $\Sigma \xi_n \pi_n x$, converge in probability. Further, for every $x \in H$ the equality $\Phi x = \Sigma_{n=0}^{\infty} \xi_n \pi_n x$ holds almost surely on Δ. If Φ has rth moment, then $\mathbb{E}|\xi_n|^r \leqslant \mathbb{E}\| \Phi u_n \|^r < \infty$. Further $\{\Phi u_n\}$ is bounded in r-mean, since otherwise we could choose a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\rho_k = \mathbb{E}\| \Phi u_{n_k} \|^r \to +\infty$. Accordingly $\{u_{n_k}/\rho_k^{1/k}\} \to 0$ and $\mathbb{E}\| \Phi(u_{n_k}/\rho_k^{1/k}) \|^r = \mathbb{E}\| \Phi u_{n_k} \|^r / \rho_k = 1$ which contradicts the continuity of Φ. □

Theoren 5.2. *Let $\{\pi_n\}$ be a sequence of pairwise orthogonal projections on a Hilbert space H and let $\{\xi_n\}$ be a sequence of complex-valued random variables having rth moment with $r \geqslant 2$. Then the following assertions are equivalent:*

1. *For each $x \in H$ the random series $\Sigma \xi_n \pi_n x$, converges in r-mean.*
2. *The sequence $\{\xi_n\}$ is bounded in r-mean.*

Furthermore, in such a case the pointwise sum of the series $\Sigma \xi_n \pi_n$ defines a continuous linear random operator on H having rth moment.

Proof. Assume that the sequence $\{\xi_n\}$ is bounded in r-mean. We show that, for each $x \in H$, the series $\Sigma |\xi_n|^2 \| \pi_n \|^2$ in $L_{r/2}(\mathbb{K})$ converges. To this end we note that

$$\sum_{n=1}^{\infty} \| |\xi_n|^2 \| \pi_n x \|^2 \|_{r/2} = \sum_{n=1}^{\infty} \| \pi_n x \|^2 \| \xi_n \|_r^2 \leqslant (\sup \| \xi_n \|_r^2) \sum_{n=1}^{\infty} \| \pi_n x \|^2$$

$$= (\sup \| \xi_n \|_r^2) \| x \|^2.$$

Therefore the series $\Sigma |\xi_n|^2 \| \pi_n \|^2$ converges absolutely in $L_{r/2}(\mathbb{K})$. Furthermore

$$\left\| \sum_{n=1}^{\infty} |\xi_n|^2 \| \pi_n \|^2 \right\|_{r/2} \leqslant (\sup \| \xi_n \|_r^2) \| x \|^2.$$

Given m. $n \in \mathbb{N}$ we have $\mathbb{E}\| \Sigma_{k=n}^{n+m} \xi_k \pi_k x \|^r = \mathbb{E}(\Sigma_{k=n}^{n+m} |\xi_k|^2 \| \pi_k x \|^2)^{r/2}$, which converges to zero when $n \to \infty$, for every $m \in \mathbb{N}$. So the series $\Sigma \xi_n \pi_n x$ converges in $\mathscr{L}_r(\mathbb{P}, H)$

and

$$\mathbb{E}\left\| \sum_{n=1}^{\infty} \xi_n \pi_n x \right\|^r \leq (\sup \|\xi_n\|_r^r) \|x\|^r,$$

which shows that the linear random operator given by the pointwise sum of the series $\Sigma \xi_n \pi_n$ is continuous.

Conversely assume the series $\Sigma \xi_n \pi_n$ to be pointwise convergent. Then $\{\xi_n \pi_n\}$ gives a sequence of continuous linear operators from H into $L_r(\mathbb{P}, H)$ which is pointwise bounded. Banach–Steinhaus theorem shows that there is a positive number M such that $\sup_{\|x\|=1} \|\xi_n \pi_n x\|_r \leq M \ \forall n \in \mathbb{N}$. Since $\sup_{\|x\|=1} \|\xi_n \pi_n x\|_r = \|\xi_n\|_r$, it follows that the sequence $\{\xi_n\}$ is bounded in r-mean. □

Unfortunately there exist sequences of complex-valued random variables $\{\xi_n\}$ bounded in r-mean with $0 < r < 2$, and sequences $\{\pi_n\}$ of pairwise orthogonal projections for which the random series $\Sigma \xi_n \pi_n x$ does not converge in probability for a suitable $x \in H$. We illustrate this fact in the following.

Examples 5.3. Let e_n be an orthonormal sequence in a Hilbert space H. Consider the sequence $\{\pi_n\}$ of pairwise orthogonal projections on H given by $\pi_n(x) = (x|e_n)e_n$ for every $x \in H$. Also consider the interval [0, 1] endowed with the Lebesgue measure. Given a measurable set Δ, χ_Δ stands for the characteristic function of Δ. If $0 < r < 2$, then we consider the sequence $\{\xi_n\}$ of random variables, given by $\xi_{2^k+m} = 2^{k/r} \chi_{[m/2^k,(m+1)/2^k[}$, $m = 0, \ldots, 2^k - 1$, $k \geqslant 0$. The sequence $\{\xi_n\}$ is bounded in r-mean. Further the sequence $\{\alpha_n\}$ given by $\alpha_{2^k+m} = 2^{-k/r}$, $m = 0, \ldots, 2^k - 1, k \geqslant 0$, satisfies that $\Sigma_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and therefore the series $\Sigma_{n=1}^{\infty} \alpha_n e_n$ defines an element $x \in H$. The random series $\Sigma \xi_n \pi_n x$ satisfies

$$\left\| \sum_{n=1}^{2^{k+1}-1} \xi_n \pi_n x \right\| = \sum_{l=0}^{k} \sum_{m=0}^{2^l-1} |2^{l/r} \chi_{[m/2^l,(m+1)/2^l[} 2^{-l/r}|^2 = \sum_{l=0}^{k} \chi_{[0,1[} = (k+1)\chi_{[0,1[}$$

and therefore it does not even converge in probability.

The random series $\Sigma \varphi(\lambda_n) \pi_n$ defines (up to equivalence) a continuous linear random operator having rth moment whenever φ is a random function acting on sp (K) bounded in r-mean with $r \geq 2$. We write $\varphi(K)$ to denote it.

COROLLARY 5.4

Let Φ be a continuous linear random operator having 2nd moment. Then the following assertions are equivalent:

1. *For every continuous linear deterministic operator F on H commuting with K we have* $\mathbb{P}[\Phi F \equiv F \Phi] > 0$.
2. *There exists a random function φ on* sp(K) *bounded in mean square such that* $\mathbb{P}[\Phi \equiv \varphi(K)] > 0$.

Proof. Assume that assertion 1 holds. By Theorem 5.1 there exists a random function φ acting on sp(K) bounded in mean square such that $\mathbb{P}[\Phi \equiv \varphi(K)] > 0$.

Conversely if 2 is fulfilled, then there exists a measurable set Δ with $\mathbb{P}[\Delta] = \mathbb{P}[\Phi \equiv \varphi(K)]$ and $\Phi_\Delta \equiv \varphi_\Delta(K)$. Given a continuous linear deterministic operator F commuting with K it is known that $\pi_n F = F \pi_n$ for every $n \in \mathbb{N}$. From this it is easy to check

that $\varphi_\Delta(K)Fx = F\varphi_\Delta(K)x$ almost surely for every $x \in H$. Thus $\mathbb{P}[\Phi F \equiv F\Phi] \geq \mathbb{P}[\Delta]$. Therefore assertion 1 follows. □

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Surface instability in a finite thickness fluid saturated porous layer

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Abstract. The Rayleigh–Taylor (RT) instability at the interface between fluid and fluid saturated sparsely packed porous medium has been investigated making use of boundary layer approximation and Saffmann [8] boundary condition. An analytical solution for dispersion relation is obtained and is numerically evaluated for different values of the parameters. It is shown that RT instability can be controlled by a suitable choice of the thickness of porous layer, ratio of viscosities and the slip parameter.

Keywords. RT instability; flow past porous media; Brinkmann model; Saffmann boundary condition.

1. Introduction

Rayleigh–Taylor instability (RT) in viscous fluids has been extensively investigated in the last few years [3], because of its importance in inertial fusion target design, astrophysics and geophysics. Diffusional instabilities of the form first discussed by Saffmann and Taylor [9] which are normally encountered in viscous creeping flow limit are also studied because of their applications in areas of failure of materials. Recently, Brown [2] has studied RT instability in a finite thickness layer of a viscous fluid under the assumption of creeping flow. Rudraiah *et al* [7] have investigated the effect of oblique magnetic field on the RT instability of a finite conducting fluid layer. However, there is a rather different subject area of interest, namely, instability between fluid and fluid saturated porous media where RT instability is relevant. Examples of this may be found in the areas of the failure of metallic glasses, grain boundaries, failure in metals, failure of polymers and so on.

It is also important for the study of motion of contact line where the effective slip of the interface reverses the singularity in the rate of strain which was otherwise introduced by the no-slip condition. In the present situation the interface may be a smooth surface or a rough surface depending upon the solid matrix of the porous material. In the former case the surface can be considered as a nominal surface (postulated by Beaver and Joseph [1] hereafter called as BJ) at which BJ slip condition exists due to transfer of momentum from fluid to fluid saturated porous media. Later Saffmann [8] used a modified version of BJ slip condition to study the flow past a fluid saturated porous media. The BJ slip condition is independent of the thickness of the porous layer and hence valid when the thickness of the porous layer is very much larger than the thickness of the fluid layer. In many industrial and biochemical applications the thickness of the porous layer is comparable to that of fluid layer and hence the slip condition should involve the thickness of

the layer. Later Rudraiah [6] has derived the slip condition known as BJR slip condition involving the thickness of the layer. This BJR slip condition reduces to BJ condition for large thickness of the layer. In the case of rough interface the effective slip condition can replace the rough boundary by a smooth surface and this introduces an effective slip condition applied in the mean position of the interface. In this situation the velocity slip is proportional to tangential stress along the surface; the constant of proportionality is called the slip coefficient. This slip coefficient can be determined using the asymptotic analysis as in the case of fluid in the absence of porous material [4].

In this paper, we study the RT instability at the interface between fluid layer and porous layer of finite thickness using Saffmann slip condition [8]. It is shown that the thickness of the layer, the slip coefficient and the ratio of viscosities of fluid greatly influence the RT instability.

2. Formulation of the problem

The physical configuration considered in this investigation is shown in figure 1. It consists of two regions 1 and 2. The region 1 (i.e., $y > h$) is concerned with a viscous fluid of viscosity μ_{eff}, bounded below at $y = 0$ by a rigid impermeable material with an interface at $y = h$. To derive the basic equations, for this physical configuration considered here, we make the following approximations:

i) The fluid in region 1 is quiescent.

ii) The fluid saturated porous medium in the region 2 is incompressible and performs steady, two-dimensional boundary layer motion, governed by Brinkmann's equation.

Under these approximations the basic equations governing the flow are (Nield [5])

$$\frac{\partial p}{\partial x} = \mu_{\text{eff}}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - \left(\frac{\mu}{k}\right)u, \tag{2.1}$$

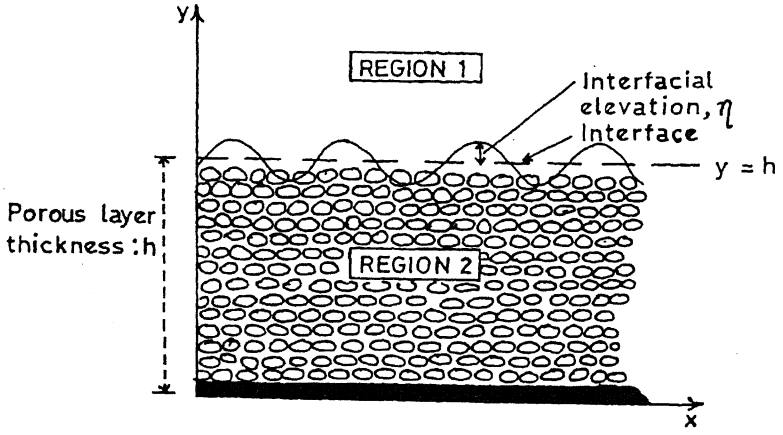

Figure 1. Physical configuration.

and

$$\frac{\partial p}{\partial y} = \mu_{\text{eff}}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - \left(\frac{\mu}{k}\right)v, \tag{2.2}$$

where p is the pressure and u and v are the two-dimensional velocity vector components of the fluid saturated porous medium with components u and v along x and y directions, respectively; k is the permeability of the porous material. Since the fluid is incompressible the continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.3}$$

These equations are governed by the following boundary conditions.

The no slip condition is

$$u = v = 0| \text{ at } y = 0, \tag{2.4}$$

which on using the continuity equation becomes

$$v = Dv = 0| \text{ at } y = 0, \tag{2.5}$$

where $D = \mathrm{d}/\mathrm{d}y$.

At the interface, we use the Saffmann condition [8]

$$\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \left[\frac{\Omega}{\sqrt{k}}\right]u \text{ at } y = h. \tag{2.6}$$

Physically, this represents the continuity of the tangential stress at the interface. Here Ω is the slip parameter which depends on the structure of the porous material. The continuity of normal stress at the interface leads to

$$p = -\delta\eta - \gamma\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \mu_{\text{eff}}\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\mu}{\sqrt{k}}\right)v. \tag{2.7}$$

Here, δ is the stress gradient, γ is the surface tension and $\partial^2\eta/\partial x^2$ is the curvature of the interface, η is the elevation of the interface satisfying the dynamic equation

$$\frac{\partial \eta}{\partial t} + (q\cdot\nabla)\eta = 0. \tag{2.8}$$

This leads to the condition

$$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}. \tag{2.9}$$

In the regime of linear theory eq. (2.9) may be written as

$$v = \frac{\partial \eta}{\partial t}. \tag{2.10}$$

We invoke solutions for the quantities η, u, v, p varying as

$$[\eta(y), u(y), v(y), p(y)] \exp(i\alpha x + nt), \tag{2.11}$$

respectively, where n is the growth rate of the interfacial disturbances and α is the wavevector. From eqs (2.3), (2.6) and (2.11), we get

$$i\alpha u + Dv = 0, \tag{2.12}$$

$$D^2v + \alpha^2 v = \left(\frac{\Omega}{\sqrt{k}}\right) Dv \text{ at } y = h. \tag{2.13}$$

Eliminating pressure term between eqs (2.1)–(2.3) and using eq. (2.12), we get the stability equation

$$D^4v + (2\alpha^2 + \alpha_0^2)D^2v + \alpha^2(\alpha^2 + \alpha_0^2)v = 0, \tag{2.14}$$

where $\alpha_0^2 = 1/Mk$ and $M = \mu_{\text{eff}}/\mu$.

Equation (2.1), using (2.11) leads to

$$p = (\mu_{\text{eff}}/\alpha^2)(D^3v - [\alpha^2 + \alpha_0^2]Dv). \tag{2.15}$$

Equating (2.15) with (2.7) and rearranging the terms we get for the growth rate n the expression

$$n = \frac{v[\gamma\alpha^2 - \delta]}{(\mu_{\text{eff}}/\alpha^2)[D^3v - \{2\alpha^2 + \alpha_0^2\}Dv] + (\mu/\sqrt{k}v)} \text{ at } y = h \tag{2.16}$$

Making eq. (2.16) dimensionless, using $\lambda = \sqrt{(\gamma/\delta)}$ the scale for length

$$\alpha^* = \alpha\lambda,\ D^* = D\lambda,\ h^* = h/\lambda,\ n^* = n\mu_{\text{eff}}(\sqrt{\gamma\,\delta})$$

we get,

$$n^* = \frac{v\alpha^{*2}(1 - \alpha^{*2})}{[(2\alpha^{*2} + \alpha_0^{*2})D^*v - D^{*3}v] - \left[\frac{v\alpha_0^*\alpha^{*2}}{M^{1/2}}\right]} \text{ at } y = h, \tag{2.17}$$

where asterisk denotes the dimensionless quantities. The solution of eq. (2.14) is

$$v(y) = A_1\,\text{ch}(\alpha y) + A_2\,\text{sh}(\alpha y) + A_3\,\text{ch}(ay) + A_4\,\text{sh}(ay), \tag{2.18}$$

where $a = \sqrt{(\alpha^2 + \alpha_0^2)}$ and for brevity, we have denoted $\cosh(x)$ and $\sinh(x)$ by $\text{ch}(x)$ and $\text{sh}(x)$, respectively. The coefficients $A_i (i = 1 \text{ to } 4)$ are determined using the boundary conditions (2.4) to (2.6) and (2.13) and they are given by

$$A_1 + A_3 = 0, \tag{2.19}$$

$$A_4 + A_2\frac{\alpha}{a} = 0, \tag{2.20}$$

$$A_2 + RA_1 = 0, \tag{2.21}$$

$$A_4 - A_1R(\alpha/a) = 0, \tag{2.22}$$

where

$$R = \frac{R_1}{R_2},$$

$$R_1 = [2\alpha^2 \mathrm{ch}(\alpha h) - (\alpha^2 + a^2)\mathrm{ch}(ah)] + \left(\frac{\Omega}{\sqrt{k}}\right)[a\,\mathrm{sh}(ah) - \alpha\,\mathrm{sh}(\alpha h)], \tag{2.23}$$

$$R_2 = \left[2\alpha^2 \mathrm{sh}(\alpha h) - \left(\frac{\Omega\alpha}{\sqrt{k}}\right)\mathrm{ch}(\alpha h) + \left(\frac{\alpha\Omega}{\sqrt{k}}\right)\mathrm{ch}(ah) - \left(\frac{\alpha}{a}\right)(\alpha^2 + a^2)\,\mathrm{sh}(ah)\right]. \tag{2.24}$$

The constants A_i ($i = 1$ to 4) in (2.19) to (2.24) are expressed in terms of A_1 and obtain,

$$v(h) = A_1\left[(\mathrm{ch}(\alpha h) - \mathrm{ch}(ah)) + R\left(\left[\frac{\alpha}{a}\right]\mathrm{sh}(ah) - \mathrm{sh}(\alpha h)\right)\right], \tag{2.25}$$

$$Dv(h) = A_1\left[(\mathrm{sh}(\alpha h) - a\,\mathrm{sh}(ah)) + R\,\mathrm{ch}(ah) - \alpha\,\mathrm{ch}(\alpha h))\right], \tag{2.26}$$

$$D^3 v(h) = A_1\left[\alpha^3 \mathrm{sh}(\alpha h) - a^3 \mathrm{sh}(ah)) + R(\alpha a^2 \mathrm{ch}(ah) - \alpha^3 \mathrm{ch}(\alpha h))\right]. \tag{2.27}$$

The expression for the growth rate of instability of the interface given by eq. (2.17) may be computed numerically for typical values of thickness h, viscosity ratio M, the porous parameter D_A and slip parameter Ω. This aspect is given in the next section.

3. Numerical computation and discussion

Analytical expressions for the dispersion relation in the case of RT instability past a porous layer is obtained and several interesting conclusions are made.

In the limit of $\Omega \to 0$ and $M \to 1$, the dispersion relation (2.17) tends to that given by Brown [2]. This can be seen from the numerical results depicted in figure 2. For different values of Ω, M and h^*, the dispersion relation (2.17) is numerically evaluated and the results are shown in figure 3, 4.

For $h^* < 1$, $M < 1$, $D_A < 1$ and $\Omega < 1$, the growth rate n^* is always positive for all values of α^* as can be seen in figure 3 (curves 1 and 2) exhibiting instability as in the case of RT instability in the absence of porous medium, discussed by Brown [2]. For these values of parameters although the magnitude of growth is one order less than that of Brown [2], the slip at the porous media has a significant effect on the qualitative nature of RT instability.

For $D_A = 1$, $M = 1$, $h^* = 1$ and $\Omega < 1$, n^* is always positive having minimum value at $\alpha^* = 0{\cdot}5$ and maxima at $\alpha^* = 0{\cdot}2$ or $0{\cdot}9$ as seen in the figure 3 (curve 3). This oscillatory nature of n^* is due to the drag at the interface. The same behaviour is also true for increasing values of h^* and D_A.

For $h^* > 1$, $M = 1$, $D_A < 1$ and $\Omega < 1$, n^* takes positive or negative values i.e., the interface is stable or unstable depending on the values of α^* as is seen in figure 4 (curve 1). For an increase in both h^* and M for $D_A < 1$ and $\Omega < 1$, n^* is positive for all α^* as shown in figure 3 (curve 4), indicating the instability.

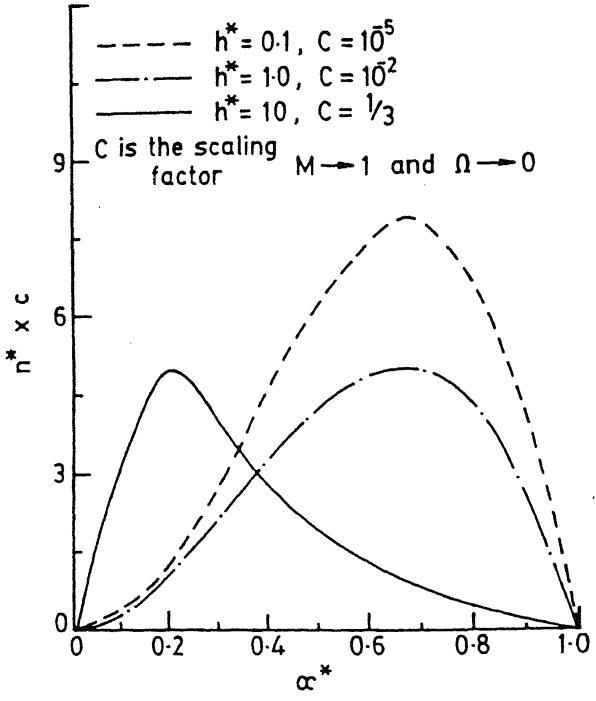

Figure 2. Plot of reduced growth n^* vs. α^*.

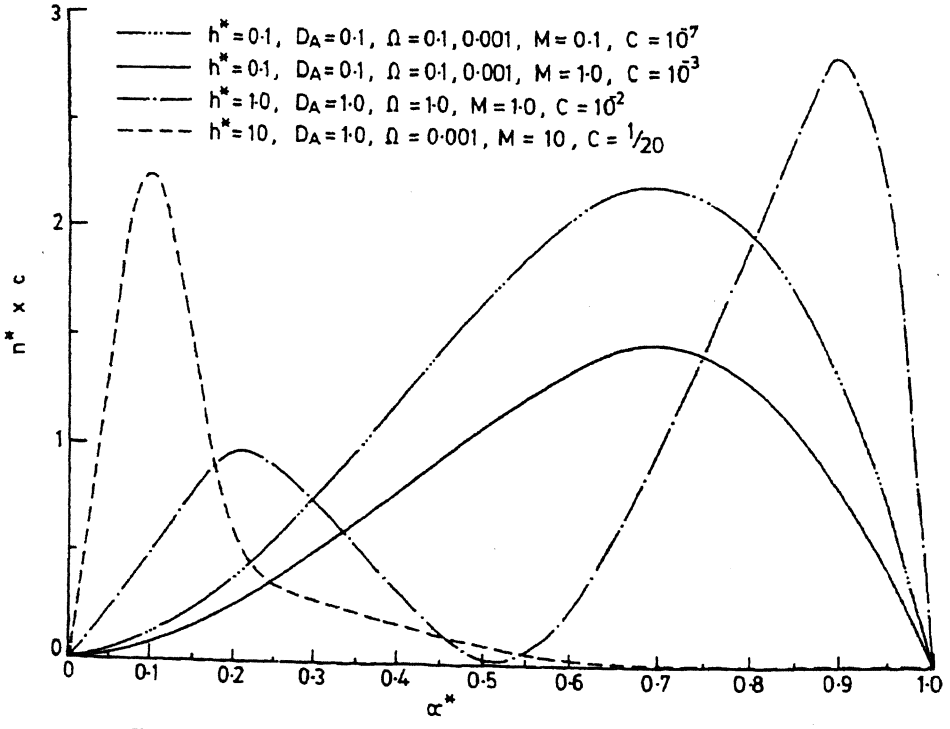

Figure 3. Plot of reduced growth rate vs. wave number, C is the convenient scaling factor.

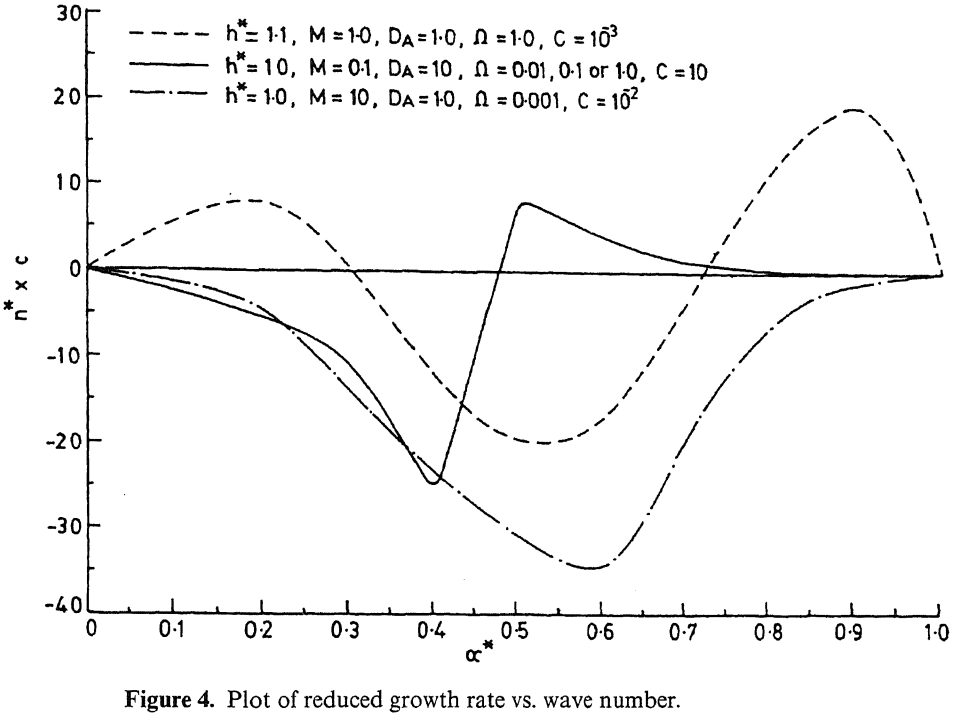

Figure 4. Plot of reduced growth rate vs. wave number.

For $h^* = 10$, $M = 0{\cdot}1$, $D_A = 10$ and $\Omega < 1$, the oscillatory behaviour of n^* prevails for some values of α^* and is shown in figure 4 (curve 2). For a suitable thickness of the porous layer the increase in the value of M make the system stable which was otherwise unstable as seen in figure 4 (curve 3).

From these numerical calculations we conclude that RT instability can be controlled by controlling the thickness of the porous layer and the slip parameter (i.e. structure of porous media). This can be effectively used in materials science processing.

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Erratum

Quasi-parabolic Siegel formula

NITIN NITSURE

(Proc. Indian Acad. Sci. (Math. Sci.), 106, pp. 133–137, May 1996)

Abstract. The main result of the above paper is mistaken, because of a defective lemma. Here we replace the defective lemma, and derive the corrected quasi-parabolic analogue of the Siegel formula.

In my paper 'Quasi-parabolic Siegel formula' (see[N]), the main result is the following formula (Theorem 3.4).

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|} = f(q, r_{i,j})(q-1)^{s-1} q^{(r^2-1)(g-1)-s} Z_{X-S}(q^{-2}) \cdots Z_{X-S}(q^{-r}).$$

The above formula is incorrect. The mistake occurs in Lemma 3.2 of the paper, which asserted that

$$\lim_{m\to\infty} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathscr{O}_X^r, E(m))|}{q^{r\chi(E(m))}} = 1.$$

What follows is the correction to Lemma 3.2 (and its proof), and the resulting correction to the main result (Theorem 3.4 of [N]). We follow the notations and the numbering of equations and statements in the original paper.

Lemma (Corrected version of Lemma 3.2 and eq. (10))

$$\lim_{m\to\infty} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathscr{O}_X^r, E(m))|}{q^{r\chi(E(m))}} = \frac{(q^r-1)^s(q^r-q)^s\cdots(q^r-q^{r-1})^s}{q^{r^2 s}}. \tag{10}$$

If S is non-empty, the limit is already attained for all large enough m (where 'large enough' depends on E).

Proof. If S is empty, the above lemma reduces to lemma 3 in [G-L]. If S is nonempty, then any morphism of locally free sheaves on X which is injective when restricted to S is injective. Let m be large enough, so that $E(m)$ is generated by global sections, $H^1(X, E(m)) = 0$, and $h^0(X, E(m)) = \chi(E(m)) \geqslant rs$. Then $H^0(X, E(m))$ has a basis consisting of sections σ_{i,P_j}, τ_l for $i = 1, \ldots, r, j = 1, \ldots, s$, and $l = 1, \ldots, \chi(E(m)) - rs$, such that

(1) the sections τ_l are zero on S,
(2) the sections σ_{i,P_j}, are zero at all other points of S except P_j (and hence σ_{i,P_j} restrict at P_j to a basis of the fiber of $E(m)$ at P_j.

Any element of $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^r, E(m)) = \mathrm{Hom}_{\mathbf{F}_q}(\mathbf{F}_q^r, H^0(X, E(m)))$ is given in terms of this basis by a $r \times q^{\chi(E(m))}$ matrix A. The condition that this lies in

$$\mathrm{Hom}^S_{\mathrm{inj}}(\mathscr{O}_X^r, E(m)) \subset \mathrm{Hom}(\mathscr{O}_X^r, E(m))$$

is the condition that each of the s disjoint $r \times r$-minors, corresponding to the part $\sigma_{1,P_j}, \ldots, \sigma_{r,P_j}$ of the basis, has nonzero determinant. This contributes the factor

$$\frac{|GL_r(\mathbf{F}_q)|}{|M_r(\mathbf{F}_q)|} = \frac{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})}{q^{r^2}}$$

for each P_j, which proves the lemma.

Now using this new value for $\lim_{m\to\infty}|\mathrm{Hom}^S_{\mathrm{inj}}(\mathscr{O}_X^r, E(m))|/q^{r\chi(E(m))}$ in place of the earlier mistaken value 1 in the equation (10) of [N], but keeping the rest of [N] as it is, we immediately get the folllowing corrected form of the main result (Theorem 3.4 of [N]).

Theorem. (Quasi-parabolic Siegel formula – Corrected form)

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|} = f(q, r_{i,j}) \frac{q^{(r^2-1)(g-1)}}{q-1} Z_X(q^{-2})\cdots Z_X(q^{-r}).$$

This corrected formula has the following essential feature, which the mistaken formula lacked. If the quasi-parabolic structure at each point of S is trivial (that is, each flag consists only of the zero subspace and the whole space), then on one hand $\mathrm{ParAut}(E) = \mathrm{Aut}(E)$, and on the other hand each flag variety is a point, and so $f(q, r_{i,j}) = 1$. Hence in this situation (which includes the case when S is empty) the above formula reduces to the original Siegel formula

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{Aut}(E)|} = \frac{q^{(r^2-1)(g-1)}}{q-1} Z_X(q^{-2})\cdots Z_X(q^{-r}).$$

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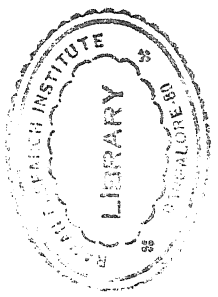
2 7 AUG 1997

Constructing irreducible representations of discrete groups



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Abstract. The decomposition of unitary representations of a discrete group obtained by induction from a subgroup involves commensurators. In particular Mackey has shown that quasi-regular representations are irreducible if and only if the corresponding subgroups are self-commensurizing. The purpose of this work is to describe general constructions of pairs of groups $\Gamma_0 < \Gamma$ with Γ_0 its own commensurator in Γ. These constructions are then applied to groups of isometries of hyperbolic spaces and to lattices in algebraic groups.

Keywords. Commensurator subgroups; unitary representations; quasi-regular representations; Gromov hyperbolic groups; arithmetic lattices.

1. Introduction

Let G be a separable locally compact group. The *unitary dual* $\hat{G}$ of G is the set of equivalence classes of irreducible representations of G, together with its Mackey Borel structure. In this paper, "representation" means "continuous unitary representation in a separable Hilbert space".

Let us recall the definition of this structure [Dix, 18.5]. For each $n \in \{1, 2, \ldots, \infty\}$, let $\mathrm{Irr}_n(G)$ denote the space of all irreducible representations of G in a given Hilbert space of dimension n. The set $\mathrm{Irr}_n(G)$ is endowed with the topology of the weak simple convergence on G (making the functions $\pi \mapsto \langle \pi(g)\xi | \eta \rangle$ continuous for all $g \in G$ and ξ, η in the Hilbert space of dimension n), and with the corresponding Borel structure. The dual $\hat{G}$ is the quotient of $\coprod_{1 \le n \le \infty} \mathrm{Irr}_n(G)$ by unitary equivalence, and the Mackey Borel structure on $\hat{G}$ is the quotient of the previously defined Borel structure.

In case of a countable group Γ, it follows from results of Glimm and Thoma that $\hat{\Gamma}$ is a standard Borel space if and only if Γ is virtually abelian (see [Dix], numbers 9.1, 9.5.6 and 13.11.12, or [Ped, 6.8.7]); in this case the representation theory of Γ is well understood. In all other cases there is no natural Borel coding of $\hat{\Gamma}$, i.e. $\hat{\Gamma}$ is not countably separated; for lack of a systematic procedure of constructing all irreducible representations of Γ, a natural problem is to construct large classes of irreducible representations.

Recall that two subgroups G_0 and G_1 of a group G are *commensurable* if $G_0 \cap G_1$ is of finite index in both G_0 and G_1. The *commensurator* of G_0 in G is defined to be

$$\mathrm{Com}_G(G_0) = \{g \in G \mid G_0 \text{ and } gG_0g^{-1} \text{ are commensurable}\}.$$

Let $(\Gamma_i)_{i \in I}$ be a family of pairwise non conjugate subgroups of a countable group Γ such that $\mathrm{Com}_\Gamma(\Gamma_i) = \Gamma_i$ for all $i \in I$. It follows from work of Mackey (see e.g. [Mac], and § 2

below) that unitary induction provides a well defined and *injective* map

$$\coprod_{i\in I} \widehat{\Gamma_i^{fd}} \subsetneq \hat{\Gamma},$$

where $\widehat{\Gamma_i^{fd}}$ denotes the subset of $\hat{\Gamma}_i$ consisting of finite dimensional representations.

Our aim in this paper is to construct actions with noncommensurable stabilizers and pairs of groups $\Gamma_0 < \Gamma$ such that $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$. More generally, we construct also pairs $\Gamma_0 < \Gamma$ such that Γ_0 is a subgroup of finite index in $\mathrm{Com}_\Gamma(\Gamma_0)$; in this case, the quasiregular representation of Γ in $l^2(\Gamma/\Gamma_0)$ is a *finite* direct sum of irreducible representations.

In § 2, we recall some classical results on unitary representations. Section 3 provides elementary examples of pairs of groups $\Gamma_0 < \Gamma$ with Γ_0 its own commensurator in Γ. We consider groups of isometries of Gromov hyperbolic spaces in § 4. Then, for a lattice Γ in the group of real points of a linear algebraic group $\mathbb{G}$ defined over $\mathbb{R}$, we consider actions of Γ on appropriate sets of maximal tori in § 5 and on other sets of subgroups of $\mathbb{G}$ in § 6; in each case, we find classes of irreducible quasi-regular representations of Γ.

Note on terminology. Commensurators have been known under various names, such as quasinormalizers [Cor], commensurizers [KrR] and commensurability subgroups [Mar]. We follow the terminology of [Shi, Chapter 3] and [A'B].

2. Commensurators and induced representations

Let Γ be a discrete group, $\Gamma_0 < \Gamma$ a subgroup and $\lambda_{\Gamma/\Gamma_0}$ the left regular representation of Γ in $l^2(\Gamma/\Gamma_0)$.

A double class $\dot{x} \in \Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)/\Gamma_0$ represented by some $x \in \mathrm{Com}_\Gamma(\Gamma_0)$ corresponds to a *finite* Γ_0-orbit $\Gamma_0 x \Gamma_0$ in Γ/Γ_0, and the mapping $\Gamma_0 \to \Gamma/\Gamma_0$ applying z to $zx\Gamma_0$ induces a bijection of $\Gamma_0/(\Gamma_0 \cap x\Gamma_0 x^{-1})$ onto $\Gamma_0 x \Gamma_0$. Consequently, $\dot{x}$ gives rise to a *bounded* intertwining operator $T_{\dot{x}}$ of $\lambda_{\Gamma/\Gamma_0}$, which is defined by

$$(T_{\dot{x}} f)(y\Gamma_0) = \sum_{\zeta \in \Gamma_0/(\Gamma_0 \cap x\Gamma_0 x^{-1})} f(y\zeta x \Gamma_0)$$

for all $f \in l^2(\Gamma/\Gamma_0)$ and for all $y\Gamma_0 \in \Gamma/\Gamma_0$.

It is then a fact (see [Bin], Theorem 2.2) that the linear space generated by

$$\{T_{\dot{x}} : l^2(\Gamma/\Gamma_0) \to l^2(\Gamma/\Gamma_0) \mid \dot{x} \in \Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)/\Gamma_0\}$$

is weakly dense in the space $\mathrm{Int}(\lambda_{\Gamma/\Gamma_0})$ of bounded intertwining operators of $\lambda_{\Gamma/\Gamma_0}$. Hence, if $\Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)$ is finite, we have

$$\dim \mathrm{Int}\,(\lambda_{\Gamma/\Gamma_0}) = \mathrm{Card}\,(\Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)/\Gamma_0)$$

and $\lambda_{\Gamma/\Gamma_0}$ is a finite direct sum of irreducible representations. In particular $\lambda_{\Gamma/\Gamma_0}$ is irreducible if and only if $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$.

The above considerations then lead to the following theorem. Here and in the sequel we call two subgroups Γ_0, Γ_1 of Γ *quasiconjugate* if there exists $\gamma \in \Gamma$ such that Γ_0 and $\gamma\Gamma_1\gamma^{-1}$ are commensurable.

Theorem 2.1 [Mackey]. *Let Γ be a discrete group and let Γ_0, Γ_1 be subgroups of Γ.*
(1) *The representation $\lambda_{\Gamma/\Gamma_0}$ is irreducible if and only if $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$, in which case $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi)$ is irreducible for any $\pi \in \widehat{\Gamma_0^{fd}}$, and unitary induction*

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}: \widehat{\Gamma_0^{fd}} \longrightarrow \hat{\Gamma}$$

is an injective map.

(2) *If* $\mathrm{Com}_{\Gamma}(\Gamma_i) = \Gamma_i$, $i = 0, 1$, *then* $\lambda_{\Gamma/\Gamma_0}$ *and* $\lambda_{\Gamma/\Gamma_1}$ *are unitarily equivalent if and only if* Γ_0 *and* Γ_1 *are quasiconjugate in* Γ.

In case Γ_0 *and* Γ_1 *are not quasiconjugate in* Γ, *if* π_0, *respectively* π_1, *are finite dimensional irreducible unitary representations of* Γ_0, *respectively* Γ_1, *then* $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi_0)$ *and* $\mathrm{Ind}_{\Gamma_1}^{\Gamma}(\pi_1)$ *are not equivalent.*

Remark. We do not know whether the condition $\pi \in \widehat{\Gamma_0^{fd}}$ in (1) can be replaced by $\pi \in \hat{\Gamma}_0$.

Let us restate the previous Theorem in a slightly different way. Let Γ be a discrete group acting on a set A, and denote by

$$\mathscr{Z}_{\Gamma}(a) \doteqdot \{\gamma \in \Gamma \mid \gamma a = a\}$$

the stabilizer of a point $a \in A$; if more precision is needed, we write $\mathscr{Z}_{\Gamma,A}(a)$ for $\mathscr{Z}_{\Gamma}(a)$.

DEFINITION

The action $\Gamma \times A \longrightarrow A$ has *noncommensurable stabilizers* (N.C.S.) if any two points a_1, $a_2 \in A$ with commensurable stabilizers coincide.

The following lemma is an easy observation.

Lemma 2.2. (1) *Let* $\Gamma \times A \longrightarrow A$ *be a N.C.S. action. For* a_1, $a_2 \in A$ *and* $\gamma \in \Gamma$, *we have* $\gamma a_1 = a_2$ *if and only if* $\gamma \mathscr{Z}_{\Gamma}(a_1)\gamma^{-1} = \mathscr{Z}_{\Gamma}(a_2)$, *if and only if* $\gamma \mathscr{Z}_{\Gamma}(a_1)\gamma^{-1}$ *and* $\mathscr{Z}_{\Gamma}(a_2)$ *are commensurable.*

In particular $(\mathscr{Z}_{\Gamma}(a))_{a \in A}$ *is a set of self-commensurizing subgroups of* Γ, *two subgroups* $Z_{\Gamma}(a_1), Z_{\Gamma}(a_2)$ *of the set being quasiconjugate if and only if* a_1, a_2 *are in the same* Γ*-orbit.*

(2) *Let* $\mathscr{G}$ *be a set of self-commensurizing subgroups of* Γ *which is stable under conjugation. Then the action of* Γ *on* $\mathscr{G}$ *by conjugation is N.C.S.*

It follows from Theorem 2.1 and Lemma 2.2. that, for a N.C.S. action $\Gamma \times A \longrightarrow A$, unitary induction

$$\mathrm{Ind}: \coprod_{a \in \Gamma \backslash A} \widehat{\mathscr{Z}_{\Gamma}(a)^{fd}} \longrightarrow \hat{\Gamma}$$

is an injective map.

For later use we record the following general fact. Let π, ρ be unitary representations of a group Γ. We write $\pi \prec \rho$ to express that π is *weakly contained* in ρ [Dix, 18.1.3], and $\pi \sim \rho$ to express that π and ρ are *weakly equivalent* [namely that $\pi \prec \rho$ and $\rho \prec \pi$].

Lemma 2.3. *Let* Γ_0 *be a subgroup of* Γ. *Then* $\lambda_{\Gamma/\Gamma_0} \prec \lambda_{\Gamma}$ *if and only if* Γ_0 *is amenable.*

Proof. If Γ_0 is amenable, $1_{\Gamma_0} \prec \lambda_{\Gamma_0}$ and hence $\lambda_{\Gamma/\Gamma_0} = \mathrm{Ind}_{\Gamma_0}^{\Gamma}(1_{\Gamma_0}) \prec \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\lambda_{\Gamma_0}) = \lambda_{\Gamma}$.

Conversely, since 1_{Γ_0} is contained in $\mathrm{Res}_{\Gamma_0}(\lambda_{\Gamma/\Gamma_0})$ and since $\mathrm{Res}_{\Gamma_0}(\lambda_{\Gamma})$ is a multiple of λ_{Γ_0}, the assumption $\lambda_{\Gamma/\Gamma_0} \prec \lambda_{\Gamma}$ implies

$$1_{\Gamma_0} \prec \mathrm{Res}_{\Gamma_0}(\lambda_{\Gamma/\Gamma_0}) \prec \mathrm{Res}_{\Gamma_0}(\lambda_{\Gamma}) \sim \lambda_{\Gamma_0}$$

and hence Γ_0 is amenable. □

3. Elementary examples of N.C.S. actions

Define a group action $G \times A \to A$ to be *large* if, for all $a \in A$, all $\mathscr{Z}_G(a)$-orbits in $A \backslash \{a\}$ are infinite. The next lemma is a convenient tool for constructing N.C.S. actions.

Lemma 3.1. (1) *A large action is N.C.S.*
(2) *Let $G \times A \to A$ be a large transitive action and let $\Gamma < G$ be a subgroup such that $\mathrm{Com}_G \Gamma = G$. Assume that there exists a point $a_0 \in A$ such that all $\mathscr{Z}_{\Gamma,A}(a_0)$-orbits in $A \backslash \{a_0\}$ are infinite. Then the restricted action $\Gamma \times A \to A$ is large.*

Proof. (1) For a large action $G \times A \to A$ and for two points $a_1, a_2 \in A$ with $\mathscr{Z}_G(a_1)$ and $\mathscr{Z}_G(a_2)$ commensurable, the $\mathscr{Z}_G(a_1)$-orbit of a_2 is finite and hence $a_1 = a_2$.

(2) For $a \in A$ and $g \in G$ such that $ga_0 = a$, the $\mathscr{Z}_{\Gamma,A}(a)$-orbits in $A \backslash \{a\}$ are infinite if and only if the $(g^{-1}\mathscr{Z}_{\Gamma,A}(a)g)$-orbits in $A \backslash \{a_0\}$ are infinite. Since

$$g^{-1}\mathscr{Z}_{\Gamma,A}(a)g = g^{-1}\Gamma g \cap \mathscr{Z}_{G,A}(a_0)$$

and $G = \mathrm{Com}_G \Gamma$, the subgroup

$$\Delta_0 \doteq \mathscr{Z}_{\Gamma,A}(a_0) \cap g^{-1}\mathscr{Z}_{\Gamma,A}(a)g = Z_{\Gamma,A}(a_0) \cap g^{-1}\Gamma g$$

is of finite index in $\mathscr{Z}_{\Gamma,A}(a_0)$. In particular all Δ_0-orbits in $A \backslash \{a_0\}$ are infinite and the same holds therefore for $g^{-1}Z_{\Gamma,A}(a)g$. □

(Claim (1) of Lemma 3.1 is a straightforward generalization of Theorem 4 in [Oba], which delas with doubly transitive actions on infinite sets.)

Example 1. Let $\mathbb{K}$ be an infinite field and let $\mathrm{Gr}_k(\mathbb{K}^n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{K}^n$, where n, k are integers with $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$.

The natural action of $GL(n, \mathbb{K})$ on $\mathrm{Gr}_k(\mathbb{K}^n)$ is N.C.S.

If $\mathbb{K}$ is a number field and if $\mathcal{O}_{\mathbb{K}}$ denotes its ring of integers, the action of $GL(n, \mathcal{O}_{\mathbb{K}})$ on $\mathrm{Gr}_k(\mathbb{K}^n)$ is N.C.S.

Proof. For two distinct points y_1, y_2 in $\mathrm{Gr}_k(\mathbb{K}^n)$, the maximal parabolic subgroup

$$P_{y_1} \doteq \{g \in GL(n, \mathbb{K}) \mid g y_1 = y_1\}$$

acts transitively on the infinite subset

$$\{y \in \mathrm{Gr}_k(\mathbb{K}^n) \mid \dim_{\mathbb{K}}(y \cap y_1) = \dim_{\mathbb{K}}(y_2 \cap y_1)\}$$

of the Grassmannian. Hence the transitive action of $GL(n, \mathbb{K})$ on $\mathrm{Gr}_k(\mathbb{K}^n)$ is large; in particular P_y is its own commensurator in $GL(n, \mathbb{K})$ for all $y \in G_k(\mathbb{K}^n)$.

Let $\mathbb{K}$ be now a number field. If $y_0 \in \mathrm{Gr}_k(\mathbb{K}^n)$ denote the subspace spanned by the first k vectors of the canonical basis of $\mathbb{K}^n$ and if $\Gamma = GL(n, \mathcal{O}_{\mathbb{K}})$, one has

$$\mathscr{Z}_\Gamma(y_0) = \left\{ \gamma \in \Gamma \mid \gamma \text{ of the form } \begin{pmatrix} * & \dots & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \dots & * & * & \dots & * \\ 0 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * & \dots & * \end{pmatrix} \right\}$$

(with the block of zeros having $n-k$ rows and k columns). Let $y_1 \in \mathrm{Gr}_k(\mathbb{K}^n)\backslash\{y_0\}$; set $l = k - \dim_{\mathbb{K}}(y_0 \cap y_1)$. We identify $\mathbb{K}^n/y_0$ with the vector space $\mathbb{K}^{n-k}$. The actions of P_{y_0} on $\mathbb{K}^n$ and on $\{g \in \mathrm{Gr}_k(\mathbb{K}^n) | \dim(y \cap y_0) = \dim(y_1 \cap y_0)\}$ factor as actions of $GL(n-k, \mathbb{K})$ on $\mathbb{K}^{n-k}$ and $\mathrm{Gr}_l(\mathbb{K}^{n-k})$ respectively, so that the action of $\mathscr{Z}_\Gamma(y_0)$ on $\mathrm{Gr}_k(\mathbb{K}^n)\backslash\{y_0\}$ factors as an action of $GL(n-k, \mathcal{O}_{\mathbb{K}})$ on $\mathrm{Gr}_l(\mathbb{K}^{n-k})$. The latter action has clearly all its orbits infinite, since the Zariski closure of $GL(n-k, \mathcal{O}_{\mathbb{K}})$ contains that of $GL(n-k, \mathbb{Z})$ and thus contains $SL(n-k, \mathbb{C})$. It follows first that all orbits of $\mathscr{Z}_\Gamma(y_0)$ on $\mathrm{Gr}_k(\mathbb{K}^n)\backslash\{y_0\}$ are infinite, and second that $\mathscr{Z}_\Gamma(y) = \Gamma \cap P_y$ is its own commensurator in $\Gamma = GL(n, \mathcal{O}_{\mathbb{K}})$ for all $y \in \mathrm{Gr}_k(\mathbb{K}^n)$. □

We observe the following consequence of Example 1.

PROPOSITION 3.2

The unitary representation π of $SL(n, \mathbb{Z})$ in $L^2(\mathbb{R}^n/\mathbb{Z}^n)$ is an orthogonal direct sum of irreducible representations.

Proof. By Fourier transform, π is equivalent to the permutation representation of $SL(n, \mathbb{Z})$ in $l^2(\mathbb{Z}^n)$; the latter is a direct sum of quasi-regular representations $\pi_k \doteqdot \lambda_{SL(n,\mathbb{Z})/\Gamma_k}$, where Γ_k denotes the stabilizer of $(k, 0, \ldots, 0) \in \mathbb{Z}^n$ in $SL(n, \mathbb{Z})$, for all $k \geqslant 0$. The one-dimensional representation π_0 is irreducible. For $k \geqslant 1$, and Γ'_k the stabilizer of $(k:0:\cdots:0) \in \mathbb{P}^{n-1}(\mathbb{Q})$, Mackey's result and Example 1 imply that $\lambda_{SL(n,\mathbb{Z})/\Gamma'_k}$ is irreducible. As Γ_k is of index 2 in Γ'_k, the representation π_k is either irreducible or sum of 2 irreducibles. □

For a group action $G \times A \to A$ and subsets $B \subset A$, $S \subset G$ we set

$$\mathscr{Z}_{G,A}(B) \doteqdot \bigcap_{b \in B} \mathscr{Z}_{G,A}(b)$$

$$\mathscr{N}_{G,A}(B) \doteqdot \{g \in G | g(B) = B\}$$

and $\mathscr{F}_A(S)$ the set of common fixed points of elements in S. Observe that

$$\mathscr{N}_{G,A}(B) = \mathscr{Z}_{G,\mathscr{P}(A)}(B),$$

where $\mathscr{P}(A)$ denotes the power set of A.

Lemma 3.3. *Let $G \times A \to A$ be an action and let $S \subset G$ be a union of conjugacy classes of G such that*

$$\mathscr{F}_A(g) = \mathscr{F}_A(g^n) \quad \textit{and} \quad |\mathscr{F}_A(g)| < \infty$$

for all $g \in S$ and for all $n > 1$. Then the action of G on the set

$$\{F \in \mathscr{P}(A) | F = \mathscr{F}_A(g) \textit{ for some } g \in S\}$$

is N.C.S.

Proof. Let $g, h \in S$ be such that the subgroups $\mathscr{N}_{G,A}(\mathscr{F}_A(g))$ and $\mathscr{N}_{G,A}(\mathscr{F}_A(h))$ are commensurable in G. Since $\mathscr{F}_A(g)$ and $\mathscr{F}_A(h)$ are both finite subsets of A, the subgroup

$$K \doteqdot \mathscr{Z}_{G,A}(\mathscr{F}_A(g)) \cap \mathscr{Z}_{G,A}(\mathscr{F}_A(h))$$

is of finite index in $\mathscr{Z}_{G,A}(\mathscr{F}_A(g))$ and $\mathscr{Z}_{G,A}(\mathscr{F}_A(h))$.

Hence there exists an integer $N \geqslant 1$ such that g^N and h^N are in K. One has

$$\mathscr{F}_A(g) = \mathscr{F}_A(g^N) \supset \mathscr{F}_A(K) \supset \mathscr{F}_A(\mathscr{L}_{G,A}(\mathscr{F}_A(h))) = \mathscr{F}_A(h)$$

and similarly $\mathscr{F}_A(h) \supset \mathscr{F}_A(g)$, so that $\mathscr{F}_A(h) = \mathscr{F}_A(g)$. □

Example 2. Consider a subgroup Γ of $SL(n, \mathbb{C})$ and an element $\gamma \in \Gamma$ which is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and which is regular in the following sense: one has $\lambda_j^N \neq \lambda_k^N$ for each integer $N \geqslant 1$ whenever j, k are distinct in $\{1, \ldots, n\}$; in other words, the fixed point set $\mathscr{F}(\gamma)$ of γ in $\mathbb{P}^{n-1}(\mathbb{C})$ has cardinality n and $\mathscr{F}(\gamma^N) = \mathscr{F}(\gamma)$ for all integers $N \in \mathbb{Z}$, $N \neq 0$. Then the subgroup

$$\mathscr{N}_{\Gamma, \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma)) = \{\gamma' \in \Gamma \mid \gamma' \text{ permutes the eigen-directions of } \gamma\}$$

of Γ is its own commensurator in Γ by Lemma 3.3. (This subgroup of Γ is distinct from Γ itself as soon as Γ is not virtually abelian.)

Observe that the group

$$\mathbb{T} \doteq \mathscr{L}_{SL(n, \mathbb{C}), \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma))$$

is a maximal torus in $SL(n, \mathbb{C})$ and that $\mathscr{N}_{\Gamma, \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma))$ is the intersection with Γ of the normalizer of $\mathbb{T}$ in $SL(n, \mathbb{C})$. More on this in § 5 below.

Example 3. Consider an integer $n \geq 2$, the group $\Gamma = SL(n, \mathbb{Z})$ and the subgroup Γ_0 of upper triangular matrices in Γ (with diagonal entries ± 1).

Then Γ_0 is its own commensurator in Γ.

Proof. Let $\mathrm{Flag}(\mathbb{C}^n)$ be the set of complete flags in $\mathbb{C}^n$. Let S be the subset of Γ consisting of matrices which have precisely one Jordan block. Then, for the action of Γ on $\mathrm{Flag}(\mathbb{C}^n)$, one has $\mathscr{F}(\gamma) = \mathscr{F}(\gamma^n)$ and $|\mathscr{F}(\gamma)| = 1$ for all $\gamma \in S$. This ends the proof because Γ_0 is the stabilizer of the flag $\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n-1}$ associated to the canonical basis of $\mathbb{C}^n$. □

Consider the group $\Gamma = SL(3, \mathbb{Z})$. For a subgroup $\Gamma_0 = \Gamma \cap P_y$ as in Example 1, it follows from Lemma 2.3 that the irreducible representation $\lambda_{\Gamma/\Gamma_0}$ is not weakly contained in λ_Γ. But for a subgroup $\Gamma_0 = \mathscr{N}_{\Gamma, \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma))$ as in Example 2 or for the triangular subgroup Γ_0 of Example 3, one has $\lambda_{\Gamma/\Gamma_0} \prec \lambda_\Gamma$ by Lemma 2.3, and consequently $\lambda_{\Gamma/\Gamma_0} \sim \lambda_\Gamma$ by [BCH].

There are examples of self-commensurizing subgroups of braid groups and of related groups in [FRZ] and in [Par].

4. Groups of isometries of hyperbolic spaces

4.1. Let X be a Gromov hyperbolic space; let $X(\infty)$ be its Gromov boundary and $\mathrm{Is}(X)$ its group of isometries. Then $\mathrm{Is}(X)$ acts on $X(\infty)$ and on $S^2X(\infty)$, the set of unordered pairs of points in $X(\infty)$.

Let Γ be a subgroup of $\mathrm{Is}(X)$. Denote by $X(\infty)_p \subset X(\infty)$ the set of fixed points of parabolic elements in Γ and by $S^2X(\infty)_h \subset S^2X(\infty)$ the set of fixed point sets of hyperbolic elements in Γ.

PROPOSITION 4.1

The action of Γ *on*

$$X(\infty)_p \coprod S^2 X(\infty)_h$$

has noncommensurable stabilizers.

Proof. Let Γ_{ne} denote the set of non elliptic elements in Γ. For the Γ-action on $X(\infty)$ and for each $\gamma \in \Gamma_{ne}$, one has

$$\mathscr{F}_{X(\infty)}(\gamma) = \mathscr{F}_{X(\infty)}(\gamma^n) \text{ for all } n \geq 1$$

and $\mathscr{F}_{X(\infty)}(\gamma)$ is of cardinality 1 or 2 depending on whether γ is parabolic or hyperbolic. Thus Proposition 4.1 follows from Lemma 3.3. □

Remark. For each hyperbolic element $\gamma \in \Gamma$, recall that the cyclic group $\gamma^{\mathbb{Z}}$ is of finite index in the group $\mathscr{Z} = \mathscr{Z}_{\Gamma, S^2 X(\infty)}(\mathscr{F}_{X(\infty)}(\gamma))$; see e.g. [GhH, chap. 8, n° 33]; in particular, the group $\mathscr{Z}$ is amenable. By Lemma 2.3, the quasi-regular representation $\lambda_{\Gamma/\mathscr{Z}}$ is weakly contained in the regular representation λ_Γ.

Assume moreover that X is a discrete space which has at most exponential growth and that $\Gamma \subset \mathrm{Is}(X)$ is a discrete subgroup. For each parabolic element $\gamma \in \Gamma$, the group $\mathscr{Z} = \mathscr{Z}_{\Gamma, X(\infty)}(\mathscr{F}_{X(\infty)}(\gamma))$ is amenable (see Proposition 1.6 in [BuM]), so that one has also $\lambda_{\Gamma/\mathscr{Z}} \prec \lambda_\Gamma$. Indeed, the set

$$\{\mathscr{Z}_{\Gamma, X(\infty)}(\omega) \mid \omega \in X(\infty)_p \coprod S^2 X(\infty)_h\}$$

coincides with the set of all maximal amenable infinite subgroups of Γ [Ada].

In case Γ is a Gromov hyperbolic group, the set $X(\infty)_p$ is empty because there is no parabolic. If Γ is moreover torsion free, then $\mathscr{Z}_\Gamma(\omega)$ is infinite cyclic for all $\omega \in S^2 X(\infty)_h$.

It is known that the reduced C^*-algebra of a torsion free Gromov hyperbolic group Γ is simple [Har]. From this and Lemma 2.3, it follows that the quasi-regular representation $\lambda_{\Gamma/\mathscr{Z}_\Gamma(\omega)}$ is quasi-equivalent to the regular representation λ_Γ for each $\omega \in S^2 X(\infty)_h$.

For a nonabelian free group, this is Proposition 1 of [Boz], itself a paper strongly motivated by [Yos].

4.2. Let now X be a proper CAT(-1)-space and let

$$\mathscr{G}X = \{c : \mathbb{R} \longrightarrow X \mid c \text{ is isometric}\}$$

be the space of parametrized geodesics in X with the topology of uniform convergence on compactas. The action of $\mathbb{R}$ on $\mathscr{G}X$ via reparametrizations

$$g_t c(s) = c(s+t), \quad c \in \mathscr{G}X, \quad s, t \in \mathbb{R}$$

commutes with that of $\mathrm{Is}(X)$ and defines for any discrete subgroup $\Gamma < \mathrm{Is}(X)$ a flow on $\Gamma \backslash \mathscr{G}X$, called the *geodesic flow*. We recall that, for a discrete divergence group $\Gamma < \mathrm{Is}(X)$, there is a canonical *Patterson–Sullivan measure* m_{PS} on $\Gamma \backslash \mathscr{G}X$ which is invariant and ergodic for the geodesic flow. The notion of a divergence group is borrowed from Patterson–Sullivan theory of Kleinian groups ([Pat], [Sul]; see also [Bou], [Coo], [CoP] which is generalized to CAT(-1)-spaces in [BuM]).

PROPOSITION 4.2

Let $\Lambda < \mathrm{Is}(X)$ be a discrete subgroup. Let

$$\mathscr{S}(\Lambda) = \{\Gamma < \Lambda \mid \Gamma \text{ is a divergence group with } m_{PS}(\Gamma\backslash\mathscr{G}X) < \infty\}$$

be endowed with the ordering given by inclusion and let $\mathscr{C} \subset \mathscr{S}(\Lambda)$ be a commensurability class.

Then $\mathscr{C}$ has a unique maximal element $\Gamma_{\mathscr{C}}$, and this subgroup $\Gamma_{\mathscr{C}}$ satisfies $\mathrm{Com}_\Lambda \Gamma_{\mathscr{C}} = \Gamma_{\mathscr{C}}$. Moreover, if $\sim$ denotes the relation of commensurability on $\mathscr{S}(\Lambda)$, the action of Λ on $\mathscr{S}(\Lambda)/\sim$ by conjugation is N.C.S.

In particular, for each $\Gamma < \mathscr{S}(\Lambda)$, the quasi-regular representation $\lambda_{\Lambda/\Gamma}$ is a finite sum of irreducible representations; if $\Gamma_+ = \mathrm{Com}_\Lambda(\Gamma)$, then Γ is of finite index in Γ_+ and $\lambda_{\Lambda/\Gamma_+}$ is irreducible.

Remarks. (i) Let $\Gamma < \mathrm{Is}(X)$ be a non-elementary discrete subgroup, $\mathscr{L}_\Gamma \subset X(\infty)$ its limit set and $Q_\Gamma = \mathrm{Co}(\mathscr{L}_\Gamma) \subset X$ the convex hull of the latter. If $\Gamma\backslash Q_\Gamma$ is compact (that is, if Γ is convex-cocompact) then Γ is a divergence group with $m_{PS}(\Gamma\backslash\mathscr{G}X) < \infty$; see [Bou].

(ii) Let X be a symmetric space of rank 1 and $\Gamma < \mathrm{Is}(X)$ a geometrically finite subgroup (see [Bow]). Then Γ is a divergence group with $m_{PS}(\Gamma\backslash\mathscr{G}X) < \infty$.

Example. Let $\Lambda < PSL(2, \mathbb{R})$ be a discrete subgroup. Then $\mathscr{S}(\Lambda)$ contains all finitely generated non virtually cyclic subgroups of Λ. Indeed, such subgroups are non-elementary and geometrically finite.

Thus, for a finitely generated infinite subgroup Γ of Λ, the quasi-regular representation $\lambda_{\Lambda/\Gamma}$ is a *finite* sum of irreducible representations: this follows from Proposition 4.1 if Γ is virtually cyclic, in which case $\lambda_{\Lambda/\Gamma} \prec \lambda_\Lambda$, and from Proposition 4.2 in other cases, for which $\lambda_{\Lambda/\Gamma} \not\prec \lambda_\Lambda$.

Proof of Proposition 4.2. It suffices to show that, given a discrete divergence group $\Gamma_0 < \mathrm{Is}(X)$ with $m_{PS}(\Gamma_0\backslash\mathscr{G}X) < \infty$ and a discrete subgroup $\Gamma < \mathrm{Is}(X)$ with $\Gamma_0 < \Gamma < \mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0)$, the subgroup Γ_0 is of finite index in Γ.

Indeed, assuming this is true, consider the commensurability class $\mathscr{C}$ of a subgroup Γ_0 of Λ which is in $\mathscr{S}(\Lambda)$. Setting $\Gamma_{\mathscr{C}} = \mathrm{Com}_\Lambda(\Gamma_0)$ one has Γ_0 of finite index in $\Gamma_{\mathscr{C}}$; one has therefore $\Gamma_{\mathscr{C}} \in \mathscr{S}(\Lambda)$ and $\mathrm{Com}_\Lambda \Gamma_{\mathscr{C}} = \Gamma_{\mathscr{C}}$. As any group commensurable with Γ_0 is in $\Gamma_{\mathscr{C}}$, the latter group is clearly the *unique* maximal element of $\mathscr{C}$. The last claim of the proposition is now obvious.

For the convenience of the reader we recall the construction of m_{PS} (see § 1.3 in [BuM]). Let δ be the critical exponent of Γ_0, let $\mu: X \to M^+(X(\infty))$ be the δ-dimensional Patterson–Sullivan density for Γ_0 and let $(\xi|\eta)_x$ denote the Gromov scalar product of ξ, $\eta \in X(\infty)$. Using the Γ-invariant measure

$$\frac{d\mu_x(\xi) \times d\mu_y(\xi)}{e^{-2\delta(\xi|\eta)_x}}$$

on $X(\infty) \times X(\infty)\backslash\{\text{diagonal}\}$, one obtains a Γ-invariant and geodesic-flow invariant measure $\tilde{m}_\mu$ on $\mathscr{G}X$; the Patterson–Sullivan measure m_{PS} is then the corresponding geodesic-flow invariant measure on $\Gamma\backslash\mathscr{G}X$.

We recall furthermore that $\gamma_* \mu_x = \mu_{\gamma x}$ for all $\gamma \in \Gamma_0$, $x \in X$, and that there exists a homomorphism χ: $\mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0) \to \mathbb{R}_+^*$ such that $\gamma_* \mu_x = \chi(\gamma)\mu_x$ for all $\gamma \in \mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0)$, $x \in X$. From this follows $\gamma_* \tilde{m}_\mu = \chi(\gamma)^2 \tilde{m}_\mu$ for all $\gamma \in \mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0)$ (see [BuM], Corollary 6.5.3).

Since Γ acts properly discontinuously on $\mathscr{G}X$, there exists a compact set $K \subset \mathscr{G}X$ of positive $\tilde{m}_\mu$-measure such that $\gamma K \cap K = \emptyset$ for all $\gamma \in \Gamma$ with $\gamma \neq e$. (We argue as if Γ was acting effectively on $\mathscr{G}X$; when it is not the case, we leave the minor appropriate changes to the reader.) For a set $\mathscr{T} \subset \Gamma$ of representatives of $\Gamma_0 \backslash \Gamma$, the set $\coprod_{\tau \in \mathscr{T}} \tau K$ injects into $\Gamma_0 \backslash \mathscr{G}X$ and therefore

$$\left(\sum_{\tau \in \mathscr{T}} \chi(\tau)^2 \right) \tilde{m}_\mu(K) = \tilde{m}_\mu \left(\coprod_{\tau \in \mathscr{T}} \tau K \right) \leqslant m_{\mathrm{PS}}(\Gamma_0 \backslash \mathscr{G}X) < \infty.$$

Hence, since $\chi | \Gamma_0 = 1$, we obtain

$$\sum_{\tau \in \Gamma_0 \backslash \Gamma} \chi(\tau)^2 < \infty.$$

For every $\gamma \in \Gamma$, we have thus

$$\left(\sum_{\tau \in \Gamma_0 \backslash \Gamma} \chi(\tau)^2 \right) \chi(\gamma)^2 = \sum_{\sigma \in \Gamma_0 \backslash \Gamma} \chi(\sigma)^2$$

which shows first that $\chi(\gamma)^2 = 1$ for all $\gamma \in \Gamma$ and second that $|\Gamma_0 \backslash \Gamma| < \infty$. □

5. Maximal tori and actions of lattices with noncommensurable stabilizers

Let $\mathbb{G}$ be a linear algebraic group defined over $\mathbb{R}$, let $\Gamma < \mathbb{G}(\mathbb{R})$ be a discrete subgroup and set

$$\mathscr{T}(\Gamma) = \{ \mathbb{T} \subset \mathbb{G} | \mathbb{T} \text{ is a maximal } \mathbb{R}\text{-split torus such that } \mathbb{T}(\mathbb{R})/(\mathbb{T}(\mathbb{R}) \cap \Gamma) \text{ is compact} \}.$$

PROPOSITION 5.1

The Γ-action by conjugation on $\mathscr{T}(\Gamma)$ is N.C.S.

Here and in the sequel, we will use the following simple lemma.

Lemma 5.2. *Let $\mathbb{G}$ be a linear algebraic group and let A_0, A_1 be two commensurable subgroups of $\mathbb{G}$. Then $\overline{(A_0)}^0 = \overline{(A_1)}^0$.*

Proof of Proposition 5.1. We have to show that, given $\mathbb{T}, \mathbb{T}' \in \mathscr{T}(\Gamma)$ such that $\mathscr{N}_{\mathbb{G}}(\mathbb{T}) \cap \Gamma$ and $\mathscr{N}_{\mathbb{G}}(\mathbb{T}') \cap \Gamma$ are quasiconjugate in Γ, then $\mathbb{T}$ and $\mathbb{T}'$ are Γ-conjugate.

First we observe that, for $\mathbb{T} \in \mathscr{T}(\Gamma)$, the group $(\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R}) \cap \Gamma)/(\mathbb{T}(\mathbb{R}) \cap \Gamma)$ is finite. Indeed, since $\mathbb{T}(\mathbb{R})/(\mathbb{T}(\mathbb{R}) \cap \Gamma)$ is compact, the canonical map

$$\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R})/(\mathbb{T}(\mathbb{R}) \cap \Gamma) \longrightarrow \mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R})/\mathbb{T}(\mathbb{R})$$

is proper and therefore $(\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R}) \cap \Gamma)/(\mathbb{T}(\mathbb{R}) \cap \Gamma)$ is a discrete subgroup of the compact group $\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R})/\mathbb{T}(\mathbb{R})$.

If now $\mathscr{N}_{\mathbb{G}}(\mathbb{T}) \cap \Gamma$ and $\mathscr{N}_{\mathbb{G}}(\mathbb{T}') \cap \Gamma$ are quasiconjugate in Γ, there exist $\Delta < \mathbb{T}(\mathbb{R}) \cap \Gamma$ of finite index and $\gamma \in \Gamma$ such that $\gamma \Delta \gamma^{-1}$ is of finite index in $\Gamma \cap \mathbb{T}'(\mathbb{R})$. Passing to Zariski closure, we obtain $\mathbb{T}' = \gamma \overline{\Delta} \gamma^{-1} = \gamma \mathbb{T} \gamma^{-1}$. □

Examples. (1) Let $\mathbb{G}$ be a semisimple $\mathbb{R}$-group and $\Gamma < \mathbb{G}(\mathbb{R})$ a lattice. Then $\mathscr{T}(\Gamma) \neq \emptyset$; this follows from the existence of $\mathbb{R}$-hyper-regular elements in Γ [PrR]. Indeed, for such a $\gamma \in \Gamma$, the centralizer $\mathscr{Z}_{\mathbb{G}}(\gamma)$ contains an $\mathbb{R}$-split torus $\mathbb{T}$ which is maximal in $\mathbb{G}$ and such that $\mathbb{T}(\mathbb{R})/(\Gamma \cap \mathbb{T}(\mathbb{R}))$ is compact.
(2) Let $\mathscr{P}$ be the set of primitive indefinite integral binary forms

$$Q(X, Y) = aX^2 + bXY + cY^2$$

with $a > 0$. Then the map which to every $Q \in \mathscr{P}$ associates $SO(Q)^0$ gives a bijection between $\mathscr{P}$ and the set of $\mathbb{R}$-split tori $\mathbb{T} \subset SL(2)$ for which $SL(2, \mathbb{Z}) \cap \mathbb{T}(\mathbb{R})$ is a lattice in $\mathbb{T}(\mathbb{R})$:

$$\mathscr{P} \cong \mathscr{T}(SL(2, \mathbb{Z})).$$

(3) It is a general fact due to Ono [Ono] that, for a $\mathbb{Q}$-torus $\mathbb{T}$ with $X_{\mathbb{Q}}(\mathbb{T}) = 1$, the group $\mathbb{T}(\mathbb{R})/\mathbb{T}(\mathbb{Z})$ is compact. Hence, given a semisimple $\mathbb{Q}$-group $\mathbb{G}$, the set $\mathscr{T}(\mathbb{G}(\mathbb{Z}))$ contains all $\mathbb{Q}$-torii $\mathbb{T}$ which are maximal $\mathbb{R}$-split and such that $X_{\mathbb{Q}}(\mathbb{T}) = 1$. As examples of such torii in $SL(n)$, let $\mathbb{K}/\mathbb{Q}$ be a totally real number field or degree n, let $\mathbb{H} \doteqdot \mathrm{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}L_1 \subset \mathbb{G}L_n$ and $\mathbb{T} \doteqdot \mathbb{H} \cap \mathbb{S}L(n)$. The group $\mathscr{U}_{\mathbb{K}}$ of units of $\mathbb{K}$ is abelian of rank $n-1$ and isomorphic to $\mathbb{H}(\mathbb{Z})$. As $\mathbb{T}(\mathbb{Z})$ is of index at most two in $\mathbb{H}(\mathbb{Z})$, the torus $\mathbb{T}(\mathbb{Z})$ is of rank $n-1$ and hence $\mathbb{T}(\mathbb{R})/\mathbb{T}(\mathbb{Z})$ is compact.

6. Algebraic subgroups and actions of arithmetic lattices with noncommensurable stabilizers

In this section $\mathbb{G}$ denotes a connected linear algebraic $\mathbb{Q}$-group; let

$$\mathscr{S}_{\mathbb{G}} = \{\mathbb{H} \mid \mathbb{H} \text{ is a connected } \mathbb{Q}\text{-subgroup of } \mathbb{G}, \text{ of finite index in } \mathscr{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0)\}.$$

We will show below that if $\mathbb{H}$ is a connected $\mathbb{Q}$-subgroup of $\mathbb{G}$, one always has the inclusion

$$\mathbb{H} < \mathscr{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0).$$

PROPOSITION 6.1

The action by conjugation of $\mathbb{G}(\mathbb{Z})$ on $\mathscr{S}_{\mathbb{G}}$ is N.C.S. and $\mathscr{S}_{\mathbb{G}}$ contains all parabolic $\mathbb{Q}$-subgroups of $\mathbb{G}$.

Lemma 6.2. *Let $\mathbb{H}$ be a $\mathbb{Q}$-subgroup of $\mathbb{G}$.*

(1) $\mathscr{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q}) < \mathrm{Com}_{\mathbb{G}}(\mathbb{H}(\mathbb{Z}))$

(2) $\mathscr{N}_{\mathbb{G}}(\mathbb{H})^0 < \mathscr{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0)$.

Proof of Lemma 6.2. Let us first show the implication (1) $\Longrightarrow$ (2). As $\mathscr{N}_{\mathbb{G}}(\mathbb{H})$ is defined over $\mathbb{Q}$, one has

$$\mathscr{N}_{\mathbb{G}}(\mathbb{H})^0 < \overline{\mathscr{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q})}$$

by a theorem of Rosenlicht [Bor, 18.3]. On the other hand Lemma 5.2 implies

$$\overline{\mathrm{Com}_{\mathbb{G}}(\mathbb{H}(\mathbb{Z}))} < \mathscr{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0)$$

and hence (1) implies (2).

In order to prove (1) we may assume that $\mathbb{H}$ is connected. Let $X_{\mathbb{Q}}(\mathbb{H})$ be the set of $\mathbb{Q}$-characters of $\mathbb{H}$ and set

$$\mathbb{H}_0 \doteqdot \bigcap_{\chi \in X_{\mathbb{Q}}(\mathbb{H})} \operatorname{Ker} \chi.$$

Clearly, $\mathbb{H}_0(\mathbb{Z})$ is a subgroup of finite index in $\mathbb{H}(\mathbb{Z})$ and it follows from [BHC] that $\mathbb{H}_0(\mathbb{Z})$ is a lattice in $\mathbb{H}_0(\mathbb{R})$. Observe also that $\mathcal{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q})$ acts on $X_{\mathbb{Q}}(\mathbb{H})$ and hence normalizes $\mathbb{H}_0$.

Let $\mathbb{G} < GL(n, \mathbb{C})$ for some n, fix $g \in \mathcal{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q})$ and choose an integer $m \geq 1$ such that mg and mg^{-1} are in $M_n(\mathbb{Z})$. For the subgroup

$$\Gamma \doteqdot \{\gamma \in \mathbb{H}_0(\mathbb{Z}) \mid \gamma \equiv \mathrm{id} \bmod m^2\},$$

we have $g\Gamma g^{-1} \subset M_n(\mathbb{Z})$ and $\det(g\Gamma g^{-1}) \subset \{1, -1\}$; hence $g\Gamma g^{-1} < \mathbb{H}_0(\mathbb{Z})$. Furthermore, Γ is of finite index in $\mathbb{H}_0(\mathbb{Z})$ and since $\mathbb{H}_0(\mathbb{Z})$ is a lattice in $\mathbb{H}_0(\mathbb{R})$, the conjugate $g\Gamma g^{-1}$ is of finite index in $\mathbb{H}_0(\mathbb{Z})$ as well. Hence

$$g \in \mathrm{Com}_{\mathbb{G}}(\mathbb{H}_0(\mathbb{Z})) = \mathrm{Com}_{\mathbb{G}}(\mathbb{H}(\mathbb{Z})). \qquad \square$$

Proof of Proposition 6.1. For the first assertion, take $\mathbb{H}_1, \mathbb{H}_2 \in \mathcal{S}_{\mathbb{G}}$ such that $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_1)(\mathbb{Z})$ and $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_2)(\mathbb{Z})$ are commensurable, hence $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_1)^0(\mathbb{Z})$ and $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_2)^0(\mathbb{Z})$ are also commensurable. Since $\mathbb{H}_i$ is connected, we have $\mathbb{H}_i < \mathcal{N}_{\mathbb{G}}(\mathbb{H}_i)^0$ and since $\mathbb{H}_i \in \mathcal{S}_{\mathbb{G}}$, Lemma 6.2.2 implies that $\mathbb{H}_i$ is of finite index in $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_i)^0$, in particular $\mathbb{H}_1(\mathbb{Z})$ and $\mathbb{H}_2(\mathbb{Z})$ are commensurable. This implies $\overline{\mathbb{H}_1(\mathbb{Z})}^0 = \overline{\mathbb{H}_2(\mathbb{Z})}^0$, and hence

$$\mathbb{H}_1 = \mathcal{N}_{\mathbb{G}}((\overline{\mathbb{H}_1(\mathbb{Z})}^0)^0 = \mathcal{N}_{\mathbb{G}}((\overline{\mathbb{H}_2(\mathbb{Z})}^0)^0 = \mathbb{H}_2.$$

For the second assertion, let $\mathbb{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbb{G}$. Since $\mathbb{P} \subset \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{P}(\mathbb{Z})}^0)$, the subgroup $\mathbb{P}' \doteqdot \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{P}(\mathbb{Z})}^0)$ is $\mathbb{Q}$-parabolic and hence $\mathcal{R}_u(\mathbb{P}') \subset \mathcal{R}_u(\mathbb{P})$. Since $\overline{\mathbb{P}(\mathbb{Z})}^0$ is normal in $\mathbb{P}'$ we have

$$\mathcal{R}_u(\overline{\mathbb{P}(\mathbb{Z})}^0) \subset \mathcal{R}_u(\mathbb{P}').$$

On the other hand, $\overline{\mathcal{R}_u(\mathbb{P})(\mathbb{Z})} = \mathcal{R}_u(\mathbb{P})$ and hence $\mathcal{R}_u(\mathbb{P})$ is a (normal) subgroup of $\overline{\mathbb{P}(\mathbb{Z})}^0$, which implies $\mathcal{R}_u(\overline{\mathbb{P}(\mathbb{Z})}^0) \supset \mathcal{R}_u(\mathbb{P})$. This finally shows that $\mathcal{R}_u(\mathbb{P}) = \mathcal{R}_u(\mathbb{P}')$ and hence $\mathbb{P} = \mathbb{P}'$. $\square$

Examples. Assume $\mathbb{G}$ to be a semi-simple, defined over $\mathbb{Q}$ and $\mathbb{Q}$-simple. Let $\mathbb{H}$ be a connected semi-simple $\mathbb{Q}$-subgroup of $\mathbb{G}$ which is maximal as a $\mathbb{Q}$-subgroup. Then $\mathbb{H} = \mathcal{N}_{\mathbb{G}}(\mathbb{H})$, and hence $\mathbb{H} = \mathrm{Com}_{\mathbb{G}}(\mathbb{H})$ by Lemma 5.2. Observe that $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$ and that $\mathbb{H}(\mathbb{Z})$ is a lattice in $\mathbb{H}(\mathbb{R})$, by [BHC].

Maximal subgroups of the classical groups have been classified by Dynkin [Dyn]. In case $\mathbb{G}$ is $SL(n, \mathbb{C})$ with its standard $\mathbb{Q}$-structure, examples of subgroups $\mathbb{H}$ as above include (to quote but a few):

(i) orthogonal groups $SO(q) \subset SL(n, \mathbb{C})$ for a non degenerate quadratic form q over $\mathbb{Q}$.
(ii) the symplectic group $Sp(n, \mathbb{C}) \subset SL(n, \mathbb{C})$ (n even),
(iii) the images of the fundamental representations $SL(m, \mathbb{C}) \to SL(\binom{m}{p}, \mathbb{C})$.

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A new proof of Suzuki's formula

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Abstract. We give a different proof of a formula of Suzuki and its strengthening by Zaidenberg for the topological Euler characteristic of an affine surface fibered over a curve. We deduce this formula using the ideas for the proof of an analogous formula for a proper morphism.

Keywords. Fibration; Euler characteristic.

1. Introduction

In [5] and [6] the following result has been proved.

Theorem. *Let V be a smooth irreducible quasi-projective surface and let f: $V \to B$ be a locally Stein (with respect to B) morphism onto a smooth quasi-projective curve B such that a general fiber of f is irreducible. Then we have:*

(a) $\chi(V) = \chi(B)\cdot\chi(F) + \Sigma_s(\chi(F_s) - \chi(F))$ *where, χ denotes topological Euler characteristic, F is a general fiber of f and F_s are the finitely many singular fibers of f (cf. § 1 for the definitions). Further, $\chi(F_s) \geqslant \chi(F)$ for all s.*
(b) *Suppose that for some singular fiber F_s there is an equality $\chi(F_s) = \chi(F)$. Then F is isomorphic to either $\mathbf{C}$ or $\mathbf{C}^* := \mathbf{C} - \{\text{one point}\}$ and $(F_s)_{\text{red}}$ is isomorphic to F.*

Part (a) of the Theorem was proved by Suzuki in [5] and its strengthening in part (b) was proved by Zaidenberg in [6, Lemma 3.2].

Suzuki's formula together with the above strengthening by Zaidenberg are very useful in the theory of non-complete algebraic surfaces. But the proof given in [5] is rather terse and involves the use of plurisubharmonic functions and Milnor's results on the topology of isolated singularities, etc. Suzuki has proved this formula for a slightly more general situation. But the version we give here is enough for algebraic geometric applications. In [6], a more detailed argument is given by developing the notion of subharmonic partitions (and also Milnor's result). Therefore this result is not as accessible to algebraic geometers as is desirable. Suzuki's formula for a proper morphism f and its strengthening is well-known. Our only observation is that using some arguments from [4, ch. 4, §4], it is easy to deduce Suzuki's formula for the non-complete case. This is the aim of this short note. We will also give a proof of its strengthening by Zaidenberg, as it is not very well-known. We believe that our proof is elementary and transparent.

2. Preliminaries

Let $f\colon V \to B$ be as above. It is well-known that there exists a unique smallest finite subset $S \subset B$ such that $f\colon V - f^{-1}(S) \to B - S$ is a C^∞- locally trivial fiber bundle. Any fiber $f^{-1}(s)$ for $s \in S$ is called a singular fiber of f. A fiber of f which is not a singular fiber is called a general fiber.

Remark. If f is locally affine with respect to B then using the improvement of Suzuki's formula in [6] mentioned above, we can easily see that there is a unique smallest subset S such that for any $s \in S$ either F_s is not reduced or if it is reduced then F_s is not homeomorphic to F.

For any complex analytic variety W, unless otherwise stated, by a component of W, we mean an irreducible component of W.

For any simplicial complex K, its topological Euler characteristic is denoted by $\chi(K)$. The ith Betti number of K is denoted by $b_i(K)$.

The cardinality of a finite set T is denoted by $\# T$.

For any smooth algebraic surface Y and a (not necessarily reduced) divisor $C := \Sigma a_i C_i$ on Y where all the irreducible curves C_i are complete, the *arithmetic genus* of C is defined by $p_a(C) = C^2 + C \cdot K/2 + 1$.

The following result is a special case of a rather general result proved in [3, Lemma 1.5].

Lemma 1. *Let $g\colon W \to C$ be a morphism from a smooth irreducible algebraic surface W onto a smooth quasi-projective curve C such that a general fiber of g is irreducible.*
(i) *If every fiber of g contains a reduced component, then we have an exact sequence*

$$\pi_1(F) \to \pi_1(W) \to \pi_1(C)$$

where, F is a general fiber of g.
(ii) *If U is a small contractible neighborhood of a point in C then the image of the map $\pi_1(F) \to \pi_1(g^{-1}(U))$ has finite index in $\pi_1(g^{-1}(U))$.*

Part (ii) easily follows from part (i) by taking a suitable ramified cover $V \to U$ of U and considering the normalized fiber product $\tilde{V} := V \times_U g^{-1}(U)$ with the natural map $\tilde{g}\colon \tilde{V} \to V$. ($\tilde{V}$ can be assumed to be a finite topological covering of $g^{-1}(U)$ such that the special fiber of $\tilde{g}$ contains a reduced irreducible component. Then (ii) applies.)

Lemma 2. *Let Y be a smooth quasi-projective surface and $C := \Sigma_{i=1}^{i=l} C_i$ be a (reduced) complete, connected curve on Y. Then $\Sigma_{i<j} \#(C_i \cap C_j) \geqslant l - 1$. If equality holds then the dual graph of C is a tree.*

The proof of this easy result is left to the reader. The next result is standard (cf. [4, ch. 4, Lemma 5]).

Lemma 3. *Let $f\colon Y \to B$ be a proper morphism from a smooth quasi-projective surface to a smooth curve with connected fibers. Let $F_b := f^*(b)$ and Δ an effective divisor supported on $f^{-1}(b)$. Then $\Delta^2 \leqslant 0$ and equality occurs iff Δ is a rational multiple of F_b.*

The next result is proved in [4, ch. 4, Lemma 4].

Lemma 4. *Let (Y, C) be as in Lemma 2 above. Then $\chi(C) \geqslant -C \cdot (C + K) + l - 1$. If equality holds then the dual graph of C is a tree.*

Proof. In [4, Ch. 4, Lemma 4], the inequality $\chi(C) \geqslant -C \cdot (C + K) + \Sigma_{i<j} \#(C_i \cap C_j)$ is proved. Hence by Lemma 2, $\chi(C) \geqslant -C \cdot (C + K) + l - 1$, and equality holds implies that the dual graph of C is a tree.

We will implicitly use the following easy result whose proof is left to the reader.

Lemma 5. Let C be a (reduced) complete, connected curve on a smooth quasi-projective surface Y and let E be an exceptional curve of the first kind contained in C. Let $f: Y \to Z$ be the contraction of E to a smooth point on the smooth surface Z. Then $b_1(C) = b_1(f(C))$.

The point of this result is that the number of essential loops in C and $f(C)$ is the same.

The only non-trivial result about Stein manifolds that we need in our proof of the theorem is the following property proved first by Bishop.

Lemma 6. A connected Stein manifold of dimension > 1 is connected at infinity.

Proof. We will only give a sketch for a topological proof. Let M be a Stein manifold of dimension $n > 1$. We can find arbitrarily large compact subsets K of M having the properties:

(1) K is a C^∞-manifold of real dimension $2n$ with nice boundary and
(2) the interior K^0 of K is a Stein manifold of dimension n which has the same homotopy type as K.

Since $H_i(K^0; \mathbf{Q}) = (0)$ for $i > n$, it follows that $H^{2n-1}(K; \mathbf{Q}) = (0)$. Now consider the long exact cohomology sequence with compact supports for the pair (M, K). The end piece of the sequence is

$$\cdots \to H_c^{2n-1}(K; \mathbf{Q}) \to H_c^{2n}(M, K; \mathbf{Q}) \to H_c^{2n}(M; \mathbf{Q}).$$

By duality, $H_c^{2n}(M, K; \mathbf{Q}) \simeq H_0(M - K; \mathbf{Q})$ and $H_c^{2n}(M, \mathbf{Q}) \simeq H_0(M; \mathbf{Q}) \simeq \mathbf{Q}$. It follows that $H_0(M - K; \mathbf{Q}) \simeq \mathbf{Q}$ and hence $M - K$ is connected.

3. Proof of the Theorem

Let V, B, f be as in the statement of the Theorem. With the notation introduced in § 1, we have $\chi(V - f^{-1}(S)) = \chi(B - S) \cdot \chi(F)$. For any complex manifold M and a closed complex subvariety N of M the following relation holds $\chi(M - N) = \chi(M) - \chi(N)$. For the proof of this, it is enough to throw away the singular locus of N from both M and N and then use the long exact sequence for cohomology with compact support for the pair (M, N) and Poincaré duality with compact support for the pair (M, N). Hence we get the formula

$$\chi(V) = \chi(B) . \chi(F) + \Sigma_s(\chi(F_s) - \chi(F)).$$

Let F_s be a singular fiber of f. Next we prove the inequality $\chi(F_s) \geqslant \chi(F)$. For this, let $V \subset X$ where X is a smooth quasi-projective surface such that f extends to a proper morphism $\tilde{f}: X \to B$. We will also assume that $D := X - V$ is a divisor with simple normal crossings and the closure $\bar{F}_s$ of F_s in X intersects D transversally.

Any (irreducible) component of D which is not contained in any fiber of $\tilde{f}$ is called a *horizontal* component of D. Any (irreducible) component of D which is contained in a fiber of f is called a *vertical* component of D. We denote by $H_1, H_2, \ldots, H_l$ the horizontal components of D and $H := \cup_1^l H_i$.

Let U be a small contractible neighborhood of s in B not containing any other singular point of f. Further, we assume that $(\tilde{F}_s)_{\text{red}}$ is a strong deformation retract of $\tilde{f}^{-1}(U)$ where, $\tilde{F}_s$ is the fiber of $\tilde{f}$ containing F_s. By assumption, $f^{-1}(U)$ is Stein. Let $\tilde{F}$ denote a general fiber of $\tilde{f}$.

By Lemma 1, we have $b_1(\tilde{F}) \geqslant b_1(\tilde{f}^{-1}(U)) = b_1((\tilde{F}_s)_{\mathrm{red}})$. Since $b_2(\tilde{F}) = 1 \leqslant b_2(\tilde{F}_s)$, we get $\chi(\tilde{F}) \leqslant \chi(\tilde{F}_s)$. Equality holds if and only if $b_1(\tilde{F}) = b_1(\tilde{F}_s)$ and $\tilde{F}_s$ is irreducible.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ be all the connected components of the union of all the vertical components of D contained in $\tilde{F}_s$. We denote by $C_1, C_2, \ldots, C_n$ the irreducible components of $\cup\Gamma_i$. Since f is locally Stein with respect to B, $f^{-1}(U)$ is connected at infinity (Lemma 6). Hence for each i, $\Gamma_i \cap H_j \neq \phi$ for some j, $1 \leqslant j \leqslant l$.

Let D_j be an irreducible component $\Gamma := \cup\Gamma_i$ such that $D_j \cap H \neq \phi$. Let m_j be the multiplicity of D_j in the scheme-theoretic fiber $\tilde{F}_s$. Then for a point $p \in D_j \cap H$, a small neighborhood of p in $\tilde{f}^{-1}(U)$ contains exactly m_j points of $\tilde{F} \cap H$.

Similarly, if C is an irreducible component of F_s occuring with multiplicity m in $\tilde{F}_s$, then in a small neighborhood of any point in $C \cap H$ there are exactly m points of $\tilde{F} \cap H$. With these notations, we get from $\tilde{F}_s \cdot H = \tilde{F} \cdot H$,

$$\Sigma_j m_j D_j \cdot H + \Sigma_C m_C C \cdot H = \tilde{F} \cdot H.$$

Here, the first summation is over the irreducible components of Γ and the second over the irreducible components of F_s. Note that by assumption $\overline{F_s}$ and D have normal crossings and hence $H \cap \Gamma \cap \overline{F_s} = \phi$.

We can write

$$\chi(\tilde{F}_s) - \chi(\tilde{F}) = (b_1(\tilde{F}) - b_1(\tilde{F}_s)) + (b_2(\tilde{F}_s) - 1).$$

Since $F_s = \tilde{F}_s - (\overline{F_s} \cap H) \cup \Gamma$, we get

$$\chi(F_s) = \chi(\tilde{F}_s) - \chi(\Gamma) - \#H \cap \overline{F_s}$$

and

$$\chi(F) = \chi(\tilde{F}) - \Sigma_j m_j D_j \cdot H - \Sigma_C m_C C \cdot H.$$

This gives

$$\chi(F_s) - \chi(F) = \chi(\tilde{F}_s) - \chi(\tilde{F}) + (\Sigma_j m_j D_j \cdot H - \chi(\Gamma)) + (\Sigma_C m_C C \cdot H - \#H \cap \overline{F_s}).$$

Clearly, $b_2(\tilde{F}_s) = b_2(\overline{F_s}) + n$. Hence we have

$$\chi(F_s) - \chi(F) = b_1(\tilde{F}) - b_1(\tilde{F}_s) + (b_2(\overline{F_s}) + n - 1) + (\Sigma_j m_j D_j \cdot H - r - n + b_1(\Gamma)) + (\Sigma_C m_C C \cdot H - \#H \cap \overline{F_s}).$$

Since $\Gamma_j \cap H \neq \phi$ for each j, we certainly have $\Sigma_j m_j D_j \cdot H \geqslant r$. Similarly, $\Sigma_C m_C C \cdot H \geqslant \#H \cap \overline{F_s}$. Now it is clear that $\chi(F_s) \geqslant \chi(F)$. This proves part (a) of the Theorem.

Proof of part (b). Suppose $\chi(F_s) = \chi(F)$. Then we get the following equalities:

(1) $b_1(\tilde{F}) = b_1(\tilde{F}_s)$ and $b_2(\overline{F_s}) = 1$. In particular, F_s is irreducible.

(2) $\Sigma_j m_j D_j \cdot H = r$, $m_{F_s} \overline{F_s} \cdot H = \#H \cap \overline{F_s}$. In particular, if F_s is not reduced in $\tilde{F}_s$ then $H \cap \overline{F_s} = \phi$.

(3) $b_1(\Gamma) = 0$. Hence each Γ_j is a connected union of smooth rational curves such that the dual graph of Γ is a tree. Further, H intersects each Γ_j transversally in exactly one point and the irreducible component of Γ_j meeting H is reduced in $\tilde{F}_s$.

First we dispose off the case when F_s is not a reduced fiber.

Case 1. Suppose that F_s has multiplicity $m > 1$. Then we consider the ramified map $U \to U$ given by $z \to z^m$ and the normalized fiber product $P := f^{-1}(U) \times_U U$. This is an étale cover of $f^{-1}(U)$ of degree m. The fiber of the map $P \to U$, say P_s, satisfies by part (a) above $\chi(P_s) = m\chi(F_s) \geqslant \chi(F) = \chi(F_s)$. Hence $\chi(F_s) \geqslant 0$. If $\chi(F_s) = 0 = \chi(F)$, then clearly F is isomorphic to $\mathbf{C}^*$. If $\chi(F_s) = 1 = \chi(F)$, then F is isomorphic to $\mathbf{C}$. In either case $\tilde{f}$ is a morphism with general fiber $\mathbf{P}^1$. It is well-known that any irreducible component of any fiber of a $\mathbf{P}^1$-fibration on a smooth projective surface is smooth (cf. [2, ch. 1, Lemma 4.4.1]). Therefore F_s is smooth and hence isomorphic to F, if taken with reduced structure.

Case 2. Now we assume that $\overline{F_s}$ is a reduced irreducible component of $\tilde{F}_s$. Denote by C the reduced divisor $(\tilde{F}_s)_{\text{red}}$. By Lemma 4 and (1) above

$$\chi(C) = 1 - b_1(C) + (1 + n) \geqslant -C \cdot (C + K) + n$$

i.e. $C \cdot (C + K) + 2 \geqslant b_1(C)$ and equality holds only if the dual graph of C is a tree. By Lemma 3, $C^2 \leqslant 0$ and equality occurs iff $\tilde{F}_s$ is reduced (since $\overline{F_s}$ has multiplicity 1).

We write $\tilde{F}_s = \overline{F_s} + \Sigma_1^n m_i C_i$.

Claim. The dual graph of C is a tree and C_i is an exceptional curve of the first kind for some i.

Suppose that $K.C_i \geqslant 0$ for each i. Then we have

$$b_1(C) \leqslant C^2 + C \cdot K + 2 \leqslant \tilde{F}_s^2 + \tilde{F}_s \cdot K + 2 = \tilde{F}^2 + \tilde{F} \cdot K + 2 = b_1(\tilde{F}).$$

From $b_1(C) = b_1(\tilde{F})$, we see that all the inequalities are equalities and this implies that the dual graph of C is a tree and for any i with $C_i.K \neq 0$, $m_i = 1$. Since each Γ_i is a tree of smooth rational curves by (3), we deduce easily that $b_1(\overline{F_s}) = b_1(\tilde{F}_s)$. Hence

$$2p_a(\tilde{F}) - 2 = \tilde{F} \cdot K = (\overline{F_s} + \Sigma n_i C_i) \cdot K \leqslant 2p_a(\overline{F_s}) - 2 = \overline{F_s}^2 + \overline{F_s} \cdot K.$$

This equation implies that $(\Sigma m_i C_i) \cdot K = \overline{F_s}^2 < 0$, if $\Gamma \neq \phi$. Thus $C_i \cdot K < 0$ and $C_i^2 < 0$ for some i implies that C_i is an exceptional curve of the first kind.

We contract the exceptional curve C_i to a smooth point on a smooth surface X_1 and consider the new embedding $V \subset X_1$. For the new closure of F_s in X_1, say $(\overline{F_s})_1$, we have $p_a((\overline{F_s})_1) \geqslant p_a(\overline{F_s}) \geqslant p_a(\tilde{F})$.

On the other hand, the first Betti numbers of C and its image in X_1 are the same. Using Lemma 4 we repeat the above argument. This gives a new (-1)-curve in the fiber containing $\overline{F_{s1}}$, not equal to $\overline{F_{s1}}$. Continuing this argument we see that the dual graph of C is a tree. This proves the claim.

After contracting all the exceptional curves as above, we arrive at a smooth quasi-projective surface Y containing V as a Zariski-open subset and with a morphism $g\colon Y \to B$ extending f. The fiber G_s of g containing F_s is the closure of F_s in Y. Since each Γ_j intersects $\overline{F_s}$ only once, we see that G_s has at worst unibranch singularities. Hence G_s is homeomorphic to its normalization $\tilde{G}_s$. By flatness of g, we get the equality $p_a(G_s) = p_a(\tilde{F})$. The fundamental group of the complete fiber containing F_s remains unchanged in the sequence of blowings-down (this uses the observation that the dual graph of C is a tree). Therefore $b_1(\tilde{G}_s) = b_1(G_s) = b_1(\tilde{F})$ also holds.

Since the arithmetic genus of G_s is at least as big as the genus of $\tilde{G}_s$ = genus of $\tilde{F}$, we deduce that G_s is itself smooth. This implies that the same number of points from G_s and $\tilde{F}$ are removed to obtain F_s and F respectively. Now it follows that F_s is a general fiber of f. This completes the proof of part (b) of the Theorem.

The following result is useful in some applications of Suzuki's result. It follows from the proof of part (b) of the Theorem.

COROLLARY

With the above notations, $\chi(F_s) - \chi(F) = 1$ *if and only if exactly one of the quantities* $b_1(\tilde{F}) - b_1(\tilde{F}_s), b_2(\overline{F_s}) - 1, b_1(\Gamma), \Sigma_j m_j D_j \cdot H - r, \Sigma_C m_C C \cdot H - \#H \cap \overline{F_s}$ *is equal to* 1 *and all the other quantities are equal to* 0.

Remark. We can give a very short proof of part (a) of the Theorem by using Grauert's contraction theorem (cf. [1]). After contracting all the Γ_i to normal singular points on a normal complex surface, we come to a situation analogous to the surface Y above with the morphism g (but possibly Y may have singularities). Then firstly, $\chi(G_s) \geqslant \chi(\tilde{F})$. Next we see easily that $\#(\tilde{F} - F) \geqslant \#(G_s - F_s)$. From this (a) follows. But Grauert's proof also uses plurisubharmonic functions! Therefore we have given a somewhat longer but elementary argument above.

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Substantial Riemannian submersions of S^{15} with 7-dimensional fibres

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Abstract. In this paper we show that a substantial Riemannian submersion of S^{15} with 7-dimensional fibres is congruent to the standard Hopf fibration. As a consequence we prove a slightly weak form of the diameter rigidity theorem for the Cayley plane which is considerably stronger than the very recent radius rigidity theorem of Wilhelm.

Keywords. Riemannian submersions; Euler class; connections; holonomy groups; second fundamental form.

1. Introduction

The study of Riemannian foliations of Euclidean spheres with no restrictions on the geometry of leaves was initiated in [8] and a complete classification was obtained for all the foliations having leaves of dimensions less than or equal to three. Earlier, the classification under the very strong hypothesis of the leaves being totally geodesic was accomplished in [5, 13] and [9]. A remarkable discovery in [8] was the fact that for substantial or highly nonflat (see [8] for definition) Riemannian foliations of S^n, the horizontal holonomy group reduces to a Lie group. The low dimensional Riemannian foliations could then be shown to be the orbits of this group. Thus proving substantiality was a crucial step and it wascarried out for the leaves of dimensions less than or equal to three. As a result it is known in particular that if $\pi: S^n \to M$ is a Riemannian submersion with connected fibres, then either it is congruent to a Hopf fibration or we have $\pi: S^{15} \to M^8$. Now the reduction to a Lie group is valid for substantial foliations without restrictions on leaf dimension. Hence it is natural to ask the following question:

If the Riemannian submersion $\pi: S^{15} \to M^8$ is substantial along every leaf, is it congruent to the Hopf fibration ? Since the question is algebraic in nature, this author was under the impression that it must have been settled by now. However looking at a recent paper [14] it was realised that such was not the case. In [14] the following lemma occurs.

Lemma. Let $\pi : S^{15}(1) \to V$ be a Riemannian submersion with connected γ-dimensional fibres, and let G be the set of points $v \in V$ so that $\pi^{-1}(v)$ is totally geodesic. Theneither G is discrete or G is totally geodesic and an isomertrically embedded copy of $S^l(1/2)$ for some $1 \leqslant l \leqslant 8$.

Now in [8] itself it is implicit that if $G \neq \phi$, then π will be substantial along every leaf and it should be the case that G is all of V. This would simplify much of the subsequent work in [14]. Therefore, in the paper we prove following:

Theorem 1.1. *If $\pi: S^{15} \to M^8$ is a Riemannian submersion with connected fibres which is substantial along each leaf, then it is congruent to the Hopf fibration.*

As a corollary we also prove theorem 1.2.

Theorem 1.2. (*Almost diameter rigidity for CaP^2*). *Let M have sectional curvatures $\geqslant 1$, diam $(M) = \pi/2$. Moreover, let M admit an equilateral triangle of sides $\pi/2$ and have the integral cohomology ring same as that of Cayley plane. Then M is isometric to the standard CaP^2 with $1 \leqslant K \leqslant 4$.*

This almost ties up the loose end in [7] and both the radius rigidity theorem in [14] and the corollary II in [4] are considerably improved.

2. The group of the submersion

In this section we determine the Lie group which could possible occur as the structure group (holonomy group) of the Riemannian submersion $\pi: S^{15} \to M^8$ when we assume it to be substantial.

Theorem 2.3. *If $\pi: S^{15} \to M^8$ is a Riemannian submersion which is substantial along each leaf, then the holonomy group G of π is such that Lie $G = so(8)$. Hence G is either $SO(8)$ or* Spin (8).

Proof. By [8], the holonomy group G of π is a Lie group of dimension $\leqslant 28$ and it acts transitively on the fibres. The fibres of π are well known to be homotopy 7-spheres (see [3]). We conclude therefore that a maximal compact subgroup of G can only be one of the following: $SO(8)$, Spin(8), Spin(7), $SU(4)$, $U(4)$ and $Sp(2)$ (see [1], p. 195).

Now by the long exact sequence of homotopy of π (or directly from [3]) M^8 is a homotopy 8-sphere and by the Thom–Gysin sequence of π, the Euler class of this 7-sphere fibration is the generator of $H^8(M, \mathbb{Z}) \approx \mathbb{Z}$. This enables us to rule out Spin(7), since the Euler class of a Spin(7) bundle must be torsion (this is because $H^*(B\,\text{Spin}(7), \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3]$ is generated by Pontrjagin classes only). Since $Sp(2) \subset SU(4) \subset U(4)$, if we rule out $U(4)$ then the other two will be ruled out automatically. Now $H^*(BU(4), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, c_3, c_4]$ is generated by Chern classes and c_4 also represents the Euler class χ. Since M is a homotopy sphere of dimension 8, by Bott's integrality theorem (see [11], p. 279), c_4 is divisible by 3! in $H^8(M, \mathbb{Z})$. But then χ cannot be a generator–a contradiction! Therefore, we conclude that $G =$ Spin (8) or $SO(8)$ due to the dimension constraint stated above. q.e.d.

Remark. Since there are no Whitney classes w_1 and w_2, we can fix the group to be Spin(8) which we do from now on.

COROLLARY 2.1

Let $\chi(N)$ denote the Lie algebra of smooth vector fields on a smooth manifold N. For $b \in M$, the map

$$A: \wedge^2(T_bM) \to \chi(\pi^{-1}(b))$$

given by $A(x \wedge y) = A_x y$ is injective and its image is a Lie subalgebra isomorphic to so (8).

Proof. See [8].

3. The associated principal bundle

Having found the holonomy group $\pi: S^{15} \to M^8$ we cannot apply the methods of [8] directly at this stage. In fact the standard Hopf fibration $S^{15} \to S^8$ (1/2) does not occur via orbits of any Lie group action on S^{15}. We therefore adopt the strategy of looking at the associated principal Spin(8) bundle

$$\tilde{\pi}: E \to M$$

and study the induced connection on it. To construct the associated principal bundle we first note that the holonomy Lie algebra $so(8) \subset \chi(\pi^{-1}(b))$ for each $b \in M$ gives an additional geometric structure on the fibres which is preserved under the horizontal holonomy displacement ([8], Lemma 2.12, and Propobition 2.13). Moreover, the fibres being homogeneous under Spin(8) action (though not via isometries *a priori*) become diffeomorphic to $S^7(1)$. Now Spin(8) can act transitively on S^7 in three ways: via ρ, σ_+ and σ_-, where ρ is the natural double cover representation in $\mathbb{R}^8$ and the other two are the two spin representations of Spin(8). However, from the geometrical point of view, the three are equivalent as all three are two sheeted covers of $SO(8)$ and one can pass from one to any other using covering transformations. These are naturally outer automorphisms of Spin(8). To construct the principal bundle we take any one of these representations, say ρ. The resulting Spin(8) action on S^7 gives rise to a copy of $so(8)$ inside $\chi(S^7(1))$. Let for each $b \in M$, F_b denote the fibre $\pi^{-1}(b)$ and define E_b to be the set of all diffeomorphisms $\phi: S^7(1) \to F_b$ which send this copy of $so(8)$ onto the holonomy Lie algebra sitting in $\chi(F_b)$ isomorphically. Clearly E_b is a copy of the group Aut$(so(8))$ whose identity component is just $PSO(8)$. We now let E to be the disjoint union of these E_b as b runs through the set M. The set E has a natural smooth structure and an obvious smooth projection $\tilde{\pi}$ onto M with fibre over any point b being exactly E_b. The group Aut$(so(8))$ acts on E from the right freely and the orbits are the fibres E_b. We now do two things for the sake of definiteness: (i) reduce the group to the identity component $PSO(8)$, this is possible due to simple connectivity of M and (ii) reduce the group further to Spin(8). This is possible due to the absence of the first two Whitney classes. We continue to denote the reduced space by E only. Thus E is now the associated principal Spin(8) bundle we were looking for. If Spin(7) is the isotropy group at some chosen base point of our model sphere $S^7(1)$, then E/Spin(7) is our S^{15} that we started with and we have a tower of smooth fibrations

$$E \to S^{15} \to M.$$

The composite is the map $\tilde{\pi}$.

4. The Riemannian structure on E

In this section we describe a metric on E which comes naturally due to the geometric considerations and which makes the above tower, a tower of Riemannian submersions. This is done in several stages.

First we note that for each $\phi \in E$, there is a decomposition of the tangent space into two complementary subspaces . The first of these is the tangent space to the orbit of Spin(8) through ϕ and the other is the space generated by the holonomy of π. To elaborate this, let $\phi \in E_b$ and γ a path starting from b. Then we get a one parameter family of diffeomorphisms $\tau_{\gamma(t)}$ from F_b to $F_{\gamma(t)}$ generated by the holonomy displacement

along γ. This gives a path $\tau_{\gamma(t)}{}^{\circ}\,\phi$ in E. Its derivative at $t=0$ gives a member of the second complementary subspace. As γ varies the full subspace is obtained. We naturally declare the break-up as orthogonal and metrize the second part by the inner product on T_bM to which it projects isomorphically under $\tilde{\pi}_*$. Henceforth we will denote the second subspace $\mathscr{H}_\phi$ and call it the horizontal space (relative to $\tilde{\pi}$). We remark that this collection of horizontal spaces gives exactly the connection on the principal bundle which is associated to the connection given by the horizontal spaces in S^{15}. We also denote the space complementary to $\mathscr{H}_\phi$ by $\mathscr{V}_\phi$. It remains to define a metric on this vertical part. This is more delicate. We proceed as follows: We already have a copy of $so(7)$ inside $so(8)$ as the isotropy Lie algebra of a chosen base point in our model S^7. This naturally breaks $so(8)$ in a unique manner as

$$so(7) \oplus \mathbb{R}^7,$$

each part being an irreducible $so(7)$ module. This is also the Cartan decomposition (see [10]). Corresponding to each $X \in so(8)$, we have a vertical vector field $\bar{X}$ on E (see [2], p. 39). Thus we also have a decomposition

$$\mathscr{V}_\phi = \mathscr{V}'_\phi \oplus \mathscr{V}''_\phi$$

This too we declare to be orthogonal. On the first part we put the metric coming from the bi-invariant inner product of $so(7)$ and on the second part the metric pulled back from S^{15} to which it goes injectively under $\tilde{\pi}_*$. Note that its image is precisely the vertical space at $\tilde{\pi}$ (ϕ) in S^{15}. This completes the description of the Riemannian structure on E. With this metric on E, we have a tower of Riemannian submersions

$$E \to S^{15} \to M^8.$$

The first one is a Spin(7) principal bundle with totally geodesic fibres, each isometric to Spin(7) with the Cartan–Killing metric while the composite is the principal Spin(8) bundle whose fibres *a priori* have varying metrics. The decomposition

$$TE = \mathscr{V} \oplus \mathscr{H}$$

is preserved under the Spin(8) action while the finer decomposition

$$TE = \mathscr{V}' \oplus \mathscr{V}'' \oplus \mathscr{H}$$

is preserved under the Spin(7) action. Moreover, the Spin(7) action is evidently via isometries.

5. The metric on the fibres of $\tilde{\pi}$

In this section we will see that though the metric on the fibres E_b, $b \in M$ could be different, the variation is very limited in nature. More precisely we have the following:

Theorem 5.4 *The fibres E_b, $b \in M$ of $\tilde{\pi}$ are isometric to* Spin(8) *furnished with a left invariant metric which is also right invariant under* Spin(7) *action.*

To prove this result we need to analyse the integrability tensor of $\tilde{\pi}$. So let $\tilde{A}$ denote the integrability tensor of $\tilde{\pi}$ and $\bar{A}$ that of $\bar{\pi}$.

Lemma 5.1. *Let $x, y \in T_bM$.* The vertical field $\tilde{A}_x y$ *along the fibre E_b is $\bar{Z}$, for some $Z \in so$* (8). *Further, if $Z = Z_1 + Z_2$ corresponding to $so(8) = so(7) + \mathbb{R}^7$, then $\bar{Z}_1 = \bar{A}_x y$ and $\bar{Z} = A_x y$.*

Proof. $\tilde{A}_x y = 1/2[X, Y]^v$ is just the curvature of the connection on the Spin(8) principal bundle alluded to earlier and hence clearly comes from a suitable element of its Lie algebra. Also $[X, Y]^v = [X, Y]^{v'} + [X, Y]^{v''} = 2(\bar{A}_x y + A_x y)$ is also $2(\bar{Z}_1 + \bar{Z}_2)$. By uniqueness of decomposition into components, the result follows. q.e.d.

COROLLARY 5.2

For $x, y \in T_b M$, $\tilde{A}_x y$ is of constant length along E_b.

Proof. $\bar{Z}_1$ is of constant length since it comes from an $so(7)$ element and $\bar{Z}_2 = A_x y$ is of constant norm since it is the basic lift of the corresponding field along $F_b \subset S^{15}$. That it is of constant norm there is well known (see [8]). Since the two components are also mutually orthogonal everywhere the corollary follows. q.e.d.

Lemma 5.2. There is an orthonormal framing of E_b which is generated by a basis of $so(8)$.

Proof. Choose a basis $\{x_i : 0 \leqslant i \leqslant 7\}$ of $T_b M$ so that $\{A_{x_0} x_i : 1 \leqslant i \leqslant 7\}$ forms an orthonormal framing of F_b. Set $v_i = A_{x_0} x_i$, and consider the fields

$$\{v_i : 1 \leqslant i \leqslant 7\} \cup \{\bar{A}_{v_j} v_k : 1 \leqslant j < k \leqslant 7\}.$$

That the first set of 7 vector fields is an orthonormal frame along E_b is obvious. That it is generated by $so(8)$ elements is one of the contention of the lemma 5.1 above. As for the second set of 21 fields we note that if $v_i = \bar{X}_i$ for suitable X_i in the $\mathbb{R}^7$ component of $so(8)$, then $\bar{A}_{v_j} v_k$ is generated by $[X_j, X_k]$. There is no need to project to the $so(7)$ part since it is already there. (This is a well-known property of Cartan decomposition). That these have pairwise constant inner-products now follows from the very way the metric was defined on the $\mathscr{V}''$ part of TE. The two sets are clearly mutually orthogonal. We also note that $\{[X_i, X_j], 1 \leqslant i < j \leqslant 7\}$ is a basis of $so(7)$. An application of Gram–Schmidt orthonormalization on the second set now gives the required framing. q.e.d.

Proof of the theorem. E_b can be identified with Spin(8) after choosing some base point on it. The left invariant vector fields of Spin(8) are mapped to the $\mathbb{R}$-span of the above mentioned basis. Left invariance of the metric of E_b now follows exactly as in [8]. Right invariance under Spin(7) is from construction. q.e.d.

Caution. Left and right in our situation are opposite to those in [8].

COROLLARY 5.3

Each fibre F_b of π is a sphere of constant sectional curvature.

Proof. Each F_b is isometric to the quotient of E_b under the group Spin(7) of isometries of E_b acting from the right. Hence Spin(8) acts on F_b via isometries from the left making it a homogeneous Riemannian space. Since the isotropy group Spin(7) acts transitively on tangent two-planes the sectional curvatures are constant pointwise. It follows they are the same constant everywhere (see [12]). q.e.d.

6. Left invariant metrics on Spin(8) invariant under AdSpin(7)

Let

$$so(8) = so(7) \oplus \mathbb{R}^7$$

be the Cartan decomposition which we note is also an $so(7)$-module decomposition into its *isotypical* components. With this notation we have the following:

Theorem 6.5. *Let g denote the Cartan Killing metric on $so(8)$. Then $g = g_1 \oplus g_2$ corresponding to the Cartan decomposition and any* AdSpin(7) *invariant metric is of the type*

$$g = g_1 \oplus c g_2$$

(upto an overall scalar multiple).

Proof. The first claim is well-known (see [10]). The second claim follows since under an AdSpin(7)-invariant metric, $so(7)^{\perp}$ must be an $so(7)$ module. This forces the above decomposition to be orthogonal. Further any Spin(7) invariant metric on the natural module $\mathbb{R}^7$ is a scalar multiple of the standard Euclidean metric. q.e.d.

COROLLARY 6.4

There is a smooth positive real valued function c on M such that E_b is isometric to Spin(8) *with the left invariant metric $g_b = g_1 \oplus c(b) g_2$.*

Proof. Just observe that there is no change of scale in the $\mathcal{V}'$ part of the tangent space of E_b as the orbits of the subgroup Spin(7) are all isometric to each other. q.e.d.

COROLLARY 6.5

For any smooth path in M the hononomy displacement of fibres in E preserves the $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ decomposition. Moreover, it is an isometry on the first part and a dilatation on the second part.

Proof. Let γ be a smooth path in M starting from b_1 and ending at b_2. Let $\phi_1 \in E_{b_1}$ *and* $\phi_2 = \tau(\phi_1)$ be in E_{b_2}. On identifying these fibres with Spin(8), it is clear that the holonomy diffeomorphism gets identified with L_g for a suitable $g \in$ Spin(8). Since g_{b_2}, the metric on E_{b_2} pulls back under L_g to a left invariant metric which is also right invariant under Spin(7), so it must be of the form $g_1 \oplus (c(b_2)/c(b_1)) \cdot g_2$ (see corollary 6.4 for notation). q.e.d.

7. Proof of the main theorem

In this section we complete the proof of Theorem 1.1 stated in the introduction. We observe that a consequence of the last corollary is that the holonomy displacement in S^{15} is via isometries up to scalings factor. This forces all the operators T^x on vertical vectors arising out of the *second fundamental form* of the fibres to be scalar multiples of identity. This in turn forces the largest and the smallest fibres to be great spheres due to isoparametricity. Hence all the fibres are great spheres. Appealing now to the classification of parallel great sphere fibrations as in [13] or [9] we conclude that π is a Hopf fibration. q.e.d.

COROLLARY 7.6 (Generalized Wilhelm's Lemma)

Let

$$\pi : S^{15}(1) \to M^8$$

be a Riemannian submersion with connected 7-dimensional fibres and let

$$G = \{b \in M : \pi^{-1}(b) \text{ is a great sphere}\},$$

then either G is empty or all of M.

Proof. If G is nonempty let $b \in G$. For any $p \in F_b$ and any $x \in \mathscr{H}_p$, the linear map

$$A_x : \mathscr{H}_p \to \mathscr{V}_p$$

is surjective (see lemma 4.2, [8]). It follows that $A_{c_x}(t)$ is surjective for all t, where c_x denotes the geodesic with initial vector $x \in \mathscr{H}_p$ (see the statement preceding the corollary 2.9 in [8]). Hence π is substantial along every fibre. q.e.d.

8. Almost diameter rigidity for CaP^2

Proof. Let x, y and z be mutually distance $\pi/2$ apart. In this situation y and z both are in the dual set $\{x\}'$ of x. (see [7] and [14] for more details about dual sets). From [7] we know there is a Riemannian submersion

$$\exp_x : S_x \to \{x\}'$$

and similarly for y and z. As argued in [14] this forces at least one fibre to be totally geodesic in each case. But then $\exp_p$ is congruent to the Hopf fibration and this makes the space isometric to the standard CaP^2 as proved in [7] and [14]. q.e.d.

Remark. This rigidity is clearly stronger than even corollary II of [4] where each pair of points separated by a distance of $\pi/2$ is required to be completed into an equilateral triangle.

It is clearly an interesting problem to find reasonable conditions which will force substantiality. One such, as we saw is total geodseity of a single fibre. The author proposes to discuss this elsewhere.

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Harmonic manifolds with some specific volume densities

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Abstract. We show that a noncompact, complete, simply connected harmonic manifold (M^d, g) with volume density $\theta_m(r) = \sinh^{d-1} r$ is isometric to the real hyperbolic space and a noncompact, complete, simply connected Kähler harmonic manifold (M^{2d}, g) with volume density $\theta_m(r) = \sinh^{2d-1} r \cosh r$ is isometric to the complex hyperbolic space. A similar result is also proved for quaternionic Kähler manifolds. Using our methods we get an alternative proof, without appealing to the powerful Cheeger–Gromoll splitting theorem, of the fact that every Ricci flat harmonic manifold is flat. Finally a rigidity result for real hyperbolic space is presented.

Keywords. Harmonic manifolds; volume densities; Ricci curvature; second fundamental form.

1. Introduction

A d-dimensional Riemannian manifold (M, g) (supposed to be complete in the rest of the paper) is said to be harmonic if its volume density function $\omega_m = \sqrt{|\det(g_{ij})|}$ is a spherically symmetric function around m. In polar coordinates this density can be written as

$$\theta_m = r_m^{d-1}\, \omega_m,$$

where $r_m(n) = r(m, n)$ is the geodesic distance between m and n. Thus (M, g) is harmonic if θ_m is a radial function around m (see [1], [11] or [12] for details). Two-point homogeneous spaces are harmonic as can be easily seen from their density function. Besides these there were no known non-flat examples of harmonic spaces. Moreover Lichnerowicz proved that up to dimension 4, harmonic spaces are in fact locally isometric to rank one symmetric spaces. This led to the Lichnerowicz conjecture which asserts that *every harmonic space is locally isometric to a two-point homogeneous space.* It should be noted that even higher rank symmetric spaces are not harmonic.

Let (M, g) be a harmonic space. The well known Ledger's formula ([1], see p. 161) gives

$$\omega_p''(r)|_{r=0} = -\frac{1}{3}\mathrm{Ricci}_p.$$

Hence for harmonic manifolds, since ω_p is a function of r alone, the Ricci curvature is a constant, i.e. harmonic spaces are Einstein. Let $\mathrm{Ricci}(M) = k$. There arise three cases:

1. $k > 0$. In this case, by Myers–Bonnet theorem, M is compact with finite fundamental group and Szabo [11] proved that compact harmonic manifolds with finite fundamental group are rank one symmetric, thus settling the Lichnerowicz conjecture.
2. $k = 0$. Here one appeals to the powerful Cheeger–Gromoll splitting theorem to conclude that M is isometric to the Euclidean space, i.e. *Ricci-flat harmonic manifolds are flat.*

3. $k<0$. The Lichnerowicz conjecture is not true in this case. Damek and Ricci [3] constructed a family of nonsymmetric harmonic spaces. These spaces are called the NA spaces. In this family there are harmonic manifolds with the same density function as that of quaternionic hyperbolic spaces. All these spaces are homogeneous. Presently it is not known whether there are nonhomogeneous harmonic spaces. So it seems that the classification of harmonic spaces can be achieved only up to the determination of all density functions, i.e, density classification of harmonic spaces. So Szabo [11] asked the following question. "Which harmonic spaces are determined by their density functions?"

We call this notion as weak density equivalence. Szabo [12] defines a stronger notion of density equivalent spaces. In § 2 we answer the above question for three specific cases namely that of the real, complex and quaternionic hyperbolic spaces. More precisely we prove

Theorem 1.1. *Let (M^d, g) be a non-compact, simply connected harmonic space with density function $\theta(r) = \sinh^{d-1} r$, then (M^d, g) is isometric to the real hyperbolic space.*

Theorem 1.2. *Let (M^{2d}, g) be a non-compact, simply connected, Kähler harmonic manifold with density function $\theta(r) = \sinh^{2d-1} r \cosh r$, then M is isometric to the complex hyperbolic space.*

Theorem 1.3. *Let (M^{4d}, g) be a noncompact, simply connected, quaternionic Kähler harmonic manifold with volume density $\theta(r) = \sinh^{4d-1} r \cosh^3 r$, then M is isometric to the quaternionic hyperbolic space.*

Theorems (1.1) and (1.2) explain the lack of examples of nonsymmetric (Kähler) harmonic spaces with the same density function as that of the real (complex) hyperbolic space. Using the same methods we also give an alternative proof, without appealing to the powerful Cheeger–Gromoll splitting theorem, of the fact that every Ricci-flat harmonic manifold is isometric to the Euclidean space, which is as follows.

Theorem 1.4. *Every simply connected Ricci-flat harmonic manifold is flat, i.e, it is isometric to the Euclidean space.*

Let (M^d, g) be a noncompact harmonic manifold with density function $\theta(r)$. In § 3 we show that $\theta(r)$ satisfies

$$\frac{\theta(r)}{\sinh^{d-1} r} \to 0 \quad \text{as} \quad r \to \infty$$

or

$$\frac{\theta(r)}{\sinh^{d-1} r} \to c > 0 \quad \text{as} \quad r \to \infty.$$

In the second case we show that M is locally isometric to the real hyperbolic space.

2. Weak density equivalence

Proof of theorem (1.1). Choose m in M. Since $\theta(r) = \sinh^{d-1} r$, $\omega_m(r) = (1/r^{d-1})$ $\sinh^{d-1} r$. Hence from the formula

$$\omega_m''(0) = -\tfrac{1}{3}\,\mathrm{Ricci}_m$$

we see that Ricci $(M) = -(d-1)$. Let γ be a geodesic starting at m and $S_{m,r}$ be the distance sphere around m of radius r. Again, since $\theta(r) = \sinh^{d-1} r$, the mean curvature $\sigma_m(r)$ of $S_{m,r}$ is

$$\sigma_{m,r} = \frac{\theta'_m(r)}{\theta_m(r)} = (d-1)\coth r.$$

Let $L = L_{m,r}$ be the second fundamental form of $S_{m,r}$, then

$$\text{trace}\, L = \sigma_{m,r} = (d-1)\coth r.$$

Moreover L satisfies the Riccati equation

$$L' + L^2 + R(.,\gamma')\gamma' = 0,$$

where R is the curvature tensor of M [8]. Taking traces we get

$$\text{trace}\, L' + \text{trace}\, L^2 + \text{Ricci}(\gamma',\gamma') = 0.$$

But trace $L = (d-1)\coth r$. Therefore,

$$\text{trace}\, L^2 = (d-1)\coth^2 r = \frac{1}{d-1}(\text{trace}\, L)^2.$$

Now let us use the Cauchy–Schwarz inequality for the linear map L. We have

$$\text{trace}\, L^2 \geqslant \frac{1}{d-1}(\text{trace}\, L)^2$$

and equality holds if and only if L is a scalar operator. In our case equality holds for L, hence L is a scalar operator. But trace $L = (d-1)\coth r$, thus $L_{m,r} = \coth r\, Id$ for all points m in M and all $r > 0$. Thus M has constant sectional curvature equal to -1. Finally because M is simply connected it is isometric to the real hyperbolic space. ■

Now let (M^{2d}, g) be a Kähler harmonic manifold. Denote its complex structure by J. Assume that M has density function $\theta(r) = \sinh^{2d-1} r \cosh r$. The main difficulty in using the method of proof of the above theorem in proving that M is isometric to the complex hyperbolic space is the following: in this case the Riccati equation may not split into two equations along the plane spanned by $\{\gamma', J\gamma'\}$. The Riccati equation will split if and only if $J\gamma'$ is an eigen direction of $R(.,\gamma')\gamma'$. Equivalently this will happen if and only if for any vector field X, the subbundle generated by $\{X, JX\}$ is integrable. To overcome this difficulty we compute the index form along γ.

Proof of theorem (1.2). Let m be a arbitrary point of M. Take any geodesic $\gamma(t)$ starting at m. Denote the complex structure on M by J and let I be the index form of M. Choose any real number $T > 0$. Let $E_2(t), E_3(t), \ldots, E_{2d}(t)$ be unit orthogonal parallel fields along $\gamma(t)$ normal to $\gamma'(t)$ with $E_2(t) = J\gamma'(t)$. Choose Jacobi fields $\{J_i(t)\}_{i=2}^{2d}$ along $\gamma(t)$ such that

$$J_i(0) = 0,\ J_i(T) = E_i(T),\ i = 2, \ldots, 2d.$$

Let

$$X_2(t) = \frac{\sinh 2t}{\sinh 2T} E_2(t)$$

and

$$X_i(t) = \frac{\sinh t}{\sinh T} E_i(t), i = 3, \ldots, 2d$$

be vector fields along $\gamma(t)$. Note that $J_i(0) = X_i(0)$ and $J_i(T) = X_i(T)$ for all i. We compare the index forms of J_i and X_i for corresponding i.

By Allamigeon's theorem we know that there are no conjugate points along γ [1]. Hence we get $I(J_i, J_i) \leqslant I(X_i, X_i)$ for all i. Sum over all i to get

$$\sum_{i=2}^{2d} I(J_i, J_i) \leqslant \sum_{i=2}^{2d} I(X_i, X_i). \tag{2.1}$$

We know that $\Sigma_{i=2}^{2d} I(J_i, J_i) = (\theta_m'(T)/\theta_m(T))$. Further a simple calculation gives

$$I(X_2, X_2) = \frac{\sinh 4T + 4T}{2\sinh^2 2T} - \int_0^T \frac{\sinh^2 2t}{\sinh^2 2T} H(\gamma'(t))\, dt \tag{2.2}$$

and

$$I(X_i, X_i) = \frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T} K(\gamma'(t), E_i(t))\, dt, i = 3, \ldots, 2d.$$

Here $K(\gamma'(t), E_i(t))$ is the sectional curvature of the plane spanned by $\gamma'(t)$ and $E_i(t)$ and $H(\gamma'(t)) = K(\gamma'(t), J\gamma'(t))$ is the holomorphic sectional curvature. Also, since $\theta(r) = \sinh^{2d-1} r \cosh r$, the Ricci curvature of M is $-2(d+1)$. Now let us calculate the sum $\Sigma_3^{2d} I(X_i, X_i)$. We have

$$\begin{aligned}
&\sum_{3}^{2d} I(X_i, X_i) \\
&= (2d-2)\frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T} \left(\sum_{3}^{2d} K(\gamma'(t), E_i(t)) \right) dt \\
&= (2d-2)\frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T} (\text{Ricci}(\gamma'(t), \gamma'(t)) - H(\gamma'(t)))\, dt \\
&= (2d-2)\frac{2T + \sinh 2T}{4\sinh^2 T} + 2(d+1)\left(\frac{\sinh 2T - 2T}{4\sinh^2 T} \right) \\
&\quad + \int_0^T \frac{\sinh^2 t}{\sinh^2 T} H(\gamma'(t))\, dt \\
&= \frac{4d \sinh 2T - 8T}{4\sinh^2 T} + \int_0^T \frac{\sinh^2 t}{\sinh^2 T} H(\gamma'(t))\, dt.
\end{aligned} \tag{2.3}$$

Thus (2.2) and (2.3) gives

$$\sum_{i=2}^{2d} I(X_i, X_i) = A(T) + \int_0^T B(t) H(\gamma'(t))\, dt,$$

where

$$A(T) = \frac{\sinh 4T + 4T}{2\sinh^2 2T} + \frac{1}{4\sinh^2 T}(4d \sinh 2T - 8T)$$

and

$$B(t)=\frac{\sinh^2 t}{\sinh^2 T}-\frac{\sinh^2 2t}{\sinh^2 2T}.$$

Substituting all this information in (2.1), we get

$$\sum_{i=2}^{2d} I(J_i,J_i)=\frac{\theta_m'(T)}{\theta_m(T)}\leqslant (A(T)-4C(T))+\int_0^T B(t)(H(\gamma'(t))+4)\,dt$$

$$=\sum_{i=2}^{2d} I(X_i,X_i),$$

where

$$C(T)=\int_0^T B(t)\,dt$$

$$=\frac{\sinh 2T-2T}{4\sinh^2 T}-\frac{\sinh 4T-4T}{8\sinh^2 2T}.$$

Note that $B(t)\geqslant 0$, $t\in[0,T]$. Finally, we have

$$A(T)-4C(T)$$

$$=\frac{\sinh 4T+4T}{2\sinh^2 2T}+\frac{1}{4\sinh^2 T}(4d\sinh 2T-8T)$$

$$-\frac{\sinh 2T-2T}{\sinh^2 T}-\frac{\sinh 4T-4T}{2\sinh^2 2T}$$

$$=\frac{\sinh 4T}{\sinh^2 2T}+\frac{(d-1)\sinh 2T}{\sinh^2 T}$$

$$=2\coth 2T+2(d-1)\coth T$$

$$=(2d-1)\coth T+2\coth 2T-\coth T$$

$$=(2d-1)\coth T+\tanh T$$

$$=\frac{\theta_m'(T)}{\theta_m(T)}.$$

Thus it follows that

$$\int_0^T B(t)(H(\gamma'(t))+4)\,dt\geqslant 0. \tag{2.4}$$

Now suppose that $H(\gamma'(t))<-4$ for some $t>0$, say $t=s$. Then by continuity of H we see that $H(\gamma'(t))<-4$ in a small neighbourhood of s, say $(s-\varepsilon,s+\varepsilon)$. But the point $m=\gamma(0)$ and $0<T$ we started with, are arbitrary. Hence if we take $m=\gamma(s)$ and $T=\varepsilon$ in the above computation, then we will get a contradiction to the inequality (2.4). Thus, since $B(t)\geqslant 0$, $t\in[0,T]$, we get $H(\gamma'(t))\geqslant -4$ for all $t>0$. Again the geodesic γ is arbitrary. We conclude that

$$H(v)\geqslant -4$$

for all unit vectors v. An algebraic calculation ([1], see p. 150) yields

$$\int_{U_m M} H(v)\,dv = \frac{\mathrm{Vol}(U_m M)}{d(d+1)}\,\mathrm{Scal}_m M,$$

where Scal is the scalar curvature of M. In our case, $\mathrm{Ricci}(M) = -2(d+1)$, hence $\mathrm{Scal}(M) = -4d(d+1)$. Thus

$$\int_{U_m M} H(v)dv = -4\,\mathrm{Vol}(U_m M).$$

Combined with the conclusion $H(v) \geqslant -4$, we get $H(v) \equiv -4$ for all unit vectors v, i.e. M has constant holomorphic sectional curvature. The simple connectivity of M shows that M is isometric to the complex hyperbolic space. ∎

Remark. The above theorems explain the lack of nonsymmetric (Kähler) harmonic spaces with the same density function as the real or complex hyperbolic space. Along the same lines we prove a similar theorem for quaternionic Kähler harmonic manifolds.

Proof of Theorem (1.3). Consider a chart (U, m), $m \in M$ with two almost complex structures J_2, J_3 such that the Levi–Civita derivatives of J_2, J_3 are linear combinations of J_2, J_3 and $J_4 = J_2 J_3$. Such a chart exists because M is quaternionic Kähler [2]. Let $\gamma(t)$ be a geodesic starting at m. Choose $T > 0$, such that $\gamma[0, T] \subset U$. Let I denote the index form on γ. Consider unit orthogonal parallel fields $E_2(t), \ldots, E_{4d}(t)$ along γ which are normal to $\gamma'(t)$ such that $E_2(t)$, $E_3(t)$, $E_4(t)$ belong to the three-dimensional subbundle spanned by $\{J_2(\gamma'(t)), J_3(\gamma'(t)), J_4(\gamma'(t))\}$.

Now $\mathrm{Ricci}(M) = -(4n+8)$ since $\theta_m(r) = \sinh^{4d-1} r \cosh^3 r$. The choice of J_2 and J_3 shows that the four-dimensional space spanned by $\{\gamma'(t), J_2\gamma'(t), J_3\gamma'(t), J_4\gamma'(t)\}$ is a quaternionic line parallel along γ which has an Sp(1) action. Therefore we get a family of almost complex structures $J_2(t), J_3(t), J_4(t)$ along γ such that

$$E_i(t) = J_i(t)(\gamma'(t)), J_i(0) = J_i, \quad \text{for} \quad i = 2, 3, 4.$$

The rest of the proof is similar to that of the previous theorem, so we shall be brief. Take vector fields

$$X_i(t) = \frac{\sinh 2t}{\sinh 2T} E_i(t), i = 2, 3, 4$$

and

$$X_i(t) = \frac{\sinh t}{\sinh T} E_i(t), i = 5, \ldots, 4d.$$

Let $J_i(t)$ be Jacobi fields along $\gamma(t)$ such that $J_i(0) = X_i(0)$ and $J_i(T) = X_i(T)$ for all i. By Allamigeon's theorem there are no conjugate points along $\gamma(t)$. Therefore the following inequality holds

$$\sum_{i=2}^{4d} I(J_i, J_i) \leqslant \sum_{i=2}^{4d} I(X_i, X_i) \qquad (2.5$$

and equality holds if and only if $X_i(t) = J_i(t)$ for all i and for all t in $[0, T]$. Now

$$I(X_i, X_i) = \frac{\sinh 4T + 4T}{2\sinh^2 2T} - \int_0^T \frac{\sinh^2 2t}{\sinh^2 2T} K(\gamma'(t), E_i(t))\,dt, \; i = 2, 3, 4.$$

and

$$I(X_i, X_i) = \frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T} K(\gamma'(t), E_i(t))\,\mathrm{d}t,\ i = 5, \ldots, 4d.$$

Summing up for $i = 2, \ldots, 4d$ we get

$$\sum_2^{4d} I(X_i, X_i) = (A(T) - 12C(T)) + \int_0^T B(t)\left(\sum_2^4 K(\gamma'(t), E_i(t)) + 12\right)\mathrm{d}t, \tag{2.6}$$

where $K(x, y)$ stands for the sectional curvature of the plane spanned by vectors x and y. Moreover

$$A(T) = \frac{3(\sinh 4T + 4T)}{2\sinh^2 2T} + \frac{((8n+4)\sinh 2T - 24T)}{4\sinh^2 T}$$

$$B(t) = \frac{\sinh^2 t}{\sinh^2 T} - \frac{\sinh^2 2t}{\sinh^2 2T} > 0, t \in [0, T]$$

and

$$C(T) = \int_0^T B(t)\,\mathrm{d}t$$

$$= \frac{\sinh 2T - 2T}{4\sinh^2 T} - \frac{\sinh 4T - 4T}{8\sinh^2 2T}.$$

But we know that $\Sigma_{i=2}^{4d} I(J_i, J_i) = (\theta_m'(T)/\theta_m(T))$ and $E_i(t) = J_i(t)$. Therefore (2.5) and (2.6) gives

$$\frac{\theta_m'(T)}{\theta_m(T)} \leqslant (A(T) - 12C(T)) + \int_0^T B(t)\left(\sum_2^4 K(\gamma'(t), E_i(t)) + 12\right)\mathrm{d}t. \tag{2.7}$$

Now we use the following relation between the components of the curvature tensor of a quaternionic Kähler manifold [2]

$$\sum_2^4 K(\gamma'(t), J_i(t)(\gamma'(t))) = \frac{3}{n+2}\,\mathrm{Ricci}(M).$$

In our case $\mathrm{Ricci}(M) = -(4d + 8)$, hence

$$\sum_2^4 K(\gamma'(t), J_i(t)(\gamma'(t))) = -12.$$

Substituting this into (2.7) we finally get

$$\frac{\theta_m'(T)}{\theta_m(T)} \leqslant A(T) - 12C(T)$$

and equality holds if and only if X_i is a Jacobi field for all i. A simple computation verifies that

$$\frac{\theta_m'(T)}{\theta_m(T)} = A(T) - 12C(T)$$

$$= (4d - 1)\coth T + 3\tanh T.$$

Thus $X_i(t) = J_i(t)$, i.e, the $X_i(t)$ are Jacobi fields on M for all i. Since the point m is arbitrary and γ is any geodesic starting at m, M is isometric to the quaternionic hyperbolic space. ■

Using the above techniques we give an alternative proof of the fact that Ricci-*flat harmonic manifolds are flat*. The method of proof is same as that of Theorems (1.1) and (1.2). We study the second fundamental form of the horospheres to prove the Theorem. Some basic ideas about horospheres are needed. They are the level sets of Busemann functions. For the definitions and properties of these functions we refer to [8].

Proof of Theorem (1.4). It is enough to prove the theorem for simply connected harmonic manifolds. Let M^d be a simply connected Ricci-flat harmonic manifold. If M were compact, then by Szabo's theorem M is a rank one symmetric space and hence Ricci $(M) > 0$, a contradiction. Thus M is noncompact. By Allamigeon's theorem M is diffeomorphic to $\mathbb{R}^d$. Moreover every geodesic is a line in M. Let m be a point of M and consider a geodesic ray $\gamma(t)$ starting at m. Choose a real number $T > 0$. Let $\{E_i(t)\}_2^d$ be parallel orthonormal vector fields along γ which are normal to $\gamma'(t)$. Consider normal Jacobi fields $\{J_i(t)\}_2^d$ such that $J_i(0) = 0$ and $J_i(T) = E_i(T)$ for $i = 2, \ldots, d$. Now consider the vector fields

$$X_i(t) = \frac{t}{T} E_i(t), i = 2, \ldots, d.$$

Note that $X_i(0) = J_i(0)$ and $X_i(T) = J_i(T)$. Let us now compare the index forms $I(J_i, J_i)$ and $I(X_i, X_i)$. Since there are no conjugate points along $\gamma(t)$ for $t \in [0, T]$, the following inequality holds:

$$\sum_2^d I(J_i, J_i) \leqslant \sum_2^d I(X_i, X_i). \tag{2.8}$$

Note that

$$\sum_2^d I(J_i, J_i) = \frac{\theta_m'(T)}{\theta_m(T)} \tag{2.9}$$

and

$$I(X_i, X_i) = \frac{1}{T} - \int_0^T \frac{t^2}{T^2} K(\gamma'(t), E_i(t))\, dt. \tag{2.10}$$

Summing over $i = 2, \ldots, d$ and using the fact Ricci $(M) = 0$, we get

$$\sum_{i=2}^d I(X_i, X_i) = \frac{n-1}{T}. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.8) gives

$$\frac{\theta_m'(T)}{\theta_m(T)} \leqslant \frac{n-1}{T}.$$

But we know that

$$\theta_m'(T)$$

because the geodesic balls are convex. Therefore

$$\frac{\theta_m'(T)}{\theta_m(T)} \to 0 \quad \text{as} \quad T \to \infty. \tag{2.12}$$

Now let F_γ be the Busemann function relative to γ and consider the horospheres determined by F_γ. Let L be the second fundamental form of the horospheres. Then (2.12) implies that trace $L = 0$. But L satisfies the Riccati equation

$$L' + L^2 + R(\cdot, \gamma')\gamma' = 0.$$

Taking traces and using the fact that Ricci $(M) = 0$, we get

$$\text{trace } L' + \text{trace}(L^2) = 0.$$

But trace $L' = 0$, since trace $L = 0$. Thus trace $L = 0$ and trace $L^2 = 0$. Hence $L = 0$. This shows that all the sectional curvatures $K(\gamma'(t), u) = 0$ for all unit vectors u. But the point m in M and the geodesic γ were arbitrarily chosen. Hence $K(M) \equiv 0$ and the proof is complete. ∎

3. Rigidity of $\mathbf{R}H^d$

In this section we prove that noncompact harmonic manifolds naturally fall into two classes:

$$\frac{\theta(r)}{\sinh^{d-1} r} \to 0$$

or

$$\frac{\theta(r)}{\sinh^{d-1} r} \to c > 0$$

for $r \to \infty$.

Further we show that the second class has only one element namely, the real hyperbolic space. Let (M^d, g) be a noncompact harmonic manifold. Since the Ricci curvature of M is constant, let us normalize the metric so that Ricci$(M) = -(d-1)$. The Bishop–Gromov volume comparison theorem (see for example [5] or [10]) gives the density function to be

$$\theta(r) = \alpha(r)\sinh^{d-1} r,$$

where $\alpha(r)$ satisfies

$$0 \leqslant \alpha(r) \leqslant 1; \alpha(0) = 1, \alpha'(0) = 0 \text{ and } \alpha'(r) \leqslant 0.$$

Hence $\alpha(r)$ is a monotonically decreasing function bounded from below by 0. Two cases arise:

1. $\lim_{r \to \infty} \alpha(r) = 0$ and
2. $\lim_{r \to \infty} \alpha(r) = c$ for some constant $c > 0$.

In the second case the density of M is asymptotically the same as that of the real hyperbolic case. In this case we show that, in fact $c = 1$ and hence M is locally isometric to the real hyperbolic space. This can be treated as a rigidity result for the real hyperbolic space.

Theorem 3.5. *Let (M^d, g) be a noncompact harmonic manifold with $\lim_{r\to\infty} \alpha(r) = c > 0$. Then, in fact $c = 1$ and M is locally isometric to the real hyperbolic space of constant sectional curvature -1.*

Proof. Let γ be a geodesic ray in M and let F_γ be the Busemann function of γ. Our hypothesis

$$\lim_{r\to\infty} \alpha(r) = c > 0$$

implies that the density of the horospheres determined by F_γ is

$$\theta(r) = c \sinh^{d-1} r.$$

Hence the mean curvature of these horospheres is

$$\sigma(r) = \frac{\theta'(r)}{\theta(r)}$$

$$= (d-1)\coth r = \text{trace } L$$

where L is the *second fundamental form* of the horospheres. Now the Riccati equation $L' + L^2 + R(\cdot, \gamma')\gamma' = 0$ combined with Ricci $(M) = -(d-1)$ gives

$$\text{trace}(L^2) = (d-1)\coth^2 r,$$

but

$$\text{trace } L = (d-1)\coth r.$$

Hence by Cauchy–Schwartz inequality we get $L = (\coth r)\, Id$. Thus M is locally isometric to the real hyperbolic space of constant curvature -1.

4. Remarks

Let us calculate $\alpha(r)$ for the complex hyperbolic space $\mathbb{C}H^d$. For that normalize the standard metric on $\mathbb{C}H^d$ so that Ricci $(\mathbb{C}H^d) = -(2d-1)$, i.e., take $g = \sqrt{(2d+2)/(2d-1)}\, Std$. Let $\mu = \sqrt{(2d-1)/(2d+2)} < 1$. Thus the density becomes

$$\theta(r) = \sinh^{2d-1} \mu r \cosh \mu r.$$

Hence

$$\alpha(r) = \frac{\theta(r)}{\sinh^{2d-1} r}$$

$$= \left(\frac{\sin h\, \mu r}{\sin h\, r}\right)^{2d-1} \cosh \mu r.$$

Letting $r \to \infty$ we get

$$\alpha(r) \sim e^{(2d\mu - (2d-1))r}.$$

But $2d\mu < (2d-1)$. Thus $\alpha(r) \to 0$ as $r \to \infty$.

Let (M^d, g) be a noncompact, Kähler harmonic manifold. Normalize the metric so as to get Ricci $(M) = -(2d+2)$. Then the density θ is given by

$$\theta(r) = \beta(r) \sinh^{2d-1} r \cosh r.$$

The following question is natural

Question. If $\beta(r) \to b > 0$ as $r \to \infty$, is M locally isometric to the complex hyperbolic space?

After this paper was written our attention was drawn by Vanhecke to his paper with Gray [7]. They investigated Riemannian spaces on which the volume of the geodesic ball with suficiently small radius R is same as that of a two-point homogeneous space. Under some extra hypothesis, for e.g. Ricci(M) is constant, they prove that these spaces are in fact locally isometric to the corresponding spaces. Thus our results follow from their theorems which are purely local. Nevertheless, it should be noted that our proofs, using the global character of our hypothesis, are entirely of different nature.

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Some inequalities for the polar derivative of a polynomial

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < 1$. In this paper, we estimate the maximum and minimum moduli of the kth polar derivative of $P(z)$ on $|z| = 1$ and thereby obtain compact generalizations of some known results, which among other results, yields interesting refinements of Erdos–Lax theorem and a theorem of Ankeny and Rivlin.

Keywords. Bernstein's inequality; polar derivative of a polynomial; inequalities in the complex domain; growth of maximum modulus.

1. Introduction and statement of results

If $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)| \tag{1}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant R^n \max_{|z|=1} |P(z)|. \tag{2}$$

Inequality (1) is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial. (For reference, see [13, 6].) Inequality (2) is a simple deduction from the maximum modulus principle (see [12, p. 346], [10, p. 158, problem 269]).

Recently Aziz and Dawood [5] have shown that if $P(z)$ has all zeros in $|z| \leqslant 1$, then

$$\min_{|z|=1} |P'(z)| \geqslant n \min_{|z|=1} |P(z)| \tag{3}$$

and

$$\min_{|z|=R>1} |P(z)| \geqslant R^n \min_{|z|=1} |P(z)|. \tag{4}$$

Ineqalities (1), (2), (3) and (4) are sharp, with equality for the polynomial $P(z) = m\, e^{i\alpha} z^n$, that is, when $P(z)$ has all its zeros at the origin.

Inequalities (1) and (2) can be sharpened, if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{5}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |P(z)|. \tag{6}$$

Both these estimates are sharp, with equality for the polynomial $P(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta| = \frac{1}{2}$. Inequality (5) was conjectured by Erdös and later verified by Lax [7] (see also [4]), whereas Ankeny and Rivlin [1] used (5) to prove (6).

Inequalities (5) and (6) were further improved in [5], where under the same hypothesis, it was shown that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right\}. \tag{7}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|P(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|P(z)|. \tag{8}$$

In both (7) and (8) equality holds for the polynomial $P(z) = \alpha z^n + \beta$, where $|\beta| \geqslant |\alpha|$.

Let $D_\alpha P(z)$ denote the polar differentiation of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of the polynomial $P(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha - z} = P'(z).$$

Now corresponding to a given nth degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$D_{\alpha_1}P(z) = nP(z) + (\alpha_1 - z)P'(z),$$

$$D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z) = (n-k+1)D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_{k-1}}P(z) + (\alpha_k - z)(D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_{k-1}}P(z))',$$

$$k = 2, 3, \ldots, n.$$

The points $\alpha_1, \alpha_2, \ldots, \alpha_k$; $k = 1, 2, \ldots, n$ may be equal or unequal. Like the kth ordinary derivative $P^{(k)}(z)$ of $P(z)$, the kth polar derivative $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)$ of $P(z)$ is a polynomial of degree at most $n-k$.

More recently, Aziz [2] extended (5) and (6) to the polar derivative of a polynomial.

In the present paper we shall first extend (3) and(4) to the polar derivative and thereby present a compact generalization of these resuslts as well. Thus we prove the following.

Theorem 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $|z| \geqslant 1$,*

$$\min_{|z|=1} |D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$

$$\geqslant n(n-1)\cdots(n-k+1)\,|\alpha_1, \alpha_2 \cdots \alpha_k|\,|z|^{n-k} \min_{|z|=1}|P(z)|,$$

where α_i are real or complex numbers with $|\alpha_i| \geqslant 1$; $i = 1, 2, \ldots, k$; $k \leqslant n-1$. The result is best possible and equality holds for the polynomial $P(z) = m e^{i\beta} z^n, m > 0$.

The following corollary, which immediately follows from Theorem 1, is a compact generalization of (3) ad (4).

COROLLARY 1

If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for any real or complex number α, with $|\alpha| \geqslant 1$,

$$|D_\alpha P(z)| \geqslant n|\alpha|\,|z|^{n-1} \min_{|z|=1} |P(z)|, \quad \text{for } |z| \geqslant 1. \tag{10}$$

The result is best possible, with equality for the polynomial having all its zeros at the origin.

Remark 1. Dividing the two sides of (10) by $|\alpha|$, letting $\alpha \to \infty$ and noting that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

we get

$$|P'(z)| \geqslant n\,|z|^{n-1} \min_{|z|=1} |P(z)|, \text{ for } |z| \geqslant 1,$$

which in particular gives (3).

Taking $z = \alpha$ in (10) and noting that

$$|D_\alpha P(z)|_{z=\alpha} = n|P(\alpha)|,$$

we obtain

$$|P(\alpha)| \geqslant |\alpha|^n \min_{|z|=1} |P(z)|,$$

for every α with $|\alpha| \geqslant 1$, from which (4) follows immediately.

Next we present the following result which is a refinement of Theorem 1 of [2] and is also a compact generalization of (7) and (8).

Theorem 2. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then for $|z| \geqslant 1$*

$$\begin{aligned} &|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| \\ &\leqslant \frac{n(n-1)\cdots(n-k+1)}{2} \Big\{ (|\alpha_1 \alpha_2 \cdots \alpha_k|\,|z|^{n-k} + 1) \max_{|z|=1} |P(z)| \\ &\qquad - (|\alpha_1 \alpha_2 \cdots \alpha_k|\,|z|^{n-k} - 1) \min_{|z|=1} |P(z)| \Big\}. \end{aligned} \tag{11}$$

where $|\alpha_i| \geqslant 1$ for $i = 1, 2, \ldots, k$. The result is best possible and equality in (11) holds for the polynomial

$$P(z) = \tfrac{1}{2}(z^n + 1).$$

The following corollary which immediately follows from Theorem 2, is a compact generalization of (7) and (8).

COROLLARY 2

If $P(z)$ is a polynomial of degree n which has no zeros in the disk $|z| < 1$, then for every real or complex number α, with $|\alpha| \geqslant 1$,

$$|D_\alpha P(z)| \leqslant \frac{n}{2}\Big\{(|\alpha|\,|z|^{n-1}+1)\max_{|z|=1}|P(z)| - (|\alpha|\,|z|^{n-1}-1)\min_{|z|=1}|P(z)|\Big\}, \text{ for } |z| \geqslant 1. \tag{12}$$

The result is best possible and equality in (12) holds for the polynomial $P(z) = a z^n + b$ where $|a| = |b| = \frac{1}{2}$ and $|\alpha| \geqslant 1$.

Remark 2. By using similar argument as in Remark 1, it follows that if we divide both sides of (12) by $|\alpha|$ and let $|\alpha| \to \infty$, we get (7) and for $z = \alpha$, we get (8).

If we write

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0,$$

then

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\ &= (n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} \\ &\quad + \cdots + (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + n a_0). \end{aligned}$$

If the polynomial $P(z)$ has no zeros in the disk $|z| < 1$, then by Theorem 2, with $k = 1$, it follows that

$$\begin{aligned} &|(n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} + \cdots + (2\alpha a_2 + (n-1)a_1)z \\ &\qquad + (\alpha a_1 + n a_0)| \\ &\leqslant \frac{n}{2}\Big\{(|\alpha|\,|z|^{n-1}+1)\max_{|z|=1}|P(z)| - (|\alpha|\,|z|^{n-1}-1)\min_{|z|=1}|P(z)|\Big\}, \\ &\qquad \text{for } |z| \geqslant 1, |\alpha| \geqslant 1. \end{aligned}$$

If we divide two sides of this inequality by $|z|^{n-1}$ and let $|z| \to \infty$, we easily obtain

$$|n\alpha a_n + a_{n-1}| \leqslant \frac{n}{2}|\alpha|\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\Big\},$$

for every α with $|\alpha| \geqslant 1$. By choosing argument of α suitably, we immediately get the following result, which is the refinement of Corollary 2 of [2].

COROLLARY 3

If

$$P(z) = \sum_{j=1}^{n} a_j z^j$$

is a polynomial of degree n which does not vanaish in the disk $|z| < 1$, then

$$n|a_n| + |a_{n-1}| \leqslant \frac{n}{2}\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\Big\}.$$

2. Lemmas

For the proofs of the theorems, we need the following lemmas. The first result is Laguerre's theorem [8, p. 52] (see also [3]).

Lemma 1. *If all the zeros of the nth degree polynomial $P(z)$ lie in a circular region C, and if W is any zero of*

$$D_\alpha P(z) = nP(z) + (\alpha - z)\,P'(z),$$

the polar derivative of $P(z)$, then both points W and α may not lie outside of C.

By repeated applications of Lemma 1, we get the following result, when the circular region C is the circle $|z| \leqslant r$.

Lemma 2. *If all the zeros of an nth degree polynomial $P(z)$ lie in $|z| \leqslant r$ and if none of the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ lie in $|z| \leqslant r$, then each of the polar derivatives $D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)$, $k = 1, 2, \ldots, n-1$ has all its zeros in $|z| \leqslant r$.*

The next lemma is due to Aziz [2].

Lemma 3. *If $P(z)$ is a polynomial of degree n such that*

$$\max_{|z|=1} |P(z)| = M$$

and $\alpha_1, \alpha_2, \ldots, \alpha_k$; $k \leqslant n-1$ are complex numbers with $|\alpha_i| \geqslant 1$ for all $i = 1, 2, \ldots, k$; then for $|z| \geqslant 1$

$$\begin{aligned} &|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| + |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(z)| \\ &\leqslant n(n-1)\cdots(n-k+1)\{|\alpha_1 \alpha_2 \cdots \alpha_k||z|^{n-k} + 1\}M, \end{aligned} \tag{13}$$

where

$$Q(z) = z^n \overline{P(1/\bar{z})}.$$

3. Proofs of the theorems

Proof of theorem 1. The result is clear if $P(z)$ has a zero on $|z| = 1$, then

$$m = \min_{|z|=1} |P(z)| = 0.$$

We now suppose that all the zeros of $P(z)$ lie in $|z| < 1$, then $m > 0$ and we have $m \leqslant |P(z)|$ for $|z| = 1$. Hence for every β with $|\beta| < 1$, we have $|P(z)| > |m\beta z^n|$ for $|z| = 1$ and therefore, it follows by Rouche's theorem that the polynomial $F(z) = P(z) - m\beta z^n$ has all its zeros in $|z| < 1$. If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are complex numbers with $|\alpha_i| \geqslant 1$, $i = 1, 2, \ldots, k$; $k \leqslant n-1$, then by Lemma 2, all the zeros of

$$\begin{aligned} &D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} F(z) \\ &= D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z) - m\beta n(n-1)\cdots(n-k+1)\alpha_1 \alpha_2 \cdots \alpha_k z^{n-k} \end{aligned} \tag{14}$$

lie in $|z| < 1$, which implies, for $|z| \geqslant 1$,

$$mn(n-1)\cdots(n-k+1)|\alpha_1 \alpha_2 \cdots \alpha_k||z|^{n-k} \leqslant |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)|. \tag{15}$$

If inequality (15) is not true, then there is a point $z = z_0$, with $|z_0| \geq 1$, such that

$$[mn(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}]_{z=z_0} > [|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|]_{z=z_0},$$

and we take

$$\beta = \frac{[D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)]_{z=z_0}}{[mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}]_{z=z_0}},$$

so that $|\beta| < 1$ and from (14), with this choice of β, we get $[D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z)]_{z=z_0} = 0$, where $|z_0| \geq 1$, which contradicts the fact that the zeros of $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z)$ lie in $|z| < 1$. Therefore from (15) we conclude that

$$\min_{|z|=1}|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$

$$\geq n(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k|\min_{|z|=1}|z|^{n-k}|P(z)|$$

and this completes the proof of Theorem 1.

Proof of theorem 2. By hypothesis, the polynomial $P(z)$ has all its zeros in $|z| \geq 1$. In case it has a zero on $|z| = 1$, then

$$m = \min_{|z|=1}|P(z)| = 0$$

and the result follows by Theorem 1 of [2].

We now suppose that all the zeros of $P(z)$ lie in $|z| > 1$, then $m > 0$ and

$$m \leq |P(z)| \quad \text{for} \quad |z| = 1. \tag{16}$$

If β is any complex number such that $|\beta| < 1$, then it follows by Rouche's theorem that the polynomial $F(z) = P(z) - \beta m$ does not vanish on $|z| = 1$ also. Because, if for some $z = z_0$,

$$F(z_0) = P(z_0) - m\beta = 0, \text{ where } |z_0| = 1,$$

then we have

$$|P(z_0)| = |m\beta| = m|\beta| < m,$$

which is a contradiction to (16). Thus for every β, with $|\beta| < 1$, the polynomial $F(z) = P(z) - \beta m$ has all its zeros in $|z| > 1$. If

$$G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{P(1/\bar{z})} - \bar{\beta}mz^n = Q(z) - \bar{\beta}mz^n,$$

then all the zeros of $G(z)$ lie in $|z| < 1$ and $|G(z)| = |F(z)|$ for $|z| = 1$. Therefore, for every complex number λ with $|\lambda| > 1$, the polynomial $F(z) - \lambda G(z)$ has all its zeros in $|z| < 1$. It follows by Lemma 1, that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are complex numbers such that $|\alpha_i| \geq 1$, $1 \leq i \leq k$, the polynomial $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}(F(z) - \lambda G(z))$ has all its zeros in $|z| < 1$. Equivalently, all the zeros of $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z) - \lambda D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}G(z)$ lie in $|z| < 1$. This clearly implies that for $|z| \geq 1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z)| \leq |D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}G(z)|,$$

or

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}(P(z) - m\beta)| \leq |D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}(Q(z) - m\bar{\beta}z^n)|,$$

equivalently

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)-m\beta n(n-1)\cdots(n-k+1)|$$
$$\leqslant|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}Q(z)-m\bar{\beta}n(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|. \tag{17}$$

In view of (15) we can choose an argument of β in (17) such that for $|z|\geqslant 1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|-m|\beta|n(n-1)\cdots(n-k+1)$$
$$\leqslant|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}Q(z)|-m|\beta|n(n-1)\cdots(n-k+1)/\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}. \tag{18}$$

Letting $|\beta|\to 1$ in (18), we get

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$
$$\leqslant|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}Q(z)|-n(n-1)\cdots(n-k+1)\{|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k}-1\}m. \tag{19}$$

Inequality (19) in conjuction with Lemma 3 gives for $|z|\geqslant 1$, $|\alpha_i|\geqslant 1$, $i=1,2,\ldots,k$; $k\leqslant n-1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$
$$\leqslant\frac{n(n-1)\cdots(n-k+1)}{2}\Big\{(|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}+1)\max_{|z|=1}|P(z)|$$
$$-(|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}-1)\min_{|z|=1}|P(z)|\Big\}.$$

This proves the desired result.

4. Note

Finally, we mention that the idea of the proof of Theorem 1 comes from a paper of Bernstein [6], where he applied the Gauss–Lucas theorem to obtain (1). The central idea is that if a polynimial $P(z)$ has all its zeros in the unit disk, then so does the derivative $P'(z)$. There are operators other than differentiation for which this is true. Polar derivative with respect to a point outside the unit disk is one such operator. There are many others and just as interesting (for example, see [11, p. 55–59]). It is an interesting problem to characterize all such operators. An interested reader may consult [9] and [11] for other results related to the problem considered in this paper.

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A theorem concerning a product of two general classes of polynomials and the multivariable H-function

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Abstract. A theorem concerning a product of two general classes of polynomials and the multivariable H-function is established. Certain integrals and expansion formulae have also been derived by the application of this theorem. This general theorem yields a number of new, interesting and useful theorems, integrals and expansion formulae as its particular cases.

Keywords. Multivariable H-function; general class of polynomials; integrals; expansion formulae.

1. Introduction and the main result

Srivastava [5; p.l, eq. (1)] considered the general class of polynomials as

$$S_n^m[x] = \sum_{\alpha=0}^{[n/m]} \frac{(-n)_{m\alpha}}{\alpha!} A_{n,\alpha} x^\alpha, n = 0, 1, 2, \ldots, \tag{1}$$

where m is an arbitrary positive integer and the coefficients $A_{n,\alpha}\,(n, \alpha \geqslant 0)$ are arbitrary constants real or complex. Also $(\lambda)_\alpha$ denotes the Pochhammer symbol and defined as

$$(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)} = \begin{cases} 1 \text{ if } \alpha = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), \\ \forall \alpha = 1, 2, 3, \cdots \end{cases} \tag{2}$$

By suitably specializing the coefficients $A_{n,\alpha}$, polynomials $S_n^m[x]$ can be reduced to the classical orthogonal polynomials i.e. Jacobi, Hermite, Laguerre polynomials etc. (see [3]).

The following theorem can be derived for the H-function of several complex variables defined by Srivastava and Panda [6; also see 8, p. 251]

Main Theorem. *If*

$$(1-t)^{u+v-w}\, {}_2F_1(2u, 2v; 2w; t) = \sum_{k=0}^{\infty} a_k t^k \tag{3}$$

then

$$\int_0^1 {}_2F_1(u, v; w+1/2; t)\ {}_2F_1(w-u, w-v; w+1/2; t)$$

$$\cdot S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1}, \ldots, z_r t^{h_r})\, dt$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m']} \sum_{k=0}^{\infty} \frac{(-n)_{m\alpha}}{\alpha!} A_{n,\alpha} \frac{(-n')_{m'\beta}}{\beta!} A_{n',\beta} \frac{(w)_k}{(w+1/2)_k} a_k$$

$$H^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha-h'\beta:h_1,\ldots,h_r], \\ [-k-h\alpha-h'\beta-1:h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; z_1 \\ [(c):\psi',\ldots,\psi^{(r)}]: [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \vdots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad z_r \end{matrix} \right), \tag{4}$$

where

$$h>0, \quad h'>0, \quad h_2>0, \quad \mathrm{Re}\left(1+\sum_{i=1}^{r} h_2 d_j^{(i)}/\delta_j^{(i)}\right)>0,$$

$$-\tfrac{1}{2}<(w-u-v)<\tfrac{1}{2}, \quad |\arg(z_i)|<T_i\pi/2, \quad T_i>0,$$

$i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}$ $(n,n',\alpha,\beta \geqslant 0)$ are arbitrary constants, real or complex.

2. Proof of the main theorem

We start with [4, p. 75]

$$ {}_2F_1(u,v;w+1/2;t)\, {}_2F_1(w-u,w-v;w+1/2;t)$$

$$= \sum_{k=0}^{\infty} \frac{(w)_k}{(w+1/2)_k} a_k t^k, \tag{5}$$

where a_k is given by (3). Now, multiplying both sides of (5) by $S_n^m[t^{h'}]\ S_{n'}^{m'}[t^{h'}]$ $H(z_1 t^{h_1},\ldots,z_r t^{h_r})$ and integrating with respect to t between the limits 0 and 1, we obtain

$$\int_0^1 {}_2F_1(u,v;w+1/2;t)\, {}_2F_1(w-u,w-v;w+1/2;t)$$

$$\cdot S_n^m[t^h]\, S_{n'}^{m'}[t^h]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \int_0^1 \sum_{k=0}^{\infty} \frac{(w)_k}{(w+1/2)_k} a_k t^k\, S_n^m[t^h]\, S_{n'}^{m'}[t^h]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t. \tag{6}$$

Using the definitions for the general class of polynomials [5, p. 1, eq. (1)] and of the multivariable H-function [8, p. 251] on the right of (6) and then interchange the order of integration and summations which is permissible under the conditions needed in (4) and evaluating with the following result

$$\int_0^1 t^l\, S_n^m[t^h]\, S_{n'}^{m'}[t^h]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m]} \frac{(-n)_{m\alpha}}{\alpha!} \frac{(-n')}{\beta!} A_{n,\alpha} \cdot A_{n',\beta}$$

$$\cdot H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})} \left(\begin{matrix} [-l-h\alpha-h'\beta: h_1,\ldots,h_r], \\ [-l-h\alpha-h'\beta-1: h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; z_1 \\ [(c):\psi',\ldots,\psi^{(r)}]: [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \vdots \; z_r \end{matrix} \right), \tag{7}$$

where

$$h>0, \quad h'>0, \quad h_i>0, \quad \mathrm{Re}\left(l+1+\sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)}\right)>0,$$

$|\arg(z_i)| < T_i\pi/2, T_i>0, i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}(n,n',\alpha,\beta \geqslant 0)$ are arbitrary constants, real or complex, we get the required result.

3. Applications

Taking $u=w$ in the main theorem, the value of a_k in (3) becomes equal to $(v)_k$ and the result (4) gives the following interesting integral

$$\int_0^1 {}_2F_1(u,v;u+1/2;t)\, S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m']} \sum_{k=0}^{\infty} \frac{(-n)_{m\alpha}}{\alpha!} \frac{(-n')_{m'\beta}}{\beta!} A_{n,\alpha} \cdot A_{n',\beta} \frac{(u)_k (v)_k}{(u+1/2)_k\, k!}$$

$$H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})} \left(\begin{matrix} [-k-h\alpha-h'\beta: h_1,\ldots,h_r], \\ [-k-h\alpha-h'\beta-1: h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; z_1 \\ [(c):\psi',\ldots,\psi^{(r)}]: [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \vdots \; z_r \end{matrix} \right), \tag{8}$$

where $h>0$, $h'>0$, $h_i>0$, $\mathrm{Re}\,(v)>1/2$, $\mathrm{Re}\,(1+\sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)})>0, |\arg(z_i)| < T_i\pi/2$, $T_i>0, i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}(n,n'\,\alpha,\beta \geqslant 0)$ are arbitrary constants, real or complex.

Setting $v=u+1/2$ and $u=-e$ (e is a non-negative integer), we have

$$\int_0^1 {}_1F_0(-e;t)\, S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m]} \sum_{k=0}^{e} \frac{(-n)_{m\alpha}}{\alpha!} \frac{(-n')_{m'\beta}}{\beta!} A_{n,\alpha} \cdot A_{n',\beta} \frac{(-e)_k}{k!}$$

$$\cdot H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})} \left(\begin{matrix} [-k-h\alpha-h'\beta: h_1,\ldots,h_r], \\ [-k-h\alpha-h'\beta-1: h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; z_1 \\ [(c):\psi',\ldots,\psi^{(r)}]: [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \vdots \; z_r \end{matrix} \right), \tag{9}$$

where $h>0,\quad h'>0,\quad h_i>0, \operatorname{Re}(1+\sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)})>0, |\arg(z_i)|<T_i\pi/2, T_i>0$ $i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficient $A_{n,\alpha}$ and $A_{n',\beta}(n,n'\alpha,\beta\geqslant 0)$ are arbitrary constants, real or complex.

Now we establish the following interesting and useful expansion formula by appeal ing to the integral on the left of (9) with the help of (7).

$$\sum_{\alpha=0}^{[n/m]}\sum_{\beta=0}^{[n'/m']}\sum_{k=0}^{e}\frac{(-n)_{m\alpha}}{\alpha!}\frac{(-n')_{m'\beta}}{\beta!}A_{n,\alpha}A_{n',\beta}\frac{(-e)_k}{k!}$$

$$H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[-k-h\alpha-h'\beta:h_1,\ldots,h_r],\\ [-k-h\alpha-h'\beta-1:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];z_1\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\vdots\ z_r\end{matrix}\right),$$

$$=\Gamma(e+h\alpha+h'\beta+1)\sum_{\alpha=0}^{[n/m]}\sum_{\beta=0}^{[n'/m']}\frac{(-n)_{m\alpha}}{\alpha!}\frac{(-n')_{m'\beta}}{\beta!}A_{n,\alpha}A_{n',\beta}$$

$$H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[o:h_1,\ldots,h_r],\\ [-e-h\alpha-h'\beta-1:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]:z_1\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\vdots\ z_r\end{matrix}\right), \tag{1}$$

provided that both the sides exist.

4. Special Cases

(i) If we take $m=2=m'$ and $A_{n,\alpha}=(-1)^{\alpha}, A_{n',\beta}=(-1)^{\beta}$ in (4), we obtain the followi theorem.

Theorem 1 (a). *If*

$$(1-t)^{u+v-w}\,{}_2F_1(2u,2v;2w:t)=\sum_{k=0}^{\infty}a_k t^k$$

then

$$\int_0^1 {}_2F_1(u,v;w+1/2:t)\ {}_2F_1(w-u,w-v;w+1/2:t)\,t^{hn/2}\,t^{h'n'/2}$$

$$H_n\left(\frac{1}{2\sqrt{t^h}}\right)H_{n'}\left(\frac{1}{2\sqrt{t^{h'}}}\right)H(z_1t^{h_1},\ldots,z_rt^{h_r})\,dt$$

$$=\sum_{\alpha=0}^{n/2}\sum_{\beta=0}^{n'/2}\sum_{k=0}^{\infty}\frac{(-n)_{2\alpha}(-1)^{\alpha}}{\alpha!}\frac{(-n')_{2\beta}(-1)^{\beta}}{\beta!}\frac{(w)_k a_k}{(w+1/2)_k}$$

$$H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[-k-h\alpha-h'\beta:h_1,\ldots,h_r],\\ [-k-h\alpha-h'\beta-1:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];z_1\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\vdots\ z_r\end{matrix}\right),$$

valid under the same conditions needed for (4).

(ii) On taking $m = 1 = m'$ and

$$A_{n,\alpha} = \binom{n+f}{n} \frac{(f+g+n+1)_\alpha}{(f+1)_\alpha},$$

$$A_{n',\beta} = \binom{n'+f'}{n'} \frac{(f'+g'+n'+1)_\beta}{(f'+1)_\beta} \text{ in (4),}$$

we get the following theorem.

Theorem 1(b). *If*

$$(1-t)^{u+v-w}\, {}_2F_1(2u, 2v; 2w: t) = \sum_{k=0}^{\infty} a_k t^k$$

then

$$\int_0^1 {}_2F_1(u, v; w+1/2: t)\ {}_2F_1(w-u, w-v; w+1/2: t)$$

$$P_n^{(f,g)}(1-2t^h)\ P_{n'}^{(f',g')}(1-2t^{h'})\ H(z_1 t^{h_1}, \ldots, z_r t^{h_r})\, dt$$

$$= \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n'} \sum_{k=0}^{\infty} (-1)^\alpha (-1)^\beta \binom{n+f}{n-\alpha} \binom{n+f+g+\alpha}{\alpha}$$

$$\binom{n'+f'}{n'-\beta} \binom{n'+f'+g'+\beta}{\beta} \cdot \frac{(w)_k a_k}{(w+1/2)_k}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha-h'\beta: h_1, \ldots, h_r], \\ [-k-h\alpha-h'\beta-1: h_1, \ldots, h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a): \theta', \ldots, \theta^{(r)}]: [(b'): \phi']; \ldots; [(b^{(r)}): \phi^{(r)}]; z_1 \\ [(c): \psi', \ldots, \psi^{(r)}]: [(d'): \delta']; \ldots; [(d^{(r)}): \delta^{(r)}]; \vdots \\ z_r \end{matrix} \right), \tag{12}$$

valid under the same conditions as obtainable from (4).

(iii) Letting $m = 1 = m'$ and $A_{n,\alpha} = \binom{n+f}{n} \frac{1}{(f+1)_\alpha}, A_{n',\beta} = \binom{n'+f'}{n'} \frac{1}{(f'+1)_\beta}$ in (4), we get the following.

Theorem 1(c). *If*

$$(1-t)^{u+v-w}\, {}_2F_1(2u, 2v: 2w: t) = \sum_{k=0}^{\infty} a_k t^k$$

then

$$\int_0^1 {}_2F_1(u, v; w+1/2; t)\ {}_2F_1(w-u, w-v; w+1/2; t)$$

$$L_n^{(f)}(t^h)\ L_{n'}^{(f')}(t^{h'})\ H(z_1 t^{h_1}, \ldots, z_r t^{h_r})\, dt$$

$$= \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n'} \sum_{k=0}^{\infty} \frac{(-1)^\alpha}{\alpha!} \frac{(-1)^\beta}{\beta!} \binom{n+f}{n-\alpha} \binom{n'+f'}{n'-\beta} \frac{(w)_k a_k}{(w+1/2)_k}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha-h'\beta: h_1, \ldots, h_r], \\ [-k-h\alpha-h'\beta-1: h_1, \ldots, h_r], \end{matrix} \right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\ [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\ z_1\\ [(c):\psi',\ldots,\psi^{(r)}]:\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\ \vdots\ z_r\end{matrix}\right), \tag{13}$$

valid under the same conditions as obtainable from (4).

(iv) *If* $n' \to 0$, *the theorem given by* (4) *reduces to a theorem recently obtained by Chaurasia and Sharma* [2, *p.* 291, *eq.* (3)].

(v) *Taking* $n' \to 0$, *the results in* (7) *to* (10) *reduce to the known results obtained by Chaurasia and Sharma* [2, *eqs* (6), *p.* 292, (7) *and* (8), *p.* 293 *and* (9), *p.* 294].

(vi) *The main theorem given in* (4) *reduces to a known theorem obtained by Chaurasia* [1, *eq.* (1.2), *p.* 193], *on making* $n \to 0$ *and* $n' \to 0$.

(vii) *When* $n \to 0$ *and* $n' \to 0$ *the result* (7) *through* (10) *reduce to the known results obtained by Chaurasia* [1, *eq.* (2.3), *p.* 194, (3.1) *and* (3.2), *p.* 195 *and* (3.3), *p.* 195].

The importance of our results lies in its manifold generality. In view of its generality of the multivariable H-function, on specializing the various parameters and variables in the multivariable H-function, we can derive, from our theorems, integrals and expansion formulae etc. involving a remarkably wide variety of useful functions (or products of several such functions) which are expressible in terms of E, F, G and H-functions of one and more variables.

Secondly, by specializing the coefficients $A_{n,\alpha}$ and $A_{n',\beta}$ and making a free use of the special cases of $s_n^m[x]$ listed by Srivastava and Singh [7], our results can be reduced to a large number of theorems, integrals and expansion formulae etc. involving the product of generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and their various special cases, products of Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations. Thus, the results established in this paper would at once yield a very large number of known and new theorems, integrals and expansion formulae, involving a large variety of polynomials and various special functions appearing in the literature on mathematical analysis, applied mathematics and mathematical physics.

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Weakly analytic sets for function spaces

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Abstract. We define and study weakly analytic sets for function spaces. This concept generalizes the concept of weak analyticity for function algebras. We also define weak analyticity for the space of affine functions using the ideas of convexity. We show that it coincides with the same concept regarding it as a function space. We give several examples to illustrate the concepts.

Keywords. Weakly analytic set; function space; function algebra; space of affine functions.

1. Introduction

A function space A on a compact Hausdorff space X is a closed subspace of the space $C(X)$ of all continuous, complex-valued functions on X separating points and containing constants. If A is an algebra, it is called a function algebra.

It is well-known that the Bishop and Šilov decompositions play an important role in characterizing function algebras. Later, some other decompositions finer than the Bishop and Šilov decompositions have been defined and studied [4, 7, 8], for example, the decompositions of weakly prime sets, weakly analytic sets etc. We studied some of these decompositions for function spaces ([9, 10]). In the first part of this paper, we generalize the concept of a weakly analytic set for a function space and study its properties. We give several examples connected with the results.

Ellis [4] has studied weakly prime sets for $A(K)$, the space of continuous ffine functions on a compact convex set K of a locally convex topological vector space, using the idea of convexity. We have shown in [10] that this idea coincides with the abstract definition of weakly prime sets for function spaces. In § 2, we define weak analyticity for $A(K)$ using convexity which is finer than the idea of weak primeness according to Ellis. Then, we show that both definitions of weak analyticity for $A(K)$ coincide. Finally, we give some examples to illustrate the concepts.

2. Function space

Arenson [2] has defined a weakly analytic set for a function algebra as follows:

DEFINITION 2.1 [2]

Let A be a function algebra on a compact Hausdorff space X. A closed subset E of X is called a *weakly analytic set* for A if every peak set for $(A_{|E})^{-}$ either coincides with E or it is nowhere dense in E.

Equivalently, E is a weakly analytic set for A if $E = G \cup H$, with G a peak set for $(A_{|E})^{-}$ and H a closed set in E, then either $G = E$ or $H = E$.

Let A be a function space on X. For a closed subset E of X, we define

$$N(A_{|E}) = \{f \in C(E): fg \in A_{|E} \text{ for all } g \in A_{|E}\}.$$

Then $N(A_{|E})$ is an algebra. Also, $N(A_{|E}) \subseteq A_{|E}$ and if A is an algebra, then $N(A_{|E}) = A_{|E}$

DEFINITION 2.2

A closed subset E of X is called a *weakly analytic set* for a function space A if $E = G \cup H$ with G a generalized peak set (i.e. intersection of peak sets) for $N(A_{|E})$ and H a closed subset of E, then either $G = E$ or $H = E$.

Equivalently, E is a weakly analytic set for A if G is a proper generalized peak set for $N(A_{|E})$, then the interior of G, $G^0 = \phi$.

The function space A is called weakly analytic if X is a weakly analytic set for A.

Remark 2.3. (i) It can be shown that each weakly analytic set is contained in a maximal weakly analytic set for A. The collection of all maximal weakly analytic sets for A is denoted by $\mathscr{W}_\alpha(A)$. In general, the members of $\mathscr{W}_\alpha(A)$ are not disjoint (see Example 3.2). (ii) It is easy to check that $\mathscr{W}_\alpha(A)$ is finer than $\mathscr{P}(A)$, the family of maximal weakly prime sets [10] for A. (iii) It is known that if A is an analytic function algebra, then it is antisymmetric [6]. It can be easily checked that if A is analytic, then it is weakly analytic and hence weakly prime. But there is a weakly analytic function algebra which is not analytic (see Example 3.4).

The importance of all the decompositions is due to their (D)-property. First, we define this property.

DEFINITION 2.4 [5]

Let $\mathscr{F}$ be a family of closed subsets of X and A be a function space on X. (i) We say that $\mathscr{F}$ has the (D)-*property* for A if whenever $f \in C(X)$ and $f_{|F} \in (A_{|F})^{-}$ for all $F \in \mathscr{F}$, then $f \in A$. (ii) We say that $\mathscr{F}$ has the (GA)-*property* for A if $\mu \in b(A^{\perp})^{e}$, then Supp $\mu \subset F$ for some $F \in \mathscr{F}$, where $b(A^{\perp})^{e}$ denotes the set of extreme points of the closed unit ball of the annihilator $A^{\perp}$ of A.

It can be checked that the (GA)-property is stronger than the (D)-property.

We shall show that $\mathscr{W}_\alpha(A)$ has the (GA)-property for A.

Theorem 2.5. *$\mathscr{W}_\alpha(A)$ has the (GA)-property for A.*

Proof. Let $\mu \in b(A^{\perp})^{e}$ and $S = \text{Supp}\mu$. It is enough to show that S is a weakly analytic set for A.

Let G be a generalized peak set for $N(A_{|S})$ and H be a closed subset of S with $G \cup H = S$. Let $\mu_1 = \mu_{|G}$ and $\mu_2 = \mu - \mu_1$. First, we shall show that $\mu_1 \in A^{\perp}$. Let $\varepsilon > 0$. There exists an open set U, containing G, in X such that $|\mu|(U \setminus G) < \varepsilon$. Clearly, $S \cap U$ is open in S and $G \subset S \cap U$. Since G is a generalized peak set for $N(A_{|S})$, there is a peak set T with peaking function f in $N(A_{|S})$ such that $G \subset T \subset S \cap U$. Define h on S by $h = 1$ on T and $h = 0$ on $S \setminus T$. Then f^n converges pointwise and boundedly to h on S. Therefore, for $g \in A$,

$$\int_T g \, d\mu = \int_S gh \, d\mu = \int_S g(\lim f^n) \, d\mu = \lim \int_S g f^n d\mu.$$

But $f^n \in N(A_{|S})$ implies that $gf^n_{|S} \in A_{|S}$ and we have $\mu \in A^\perp = (A_{|S})^\perp$. Therefore, $\int_T g\,d\mu = 0$. Now,

$$\left|\int g\,d\mu_1\right| = \left|\int_G g\,d\mu\right| = \left|\int_G g\,d\mu - \int_T g\,d\mu\right|$$
$$\leqslant \|g\|(|\mu|(T\setminus G)) \leqslant \|g\|(|\mu|(U\setminus G)) < \varepsilon\|g\|.$$

Since ε is arbitrary, $\int g\,d\mu_1 = 0$. Hence $\mu_1 \in A^\perp$. Consequently, $\mu_2 \in A^\perp$. Also, $||\mu|| = 1 = ||\mu_1|| + ||\mu_2||$. Hence $\mu_1 = \mu$ or $\mu_2 = \mu$, as $\mu \in b(A^\perp)^e$, i.e. $G = S$ or $H = S$. Hence S is a weakly analytic set for A.

Unlike maximal weakly prime sets, the maximal weakly analytic sets are, in general, not generalized peak sets, even for function algebras (see Example 3.3). However, if X is metrizable then, Arenson [2] has shown that, for a function algebra A, $A_{|E}$ is closed in $C(E)$ for $E \in \mathscr{D}_\alpha(A)$ (which holds if E is a generalized peak set), where $\mathscr{D}_\alpha(A)$ denotes the decomposition of X corresponding to the family $\mathscr{W}_\alpha(A)$. We prove a stronger result that if X is metrizable, then members of $\mathscr{D}_\alpha(A)$ are p-sets for a function space A. In fact, it is a consequence of the following more general result.

PROPOSITION 2.6

Let $\mathscr{E}$ be a decomposition of a metrizable space X with the (GA)-property for a function space A. Then each $E \in \mathscr{E}$ is a p-set for A (i.e., $\mu \in A^\perp \Rightarrow \mu_E \in A^\perp$, where μ_E is the restriction of μ on E) and hence a generalized peak set for A.

Proof. Let $E \in \mathscr{E}$. It is enough to show that $\mu_E \in A^\perp$ whenever $\mu \in b(A^\perp)$. Let $\mu \in b(A^\perp)^e$. Then, there exists $F \in \mathscr{E}$ such that $\mathrm{Supp}\mu \subset F$. Since $\mathscr{E}$ is a decomposition, either $E = F$ or $E \cap F = \phi$. Hence $\mu_E = \mu$ or $\mu_E = 0$ and so $\mu_E \in A^\perp$.

Let $\mu \in b(A^\perp)$. Since X is metrizable, $C(X)$ is separable and hence the weak* topology on $M(X)$ is metrizable. Since $b(A^\perp)$ is the weak* closed convex hull of $b(A^\perp)^e$ in $M(X)$, by Choquet's theorem [11, p. 19], there exists a regular Borel measure λ supported on $b(A^\perp)^e$ such that $f(\mu) = \int_X f\,d\mu = \int_{b(A^\perp)^e} f(\nu)\,d\lambda(\nu)$ for every $f \in C(X)$. Now, E is a G_δ-set and hence

$$\int_E f\,d\mu = \int_X f\chi_E\,d\mu = \int_{b(A^\perp)^e} (f\chi_E)(\nu)\,d\lambda(\nu)$$

for every $f \in C(X)$. But for $\nu \in b(A^\perp)^e$, $\nu_E = \nu$ or $\nu_E = 0$. Hence, for $f \in A$,

$$0 = \int_X f\,d\nu = \int_E f\,d\nu = \int_X (f\chi_E)\,d\nu = (f\chi_E)(\nu).$$

It follows that $\int_E f\,d\mu = 0$ for each $f \in A$, i.e., $\mu_E \in A^\perp$. Thus E is a p-set for A. Since each p-set is a generalized peak set for a function space, the result follows.

COROLLARY 2.7

Let X be a metrizable space. Then each $E \in \mathscr{D}_\alpha(A)$ is a p-set for A.

Proof. By Theorem 2.5, $\mathscr{W}_\alpha(A)$ has the (GA)-property for A. Since each member of $\mathscr{W}_\alpha(A)$ is contained in some member of $\mathscr{D}_\alpha(A)$, it can be easily checked that $\mathscr{D}_\alpha(A)$ also has the (GA)-property for A. So, by Proposition 2.6, each $E \in \mathscr{D}_\alpha(A)$ is a p-set for A.

3. Examples

In this section, we study maximal weakly analytic sets for some function algebras and function spaces.

Example 3.1. Let X be the union of a line segment F and a sequence of disjoint solid rectangles $\{F_n : n \in \mathbb{N}\}$ converging to F. Let A be the set of all f in $C(X)$ such that $f_{|F_n}$ is a polynomial of degree atmost n. Then A is a function space on X and as in [9, Example 3.5], it can be checked that $\mathscr{W}_\alpha(A) = \{F_n : n \in \mathbb{N}\} \cup \{\{x\} : x \in F\} = \mathscr{D}_\alpha(A)$. Note that, for this function space A, $N(A) = \{f \in C(X) : f_{|F_n}$ is constant, for each $n \in \mathbb{N}\}$ and hence $\mathscr{W}_\alpha(N(A)) = \{F_n : n \in \mathbb{N}\} \cup \{F\}$.

Example 3.2. Let $D = D_1 \cup D_2$, where $D_1 = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $D_2 = \{z \in \mathbb{C} : |z - 2| \leqslant 1\}$. Let $A = A(D)$, the set of all functions in $C(D)$ which are analytic in the interior of D. Then A is a function algebra on D [4, Example 2]. It can be checked that $\mathscr{W}_\alpha(A) = \{D_1, D_2\}$. But, as $D_1 \cap D_2 \neq \phi$, $\mathscr{D}_\alpha(A) = \{D\}$.

Next, we give an example which shows that, in general, the members of $\mathscr{W}_\alpha(A)$ are not generalized peak sets.

Example 3.3. [4]. Let D be the set as considered in Example 3.2. Let X denote the quotient space of D by identifying the points 0 and 1; 2 and 3 and let $q : D \to X$ denote the quotient map. Also, let $A = \{f \in C(X); f \circ q \in A(D)\}$. Then A is a function algebra on X. Also, $\mathscr{W}_\alpha(A) = \{q(D_1), q(D_2)\}$. It can be verified that $q(D_1)$ is a peak set for A but $q(D_2)$ cannot be a peak set for A. Further, note that, A is a weakly prime algebra which is not weakly analytic.

Finally, we show that the concept of weak analyticity is really weaker than that of analyticity.

Example 3.4. Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $T = \{z \in \mathbb{C} : |z| = 1\}$. Also, let $S = T \cup \{0\}$ and $A = A(D)_{|S}$. Then A is a function algebra on S. A is not an analytic algebra, as the function $f(z) = z$ is in A and it is zero on an open subset $\{0\}$ of S. But it can be checked that A is a weakly analytic function algebra.

4. Space of affine functions

Let $A(K)$ denote the Banach space of all real-valued continuous affine functions on a compact convex set K in a locally convex topological vector space and ∂K denote the set of extreme points of K.

In this section, we define weak analyticity for $A(K)$ using only ideas of convexity. Then as in the case of weakly prime sets [10], we show that the concept coincides with the one given in Definition 2.2. Finally, we give certain examples to illustrate the concepts.

For the definitions and results regarding compact convex sets and space of affine functions, we refer to [1] and [3].

DEFINITION 4.1

A subset E of ∂K is called a *weakly analytic set for* $A(K)$ if $E = \partial F$, for some closed face F of K and if $F = co(G \cup H)$, where G is a closed split face of F and H is a closed convex subset of F, then either $G = F$ or $H = F$.

We say that $A(K)$ is weakly analytic if ∂K is a weakly analytic set for $A(K)$.

The following proposition can be easily verified.

PROPOSITION 4.2

Each weakly analytic set for $A(K)$ is contained in a maximal weakly analytic set for $A(K)$.

Let $\mathscr{W}_{\alpha c}(A(K))$ denote the family of all maximal weakly analytic sets for A(K). Then, it can be checked that $\mathscr{W}_{\alpha c}(A(K))$ is finer than $\mathscr{W}_{pc}(A(K))$, the family of maximal weakly prime sets for $A(K)$ defined by Ellis [4].

Now, $A(K)$ can also be looked upon as a function space on K. So, we can discuss $\mathscr{W}_{\alpha}(A(K))$ for $A(K)$. But, since the functions in $A(K)$ are determined by their values on ∂K, it is natural to consider the space $A(K)_{|\partial K}$. Also, note that, the members of $\mathscr{W}_{\alpha c}(A(K))$ are subsets of ∂K. So, one would like to compare the families $\mathscr{W}_{\alpha c}(A(K))$ and $\mathscr{W}_{\alpha}(A(K)_{|\partial K})$.

We shall need the following lemma which can be easily proved.

Lemma 4.3. *$Ce(A(K))_{|\partial K} = N(A(K)_{|\partial K})$, where $Ce(A(K)) = \{f \in A(K): fg_{|\partial K} \in A(K)_{|\partial K}$ for every $g \in A(K)\}$, the centre of $A(K)$.*

As a consequence of the above lemma, we get the following result.

COROLLARY 4.4

A subset E of ∂K is a facially closed subset of ∂K if and only if E is a generalized peak set for $N(A(K)_{|\partial K})$

Now, we show that the families $\mathscr{W}_{\alpha c}(A(K))$ and $\mathscr{W}_{\alpha}(A(K)_{|\partial K})$ coincide, if they consist of only one member ∂K.

PROPOSITION 4.5

$\mathscr{W}_{\alpha c}(A(K)) = \{\partial K\}$ if and only if $\mathscr{W}_{\alpha}(A(K)_{|\partial K}) = \{\partial K\}$, i.e., $A(K)$ is weakly analytic in the sense of Definition 4.1 *if and only if the function space $A(K)_{|\partial K}$ is weakly analytic in the sense of Definition* 2.2.

Proof. Assume that $A(K)$ is weakly analytic. To show that the function space $A(K)_{|\partial K}$ is weakly analytic, let G be a generalized peak set for $N(A(K)_{|\partial K})$. By Corollary 4.4, G is a facially closed subset of ∂K. Then $G = \partial E$, for some closed split face E of K. Let $H = \overline{co}(\partial K - G^0)$. Then H is a closed convex subset of K. Also, $\overline{co}\ (E \cup H) = \overline{co}(\overline{co}(\partial E) \cup \overline{co}(\partial K - G^0)) = \overline{co}(\partial E \cup (\partial K - G^0)) = \overline{co}(G \cup (\partial K - G^0)) = \overline{co}(\partial K) = K$. Since $A(K)$ is weakly analytic, by Definition 4.1, $E = K$ or $H = K$. If $E = K$, then $\partial E = \partial K$, i.e., $G = \partial K$. If $H = K$, then $\partial H = \partial K$. Now, $\overline{co}(\partial K - G^0) = H = \overline{co}(\partial H)$ and so, $\partial H \subset \partial K - G^0$. Thus, we get $\partial K = \partial H \subset \partial K - G^0 \subset \partial K$ which implies that $G^0 = \phi$. Hence, either $G = \partial K$ or $G^0 = \phi$. Consequently, $A(K)_{|\partial K}$ is weakly analytic in the sense of Definition 2.2.

Conversly, assume that the function space $A(K)_{|\partial K}$ is weakly analytic. Let $K = co(G \cup H)$, with G a closed split face of K and H a closed convex subset of K. Then ∂G is facially closed and so it is a generalized peak set for $N(A(K)_{|\partial K})$, by Corollary 4.4. Also, $H \cap \partial K$ is closed in ∂K. Now, we shall show that $\partial G \cup (H \cap \partial K) = \partial K$.

Since $\partial G = G \cap \partial K$, it is clear that $\partial G \cup (H \cap \partial K) \subset \partial K$. Let $x \in \partial K$. If $x \notin \partial G$, then $x \notin G$. As $K = co(G \cup H)$, $x = \Sigma_i r_i x_i$ with $x_i \in (G \cup H)$. Since $x \in \partial K$, $x = x_i$ for all i. But, if $x \notin G$, then $x_i \in H$ for each i and so, $x \in H$. Thus, if $x \notin \partial G$, then $x \in H \cap \partial K$, i.e. $\partial K \subset \partial G \cup (H \cap \partial K)$. Hence $\partial K = \partial G \cup (H \cap \partial K)$.

Thus, we have $\partial K = \partial G \cup (H \cap \partial K)$, where ∂G is a generalized peak set for $N(A(K)_{|\partial K})$ and $H \cap \partial K$ is closed in ∂K. Since $A(K)_{|\partial K}$ is weakly analytic, either $\partial G = \partial K$ or $H \cap \partial K = \partial K$. If $\partial G = \partial K$, then $G = K$. If $H \cap \partial K = \partial K$, then $\partial K \subset H$ and so, $K \subset H$ as H is convex, i.e. $K = H$. Thus, either $G = K$ or $H = K$. Hence $A(K)$ is weakly analytic in the sense of Definition 4.1.

We do not know, how $\mathscr{W}_{\alpha c}(A(K))$ and $\mathscr{W}_{\alpha}(A(K)_{|\partial K})$ are related in general.

Examples 4.6. (i). Let K be a triangle in $\mathbb{R}^2$ with vertices $\{a, b, c\}$. Then, it is clear that $\partial K = \{a, b, c\}$ and each vertex is a closed split face of K. Let $E = \{a, b\}$. Then $F = coE$ is a closed face of K. Also, $F = co(G \cup H)$, where $G = \{a\}$ and $H = \{b\}$ are closed split faces of F. Thus, a set containing more than one point cannot be a weakly analytic set for $A(K)$, according to Definition 4.1. Hence we get, $\mathscr{W}_{\alpha c}(A(K)) = \{\{a\}, \{b\}, \{c\}\}$. Further, since ∂K is a Bauer simplex, $A(K)_{|\partial K} = C(\partial K)$. So, one can easily check that $\mathscr{W}_{\alpha}(A(K)_{|\partial K}) = \{\{a\}, \{b\}, \{c\}\}$. (ii) Let $G = \{(x, 0) \in l^1 x \mathbb{R} : x \geqslant 0, \ ||x|| \leqslant 1\}$, $F = \{(0, t) \in l^1 x \mathbb{R} : |t| \leqslant 1\}$ and let $K = co(F \cup G)$. Then $\partial K = (\partial G \cup \partial F) - \{(0,0)\}$. It is shown in [3, p. 219] that the Bishop decomposition for $A(K)$ consists of all singleton sets. Since the decomposition of weakly prime sets is finer than the Bishop decomposition [4] and the decomposition of weakly analytic sets is finer than the decomposition of weakly prime sets, $\mathscr{W}_{\alpha c}(A(K))$ consists of all singleton sets. (iii) Let K be a square in $\mathbb{R}^2$, with vertices $\{a, b, c, d\}$. Then $\partial K = \{a, b, c, d\}$. Also, it can be checked that K has no closed split face. So, ∂K is trivially a weakly analytic set for $A(K)$. Thus, $\mathscr{W}_{\alpha c}(A(K)) = \{\partial K\}$. Hence by Proposition 4.5, $\mathscr{W}_{\alpha}(A(K)_{|\partial K}) = \{\partial K\}$.

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Failure of Palais–Smale condition and blow-up analysis for the critical exponent problem in $\mathbb{R}^2$

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Abstract. Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth non-linearity behaving like $\exp\{s^2\}$ as $s\to\infty$. Let F denote the primitive of f. Consider the functional $J:H_0^1(\Omega)\to\mathbb{R}$ given by

$$J(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,dx-\int_\Omega F(u)\,dx.$$

It can be shown that J is the energy functional associated to the following nonlinear problem:

$$-\Delta u=f(u) \quad \text{in} \quad \Omega,$$
$$u=0 \qquad \text{on} \quad \partial\Omega.$$

In this paper we consider the global compactness properties of J. We prove that J fails to satisfy the Palais–Smale condition at the energy levels $\{k/2\}$, k any positive integer. More interestingly, we show that J fails to satisfy the Palais–Smale condition at these energy levels along two Palais–Smale sequences. These two sequences exhibit different blow-up behaviours. This is in sharp contrast to the situation in higher dimensions where there is essentially one Palais–Smale sequence for the corresponding energy functional.

Keywords. Blow-up analysis; critical exponent problem in $\mathbb{R}^2$; Moser functions; Palais–Smale sequence; Palais–Smale condition.

1. Introduction

1.1 *Preliminaries*

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n\geqslant 2$. For $n\geqslant 3$, let $\mathscr{A}_n$ denote the subset of $C^1(\bar{\mathbb{R}}_+,\bar{\mathbb{R}}_+)$ consisting of functions $g(s)$ which satisfy the following growth conditions:

$$\lim_{s\to\infty} g(s)\,s^{((n+2)/(n-2))}=\infty,$$

$$\lim_{s\to\infty} g(s)\,s^{-((n+2)/(n-2))}=0.$$

When $n=2$, let $\mathscr{B}$ denote the subset of $C^1(\bar{\mathbb{R}}_+,\bar{\mathbb{R}}_+)$ consisting of functions $h(s)$ which vanish only at $s=0$ and which satisfy the following growth conditions:

For every $\delta>0$,

$$\lim_{s\to\infty} h(s)\exp\{\delta s^2\}=\infty,$$

$$\lim_{s\to\infty} h(s)\exp\{-\delta s^2\}=0.$$

In view of the Sobolev and the Trudinger [22] imbeddings, we call f a function of critical growth in $\mathbb{R}^n$, $n \geqslant 2$, if

$$f(s) = \begin{cases} s^{((n+2)/(n-2))} + g(s), & g \in \mathscr{A}_n \quad \text{if} \quad n \geqslant 3, \\ h(s) \exp\{4\pi s^2\}, & h \in \mathscr{B} \quad \text{if} \quad n = 2. \end{cases}$$

For questions related to the achievement of the best constant in the Trudinger imbedding, see [11, 16, 13, 15].

We now consider the following non-linear problem with f having critical growth in $\mathbb{R}^n$, $n \geqslant 2$:

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{P}$$

Let F denote the primitive of f. Then, the energy functional $J: H_0^1(\Omega) \to \mathbb{R}$ associated to the problem (P) is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(u)\, dx. \tag{1.1}$$

It is not difficult to show that every solution of the non-linear problem (P) is a critical point of J and vice-versa.

1.2 *A brief review of past developments*

Since problem (P) lacks compactness, J will not satisfy Palais–Smale condition at all energy levels. It is of great interest to characterize the energy levels at which J fails to satisfy the Palais–Smale condition. We explain briefly why this characterization is important. Using Rellich's identity [8], Pohazaev [19] was able to show that when $n \geqslant 3$, (P) does not admit a solution if $f(s) = s^{(n+2)/(n-2)}$ and Ω is star-shaped with respect to anyone of its points. In order to reverse this non-existence and obtain some existence results for the problem (P) there are therefore two simple prescriptions:

(i) add a perturbation $g(s)$ from $\mathscr{A}_n$ to the critical non-linearity $s^{(n+2)/(n-2)}$, or,
(ii) pose the problem (P) on a domain Ω with 'rich topology'.

When $n \geqslant 3$, the first program was carried out in Brezis–Nirenberg [10] where they prove existence results for (P) provided the perturbation g is large enough. They also show that if the perturbation is small, the Pohazaev-type non-existence persists. In order to carry out the second program, it is necessary to know precisely the energy levels at which J fails to satisfy the Palais–Smale condition. Such a characterization was obtained by Struwe [21] who showed that Palais–Smale fails only at a discrete (but infinite) set of equally spaced energy levels. Using this characterization and tools from algebraic topology, Bahri–Coron [7] were able to prove that if $n \geqslant 3$ and at least one of the homology groups of Ω (with coefficients in $\mathbb{Z}_2$) is non-trivial, (P) admits a solution even when $f(s) = s^{(n+2)/(n-2)}$. These developments underline the fact that one can exploit the topology of Ω to prove existence results to problem (P) once the energy levels at which Palais–Smale condition fails are known.

In the case $n = 2$ corresponding questions have been considered in [1–5], but the picture remains far from complete. One of the difficulties is that an effective Pohazaev-type identity does not seem to exist when $n = 2$. Hence non-existence results become

arder to obtain for general star-shaped domains. Nevertheless, in the radial situation, on-existence results for a large class of convex non-linearities have been obtained in 4] and [5]. We briefly describe these results. The critical non-linearity when $n=2$ is $(s)=h(s)\exp\{4\pi s^2\}$ with $h\in\mathscr{B}$. In analogy with the results for $n\geqslant 3$, one expects on-existence if the perturbation h is 'small'. This is indeed so if h is 'small' in the sense hat

$$\limsup_{s\to\infty} h(s)s<\infty.$$

We refer to [4] and [5] for a precise statement of these non-existence results. Therefore, o reverse this non-existence either one can consider large perturbations h, or one can ose the problem on a domain Ω in $\mathbb{R}^2$ with 'holes'. The former option was considered y Adimurthi in [1] where he proved that if the perturbation is 'large', more precisely, if im $\sup_{s\to\infty} h(s)s=\infty$, then (P) admits a solution on any bounded smooth domain Ω in $\mathbb{R}^2$ (also see remark 1.1 below). Further, he showed in the same paper that Palais–Smale olds in the infinite energy range $(-\infty,\frac{1}{2})$. As we saw earlier for the case $n\geqslant 3$, we need o characterize the energy levels at which J fails to satisfy the Palais–Smale condition if ve want to consider the latter option of exploiting the topology of Ω for proving xistence results.

.3 *Contents of the paper*

This paper deals with the two dimensional case. In §2 and §3 we exhibit two Palais–Smale sequences along which J fails to satisfy the Palais–Smale condition at the energy levels $k/2$, k any positive integer. The first Palais–Smale sequence (§2) is the sequence of 'Moser functions' whereas the second sequence (§3) consists of solutions to the problem (ref. [1]),

$$\begin{aligned} -\Delta u &= h(u)\exp\{4\pi u^2\} && \text{in } B(R),\\ u&>0 && \text{in } B(R),\\ u&=0 && \text{on } \partial B(R), \end{aligned}$$

where $B(R)\subset\mathbb{R}^2$ denotes the disc of radius R centered at the origin and $h(s)s\to\infty$ as $s\to\infty$.

A very surprising consequence that comes out of the analysis here is that these two Palais–Smale sequences exhibit different blow-up behaviours. This is in sharp contrast to the situation in higher dimensions where there is essentially only one type of Palais–Smale sequence for J (ref. [21]). The blow-up limit of the first Palais–Smale sequence solves the following equation in $\mathbb{R}^2$, for some compactly supported measure μ:

$$-\Delta u=\mu.$$

The blow-up limit of the second Palais–Smale sequence solves the equation:

$$-\Delta u=\frac{1}{2\pi}\exp\{u\} \quad \text{in} \quad \mathbb{R}^2.$$

Here we recall that any solution u of the above equation corresponds to a conformal change of metric on $\mathbb{R}^2$ from the standard metric to a metric of constant Gaussian curvature $K=1$. Results on these blow-ups are contained in §4(1) and §4(2). We

strongly believe that the existence of two distinct types of Palais–Smale sequences has important implications regarding the global compactness properties of J.

For generalizations of theorems A and B to the case of the n-Laplacian in $\mathbb{R}^n$, see §5.

Statement of the theorems. For convenience, we list the statements of all the four theorems we will prove:

Let $0 < l < R$. Define the Moser functions

$$m_{l,R}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} [\log(R/l)]^{1/2} & 0 \leqslant |x| < l, \\ \dfrac{\log(R/|x|)}{[\log(R/l)]^{1/2}} & l \leqslant |x| \leqslant R, \\ 0 & |x| \geqslant R. \end{cases} \tag{1.2}$$

Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Let $x_0 \in \Omega$ and $R < d(x_0, \partial\Omega)$. Define $m_{l,R,x_0}(x) = m_{l,R}(x - x_0)$.

Theorem A. *Let $f(s) = h(s)\exp\{4\pi s^2\}$, $s \geqslant 0$, $h \in \mathscr{B}$. Assume that h satisfies the hypothesis:*

$$|h'(t)| \leqslant ch(t)t^{1-\eta} \quad \text{and} \quad F(t) \leqslant cf(t)t^{1-\eta}$$

for some $\eta \in (0, 1]$, for some $c > 0$ and all large t.

Let R depend on l so that $\lim_{l\to 0}(\log(2/R)/\log(R/l)) = 0$. Then there exists a sequence $\rho_l \to 1$ as $l \to 0$ such that

(a) $\{\rho_l m_{l,R,x_0}\}_{l>0}$ *is a Palais–Smale sequence for J as $l \to 0$,*
(b) *J fails to satisfy the Palais–Smale condition along the above sequence at the energy levels $k/2$, k any positive integer.*

Remark 1.1. Examples of non-linearities satisfying the hypotheses in theorem A are: any $h \in \mathscr{B}$ such that $h(s) = \exp\{\pm s^\beta\}$, for all large s and some $\beta \in (0, 2)$; any $h \in \mathscr{B}$ such that $h(s) = s^\theta$ for all large s and some $\theta \in \mathbb{R}$.

Let u_R denote the solution obtained in [1] for the problem (1.1) posed on $B(x_0, R)$. Let $\tilde{u}_R$ denote the extension of u_R obtained by setting $\tilde{u}_R \equiv 0$ in $\Omega \setminus B(x_0, R)$.

Theorem B. *Let $x_0 \in \Omega$ and $R > 0$ be small enough so that $B(x_0, R) = \{x : |x - x_0| < R\} \subset \Omega$. Let $h \in \mathscr{B}$ and $f(s) = h(s)\exp\{4\pi s^2\}$, $s \geqslant 0$, satisfy the following hypotheses: There exists $s_0 > 0$ such that*

(i) $h(s)s \to \infty$ *as* $s \to \infty$,
(ii) $g(s) = \log f(s)$ *is C^3 and convex for all* $s \geqslant s_0$,
(iii) $f(s)$ *is strictly increasing for all* $s \geqslant s_0$,
(iv) $g(s) - \frac{1}{2}sg'(s) + \log(g'(s)/2) \geqslant (1 + \theta)\log s$ *for all* $s \geqslant s_0$ *and some* $\theta > -1$,
(v) $\limsup_{s\to\infty}(g(s) - \frac{1}{2}sg'(s))s^{-\beta} = O(1)$ *for some* $\beta \in [0, 2)$,
(vi) $\overline{\lim}_{s\to\infty}(sg^{(k+1)}(s)/g^{(k)}(s))$ $k = 0, 1$ *exists and is different from* 0 *(here $g^{(k)}$ denotes the k-th derivative of g).*

Then,

(a) $\{\tilde{u}_R\}_{R>0}$ *as $R \to 0$ are Palais–Smale sequences for J,*
(b) *J does not satisfy Palais-Smale condition at the energy levels $k/2$, $k = 1, 2, 3, \ldots$.*

Remark 1.2. Let $g(s) = \log f(s)$. Then the following are simple consequences of the assumptions (i)–(vi):

$$g'(s) = 8\pi s + O\left(\frac{1}{s}\right),$$

$$g''(s) = 8\pi + O\left(\frac{1}{s^2}\right),$$

$$g'''(s) = O\left(\frac{1}{s^3}\right).$$

Hence we may choose s_0 large so that g is a convex increasing C^3 function in $[s_0, \infty)$.

Remark 1.3. Examples of non-linearities satisfying the hypotheses in theorem D are: any $h \in \mathscr{B}$ such that $h(s) = \exp\{s^\beta\}$, $0 < \beta < 2$, for all large s; any $h \in \mathscr{B}$ such that $h(s) = s^\theta$ for all large s and some $\theta > -1$.

Theorem C. *Let $\rho \in (0, 1]$ be a parameter depending continuously on l such that $\rho - l \neq 0$ for all $l > 0$ and $\rho(0) = 0$. We also assume that the following limits (possibly ∞) exist:*

$$a = \lim_{l \to 0} \frac{l}{\rho},$$

$$b = \lim_{l \to 0} \frac{\log \rho}{\log l}.$$

It follows that

$$\log a = \lim_{l \to 0} \log l \left(1 - \frac{\log \rho}{\log l}\right). \tag{1.3}$$

Extending each $m_{l,1}$ by zero outside $B(1)$ we may consider $m_{l,1}$ to be members of $H^1(\mathbb{R}^2)$. Let $y \in \partial B(1)$ be arbitrary and $x \in \mathbb{R}^2$. Then,

(i) *If $a = \infty$, uniformly on compact subsets of $\mathbb{R}^2$,*

$$\lim_{l \to 0} m_{l,1}(\rho x)^2 - m_{l,1}(\rho y)^2 = 0.$$

(ii) *If $a < \infty$ and $\rho \leqslant l$ for all $l > 0$, uniformly on compact subsets of $\mathbb{R}^2$,*

$$\lim_{l \to 0} \{m_{l,1}(\rho x)^2 - m_{l,1}(\rho y)^2\} = \begin{cases} 0 & 0 \leqslant |x| \leqslant a, \\ \dfrac{b}{\pi} \log\left(\dfrac{a}{|x|}\right) & |x| > a. \end{cases}$$

(iii) *If $a > 0$ and $\rho > l$ for all $l > 0$, uniformly on compact subsets of $\mathbb{R}^2$,*

$$\lim_{l \to 0} \{m_{l,1}(\rho x)^2 - m_{l,1}(\rho y)^2\} = \begin{cases} 2\log\left(\dfrac{1}{a}\right) & |x| \leqslant a, \\ 2\log\left(\dfrac{1}{|x|}\right) & |x| > a. \end{cases}$$

(iv) *If $a = 0$, for $x \neq 0$, uniformly on compact subsets of $\mathbb{R}^2 \setminus \{0\}$,*

$$\lim_{l \to 0} \{m_{l,1}(\rho x)^2 - m_{l,1}(\rho y)^2\} = \frac{b}{\pi} \log\left(\frac{1}{|x|}\right),$$

Theorem D. *Let $h \in \mathcal{B}$ and $f(s) = h(s)\exp\{4\pi s^2\}$, $s \geq 0$, satisfy the hypotheses listed in the statement of theorem B. Let u_R be any solution of* (P) *posed on $\Omega = B(R)$ with the above choice of f. Then,*

(i) $\|\nabla u_R\|_{L^2(B(R))} \to 1$ *as* $R \to 0$,

(ii) *There exists a parameter ρ depending continuously on R, $\rho \to 0$ as $R \to 0$, such that for any $z \in \partial B(1)$ and $x \in \mathbb{R}^2$,*

$$u_R^2(\rho x) - u_R^2\left(\frac{\rho z}{2}\right) \to 2\log\left(\frac{2}{1+|x|^2}\right)$$

as $\rho \to 0$, uniformly on compact subsets of $\mathbb{R}^2$.

Remark 1.4. Statement (ii) of the theorem implies that the 'limiting equation' to our problem is

$$-\Delta u = \frac{1}{2\pi}\exp\{u\} \quad \text{in } \mathbb{R}^2,$$

$$u = 0 \qquad\qquad \text{on } \partial B(1).$$

Remark 1.5. (1) In fact, for existence of solution to (P) when $n = 2$, it is sufficient if $\limsup_{s\to\infty} h(s)s > c$, where c is a positive constant depending only on Ω and f; refer to remark 4.2 in [1].

(2) In [12], a partial result on non-existence in the case $n = 2$ is obtained, which however, happens to be a special case of theorem B in [5].

2. Proof of Theorem A: Palais–Smale fails along 'Moser functions'

We first prove a few preliminary lemmas from which the theorem will follow readily. In what follows we assume without loss of generality that $0 \in \Omega$, $x_0 = 0$ and $B(0,3) \subset \Omega$. also we equip $H_0^1(\Omega)$ with gradient norm $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$.

Lemma 2.1. *Let $u \in H_0^1(B(0,R_1)) \subset H_0^1(\Omega)$ be a function radial about the origin. Then there exists a positive constant ω depending only on Ω such that for $R_0 \in (R_1, d(0,\partial\Omega))$,*

$$\|J'(u)\|^2 \leq \frac{\omega}{(R_0 - R_1)^2}\left(\int_0^{R_0}\left|\frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u)t\,dt\right|^2 r\,dr\right). \tag{2.1}$$

Proof. For $\phi \in H_0^1(\Omega)$, define the θ-average $\tilde{\phi}$ of ϕ to be

$$\tilde{\phi}(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi(r\cos\theta,\ r\sin\theta)\,d\theta.$$

Then,

$$\langle J'(u), \phi\rangle = \int_\Omega \nabla u\cdot\nabla\phi\,dx - \int_\Omega f(u)\phi\,dx$$

$$= 2\pi\left(\int_0^{R_1}\frac{\partial u}{\partial r}\frac{\partial\tilde{\phi}}{\partial r}r\,dr - \int_0^{R_1} f(u)\,\tilde{\phi}\,r\,dr\right).$$

Let $\eta \in C_0^\infty(B(0,R_0))$ be such that η is radial about the origin, $\eta \equiv 1$ on $B(0,R_1)$,

$|\partial\eta/\partial r| \leqslant 4/(R_0 - R_1)$. Let $\phi_1 = \eta\,\tilde{\phi}$. Then,

$$-\int_0^{R_1} f(u)\,\tilde{\phi}\,r\,\mathrm{d}r = -\int_0^{R_0} f(u)\,\phi_1 r\,\mathrm{d}r$$

$$= \int_0^{R_0} f(u)\left(\int_r^{R_0} \frac{\partial\phi_1}{\partial s}\,\mathrm{d}s\right) r\,\mathrm{d}r$$

$$= \int_0^{R_0} \left(\frac{1}{r}\int_0^r f(u)\,t\,\mathrm{d}t\right)\frac{\partial\phi_1}{\partial r}\,r\,\mathrm{d}r.$$

Therefore,

$$|\langle J'(u), \phi\rangle|^2 = 2\pi\left|\int_0^{R_0}\left(\frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u)\,t\,\mathrm{d}t\right)\frac{\partial\phi_1}{\partial r}\,r\,\mathrm{d}r\right|^2$$

$$\leqslant 2\pi\left(\int_0^{R_0}\left|\frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u)\,t\,\mathrm{d}t\right|^2 r\,\mathrm{d}r\right)\left(\int_0^{R_0}\left|\frac{\partial\phi_1}{\partial r}\right|^2 r\,\mathrm{d}r\right).$$

Since $\partial\phi_1/\partial r = \eta\,(\partial\tilde{\phi}/\partial r) + (\partial\eta/\partial r)\,\tilde{\phi}$, we get, for some positive constants, ω_1, ω depending only on Ω,

$$\int_0^{R_0}\left(\frac{\partial\phi_1}{\partial r}\right)^2 r\,\mathrm{d}r \leqslant \omega_1\left\{\int_\Omega |\nabla\phi|^2\,\mathrm{d}x + \frac{1}{(R_0 - R_1)^2}\int_\Omega |\phi|^2\,\mathrm{d}x\right\}$$

$$\leqslant \frac{\omega}{(R_0 - R_1)^2}\int_\Omega |\nabla\phi|^2\,\mathrm{d}x.$$

Therefore, the last two inequalities imply that

$$\|J'(u)\|^2 \leqslant \frac{\omega}{(R_0 - R_1)^2}\int_0^{R_0}\left|\frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u)\,t\,\mathrm{d}t\right|^2 r\,\mathrm{d}r.$$

This proves the lemma. ∎

For simplicity of notation, we let $m_{l,R,0}(x) = m_{l,R}(x)$. For $\rho \in \mathbb{R}$, using the facts $\|m_{l,R}\| = 1$ and f has exponential growth, we have

$$J(\rho m_{l,R}) = \frac{\rho^2}{2} - \int_\Omega F(\rho m_{l,R})\,\mathrm{d}x$$

$$\to -\infty \text{ as } |\rho| \to \infty.$$

Hence, we can find $\rho_{l,R} \geqslant 0$ such that

$$J(\rho_{l,R} m_{l,R}) = \sup_{\rho\in\mathbb{R}} J(\rho m_{l,R}).$$

Therefore, the derivative of the function $\rho \mapsto J(\rho m_{l,R})$ vanishes at $\rho = \rho_{l,R}$ which gives, denoting $u_{l,R} = \rho_{l,R} m_{l,R}$,

$$\|\nabla u_{l,R}\|^2_{L^2(\Omega)} = \int_\Omega f(u_{l,R})\,u_{l,R}\,\mathrm{d}x. \tag{2.2}$$

That is, since $\|\nabla u_{l,R}\|_{L^2(\Omega)} = \rho_{l,R}$,

$$\rho_{l,R}^2 = \int_\Omega f(u_{l,R})\, u_{l,R}\, dx. \tag{2.3}$$

We note that the integrands in equations (2.2), (2.3) and (2.4) are radial functions supported in $B(0,2)$. Therefore, we can transform the integrals in these equations to integrals on $(0,\infty)$ via the transformation

$$|x| = r = 2\exp\{-t/2\}. \tag{2.4}$$

Let $\tau(t) = 2\exp\{-t/2\}$. For any $u \in H_0^1(\Omega)$, u radial about the origin denote $v = u \circ \tau$. Let $w_{l,R} = m_{l,R} \circ \tau$ and $v_{l,R} = \rho_{l,R} w_{l,R}$. Then it can be checked that

$$w_{l,R}(t) = \frac{1}{\sqrt{4\pi}} \begin{cases} 0 & 0 \leqslant t < t_R, \\ (t - t_R)(t_l - t_R)^{-1/2} & t_R \leqslant t < t_l, \\ (t_l - t_R)^{1/2} & t \geqslant t_l, \end{cases} \tag{2.5}$$

where $t_R = 2\log(2/R)$ and $t_l = 2\log(2/l)$.

Now, under the transformation (2.5), the equations (2.2), (2.3) and (2.4) will become respectively

$$\|J'(u)\|^2 \leqslant \frac{4\pi\omega}{R_0 - R_1} \left\| \frac{dv}{dt} - \int_t^\infty f(v)\exp\{-s\}\, ds \right\|^2_{L^2(0,\infty)}, \tag{2.6}$$

$$\int_0^\infty \left| \frac{dv_{l,R}}{dt} \right|^2 dt = \int_0^\infty f(v_{l,R})\, v_{l,R} \exp\{-t\}\, dt, \tag{2.7}$$

and

$$\frac{\rho_{l,R}^2}{4\pi} = \int_0^\infty f(v_{l,R})\, v_{l,R} \exp\{-t\}\, dt. \tag{2.8}$$

Also, if $u \in H_0^1(\Omega)$ is radial about the origin, $J(u)$ is transformed under (2.5) as

$$J(u) = 2\pi \int_0^\infty \left| \frac{dv}{dt} \right|^2 dt - 4\pi \int_0^\infty F(v)\exp\{-t\}\, dt. \tag{2.9}$$

Let $\{l_n\}$ and $\{R_n\}$ be two sequences of positive numbers with $l_n \leqslant R_n$ for all n,

$$\lim_{n\to\infty} l_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log(2/R_n)}{\log(R_n/l_n)} = 0.$$

By abuse of notation, let $u_n = u_{l_n,R_n}$, $v_n = v_{l_n,R_n}$, $m_n = m_{l_n,R_n}$, $t_n = t_{l_n}$, $\bar{t}_n = t_{R_n}$, $\rho_n = \rho_{l_n R_n}$. We prove that $\{u_n\}$ is a Palais–Smale sequence for J, and also that J does not satisfy the Palais-Smale condition at the energy level $\frac{1}{2}$. This we do by showing that

$$J(u_n) \to \tfrac{1}{2} \quad \text{as } n \to \infty \quad \text{and,}$$

$$\|J'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

Hereafter, we work only with the transformed equations (2.6)–(2.10)

Lemma 2.2.

$$\rho_n \to 1 \quad \text{as } n \to \infty.$$

Proof. We prove the lemma by showing

$$\limsup \rho_n \leqslant 1 \quad \text{and} \quad \liminf \rho_n \geqslant 1.$$

$\limsup \rho_n \leqslant 1$: We argue by contradiction. Suppose there exists a subsequence of ρ_n, which we again denote by ρ_n, and $\varepsilon > 0$, such that $\rho_n^2 \geqslant 1 + \varepsilon$ for all n. Then (2.8) gives, denoting $s_n = (\sqrt{4\pi})^{-1} (t_n - \bar{t}_n)^{1/2}$,

$$\begin{aligned}
\frac{\rho_n^2}{4\pi} &\geqslant \int_{t_n}^{\infty} f(v_n)\, v_n \exp\{-t\}\, dt \\
&= \int_{t_n}^{\infty} h(v_n)\, v_n \exp\{4\pi v_n^2 - t\}\, dt \\
&= \int_{t_n}^{\infty} h(\rho_n s_n)\, \rho_n s_n \exp\{\rho_n^2 (t_n - \bar{t}_n) - t\}\, dt \\
&= h(\rho_n s_n)\, \rho_n s_n \exp\{(\rho_n^2 - 1)\, t_n - \rho_n^2 \bar{t}_n\}. \qquad (2.10)
\end{aligned}$$

By the growth assumption on h, for any $\delta \in (0, 1)$ we may find $c_\delta > 0$ such that for all large t,

$$h(t)\, t^{-1} \geqslant c_\delta \exp\{-4\pi\delta t^2\}.$$

Therefore, the inequality (2.10) gives, for all large n,

$$\begin{aligned}
\frac{1}{4\pi} &\geqslant h(\rho_n s_n)(\rho_n s_n)^{-1} s_n^2 \exp\{(\rho_n^2 - 1)\, t_n - \rho_n^2 \bar{t}_n\} \\
&\geqslant c_\delta s_n^2 \exp\{-\delta \rho_n^2 (t_n - \bar{t}_n) + (\rho_n^2 - 1) t_n - \rho_n^2 \bar{t}_n\} \\
&= c_\delta s_n^2 \exp\{\rho_n^2 (1 - \delta)(t_n - \bar{t}_n) - t_n\}. \qquad (2.11)
\end{aligned}$$

Now, the assumption

$$\lim_{n \to \infty} \frac{\log(2/R_n)}{\log(R_n/l_n)} = 0$$

implies that

$$\lim_{n \to \infty} \frac{\bar{t}_n}{t_n - \bar{t}_n} = 0$$

and hence in particular that $\bar{t}_n = o(t_n)$. Therefore we may choose positive numbers δ, ν and η small enough so that $\bar{t}_n \leqslant \eta t_n$ for all large n and $(1 + \varepsilon)(1 - \delta)(1 - \eta) > 1 + \nu$. Therefore, the inequality (2.11) together with the assumption $\rho_n^2 \geqslant 1 + \varepsilon$ for all n implies that for all large n,

$$\frac{1}{4\pi} \geqslant c_\delta s_n^2 \exp\{\nu t_n\}$$

which is a contradiction. This contradiction shows that $\limsup_n \rho_n^2 \leqslant 1$.

$\liminf \rho_n \geq 1$: Again, suppose there is a subsequence of ρ_n, denoted still by ρ_n, and $\varepsilon > 0$ such that $\rho_n^2 \leq 1 - \varepsilon$ for all n. From (2.8),

$$\begin{aligned}\frac{1}{4\pi} &= \int_0^{t_n} \rho_n^{-2} f(v_n) v_n \exp\{-t\}\,dt + \int_{t_n}^{\infty} \rho_n^{-2} f(v_n) v_n \exp\{-t\}\,dt \\ &= I_1^n + I_2^n. \end{aligned} \tag{2.12}$$

We have,

$$\begin{aligned} I_1^n &= \int_{\bar{t}_n}^{t_n} \rho_n^{-2} h(v_n) v_n \exp\left\{\rho_n^2 \frac{(t-\bar{t}_n)^2}{(t_n-\bar{t}_n)} - t\right\} dt \\ &= \frac{1}{4\pi(t_n-\bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t-\bar{t}_n)^2 \exp\left\{\rho_n^2 \frac{(t-\bar{t}_n)^2}{(t_n-\bar{t}_n)} - t\right\} dt \\ &\leq \frac{1}{4\pi(t_n-\bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t-\bar{t}_n)^2 \exp\{(\rho_n^2-1)t - \rho_n^2 \bar{t}_n\}\,dt \\ &\leq \frac{1}{4\pi(t_n-\bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t-\bar{t}_n)^2 \exp\{-\varepsilon t - \rho_n^2 \bar{t}_n\}\,dt. \end{aligned}$$

By the growth assumption on h and the assumption $h(0) = 0$, there exists for any $\delta > 0$ a positive constant $C(\delta)$ such that,

$$h(t) t^{-1} \leq C(\delta) \exp\{4\pi\delta t^2\} \quad \text{for all } t \geq 0.$$

We now choose $\delta < \min\{\varepsilon, (\varepsilon(1-\varepsilon)^{-1})/2\}$. For all large n, since $\delta < \varepsilon$,

$$\begin{aligned} &\frac{1}{4\pi(t_n-\bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t-\bar{t}_n)^2 \exp\{-\varepsilon t - \rho_n^2 \bar{t}_n\}\,dt \\ &\qquad \leq \frac{C(\delta)}{4\pi(t_n-\bar{t}_n)} \int_{\bar{t}_n}^{t_n} (t-\bar{t}_n)^2 \exp\{\delta\rho_n^2(t-\bar{t}_n) - \varepsilon t - \rho_n^2 \bar{t}_n\}\,dt \\ &\qquad \leq \frac{C(\delta)}{4\pi(t_n-\bar{t}_n)} \int_{\bar{t}_n}^{t_n} (t-\bar{t}_n)^2 \exp\{(\delta-\varepsilon)t\}\,dt \\ &\qquad = O(t_n^{-1}). \end{aligned}$$

Hence, $I_1^n = O(t_n^{-1})$. Similarly, for all n large

$$\begin{aligned} I_2^n &= \frac{1}{4\pi} \int_{t_n}^{\infty} h(v_n) v_n^{-1} (t_n - \bar{t}_n) \exp\{\rho_n^2(t_n-\bar{t}_n) - t\}\,dt \\ &\leq \frac{C(\delta)}{4\pi} (t_n-\bar{t}_n) \exp\{(1+\delta)\rho_n^2 t_n\} \int_{t_n}^{\infty} \exp\{-t\}\,dt \\ &= \frac{C(\delta)}{4\pi} (t_n-\bar{t}_n) \exp\{((1+\delta)\rho_n^2 - 1)t_n\}. \end{aligned}$$

Since $\delta < (\varepsilon(1-\varepsilon)^{-1})/2$, the above inequality gives that $I_2^n = O(\exp\{-(\varepsilon/2)t_n\})$. Hence $I_1^n + I_2^n = O(t_n^{-1})$ which contradicts (2.12). This contradiction shows that $\liminf \rho_n \geq 1$ and this proves the lemma. ■

Let $\bar{v}_n = v_n(t_n)$. From (2.8) one gets

$$\frac{\rho_n^2}{4\pi} = \int_0^{t_n} f(v_n) v_n \exp\{-t\}\,dt + f(\bar{v}_n)\bar{v}_n \exp\{-t_n\}. \tag{2.13}$$

In the following lemma we compute the limit, as $n \to \infty$, of the two terms on the right of equation (2.13). These limits will enable us to show that $||J'(u_n)|| \to 0$ as $n \to \infty$.

Lemma 2.3.

$$\lim_{n\to\infty} \int_0^{t_n} f(v_n) v_n \exp\{-t\}\,dt = \frac{1}{8\pi},$$

$$\lim_{n\to\infty} f(\bar{v}_n)\bar{v}_n \exp\{-t_n\} = \frac{1}{8\pi}.$$

Since the proof is technical, we prove Lemma 2.3 in the appendix (§8).

Lemma 2.4.

$$J(u_n) \to \tfrac{1}{2} \quad as \quad n \to \infty,$$

Proof. Since

$$2\pi \int_0^{\infty} \left|\frac{dv_n}{dt}\right| dt = \frac{\rho_n^2}{2} \to \frac{1}{2} \quad \text{as} \quad n \to \infty,$$

(2.9) implies that the lemma is proved if we show that

$$\int_0^{\infty} F(v_n) \exp\{-t\}\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

Let N be a large fixed positive number. Noting that $F(t) \leqslant Mf(t)t^{1-\eta}$ for some M positive, $\eta \in (0, 1]$ and all large t, we have

$$\begin{aligned}
&\int_0^{\infty} F(v_n) \exp\{-t\}\,dt \\
&\quad = \int_{\{v_n \leqslant N\}} F(v_n) \exp\{-t\}\,dt + \int_{\{v_n > N\}} F(v_n) \exp\{-t\}\,dt \\
&\quad \leqslant \int_{\{v_n \leqslant N\}} F(v_n) \exp\{-t\}\,dt + M \int_{\{v_n > N\}} f(v_n) v_n^{1-\eta} \exp\{-t\}\,dt.
\end{aligned}$$

Using the dominated convergence theorem, the first integral on the right tends to zero as $n \to \infty$. Also, since $\int_0^{\infty} f(v_n) v_n \exp\{-t\}\,dt$ is bounded,

$$\begin{aligned}
\int_{\{v_n > N\}} f(v_n) v_n^{1-\eta} \exp\{-t\}\,dt &\leqslant \frac{1}{N^{\eta}} \int_{\{v_n > N\}} f(v_n) v_n \exp\{-t\}\,dt \\
&= O(N^{-\eta}).
\end{aligned}$$

Since N was an arbitrarily large positive number, it follows that

$$\int_0^{\infty} F(v_n) \exp\{-t\}\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

This proves the lemma. ∎

Lemma 2.5.

$$\|J'(u_n)\| \to 0 \quad as \quad n \to \infty.$$

Since the proof is again technical, we prove Lemma 2.5 in the appendix.

Proof of Theorem A. Lemmas 2.4 and 2.5 show that $\{u_n\}$ is a Palais–Smale sequence for J thereby yielding (a) of the theorem. Since $u_n \to 0$ in $H_0^1(\Omega)$ (up to a subsequence), and $J(u_n) \to \frac{1}{2}$, it follows that $\{u_n\}$ does not contain any subsequence that converges strongly in $H_0^1(\Omega)$. Hence J does not satisfy the Palais–Smale condition at the energy level $\frac{1}{2}$. By a standard procedure, for any positive integer k, we can construct a sequence of functions ξ_n, each of them being sum of k Moser functions with mutually disjoint supports, to get as $n \to \infty$, $J(\xi_n) \to k/2$ and

$$\|J'(\xi_n)\| \to 0.$$

This shows that J does not satisfy Palais–Smale condition at the energy levels $k/2$, k any positive integer. This proves (b) and hence the theorem. ■

3. Proof of theorem B: Palais–Smale fails along solutions to (P) on a disc

We prove a few preliminary lemmas from which the theorem will follow easily. But first we transform the problem to an ODE (ordinary differential equation). We are considering solutions u_R, obtained in [1] and [6], of the problem

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad B(R), \\ u &> 0 \quad \text{in} \quad B(R), \\ u &= 0 \quad \text{on} \quad \partial B(R). \end{aligned} \tag{P_R}$$

By regularity (see Remark 4.1 in [1]), u is in $C^{2,\alpha}(\overline{B(R)})$ for some $\alpha > 0$. We note that, by Gidas–Ni–Nirenberg [14], u is radial about the origin. Defining $y(t) = u(x)$ for any x such that $|x| = 2\exp\{-t/2\}$, we see that y solves the problem, with $\gamma = u(0)$,

$$\begin{aligned} -y'' &= \exp\{-t\} f(y), \\ y(\infty) &= \gamma > 0, \\ y'(\infty) &= 0. \end{aligned} \tag{P_γ}$$

Let $T_0(\gamma)$ denote the first zero of y. Let $g(\gamma) = \log f(\gamma)$.

Lemma 3.1.

$$y'(T_0(\gamma)) T_0^{1/2}(\gamma) \to 0 \quad as \quad \gamma \to \infty.$$

Proof. Let $T_2(\gamma)$ be the point at which $y(T_2(\gamma)) = s_0$ for all $\gamma > s_0$ (s_0 as in the hypotheses of theorem). We will require the following asymptotics:

$$T_0(\gamma) \geqslant \left(g(\gamma) - \frac{\gamma g'(\gamma)}{2}\right) + \log\left(\frac{g'(\gamma)}{2}\right) + O(1),$$

$$T_2(\gamma) = \frac{s_0 g'(\gamma)}{2} + \left(g(\gamma) - \frac{\gamma g'(\gamma)}{2}\right) + \log\left(\frac{g'(\gamma)}{2}\right) + O(1),$$

$$y'(T_2(\gamma)) = \frac{2}{g'(\gamma)}\left[1 + O\left(\frac{\log^2(\gamma)}{g(\gamma)}\right)\right].$$

(For the proof of the first asymptotic we refer to lemma 2 in [6]. The second and third asymptotics have been proved in lemma 4.4 in the next section.) Define $\delta(\gamma) = g(\gamma) - \frac{1}{2}\gamma g'(\gamma) + \log(\frac{1}{2}g'(\gamma))$. Let $\gamma \to \infty$ be a given sequence. To prove the lemma we distinguish two cases:

Case (i). *For some subsequence* $\gamma_k \to \infty$, $\delta(\gamma_k) \geqslant \log \gamma_k$: In this case $T_0(\gamma_k) \geqslant \log \gamma_k + O(1)$. Hence,

$$\begin{aligned} y'(T_0(\gamma_k)) &= y'(T_2(\gamma_k)) + \int_{T_0(\gamma_k)}^{T_2(\gamma_k)} f(y)\exp\{-t\}\,dt \\ &= O\left(\frac{1}{\gamma_k}\right) + O\left(\exp\{-T_0(\gamma_k)\}\right) \\ &= O\left(\frac{1}{\gamma_k}\right). \end{aligned}$$

Case (ii). $\delta(\gamma) \leqslant \log \gamma$ *for all large* γ: Define $T_3(\gamma) = 4\log\gamma$. Then,

$$\begin{aligned} y(T_3(\gamma)) &= y(T_2(\gamma)) - y'(T_2(\gamma))(T_2(\gamma) - T_3(\gamma)) + O\left(\exp\{-T_3(\gamma)\}\right) \\ &= s_0 - \frac{2}{g'(\gamma)}\left[1 + O\left(\frac{\log^2(\gamma)}{g(\gamma)}\right)\right]\left[\frac{s_0 g'(\gamma)}{2} + \delta(\gamma) - T_3(\gamma)\right] + O\left(\frac{1}{\gamma^4}\right) \\ &= O\left(\frac{\log\gamma}{\gamma}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} y'(T_3(\gamma)) &= y'(T_2(\gamma)) + \int_{T_3(\gamma)}^{T_2(\gamma)} f(y)\exp\{-t\}\,dt \\ &= O\left(\frac{1}{\gamma}\right) + O(\exp\{-T_3(\gamma)\}) \\ &= O\left(\frac{1}{\gamma}\right). \end{aligned}$$

Hence, with θ as in the hypothesis of the theorem,

$$\begin{aligned} y'(T_0(\gamma)) &= y'(T_3(\gamma)) + \int_{T_0(\gamma)}^{T_3(\gamma)} f(y)\exp\{-t\}\,dt \\ &= O\left(\frac{1}{\gamma}\right) + O\left(\int_{T_0(\gamma)}^{T_3(\gamma)} y(t)\exp\{-t\}\,dt\right. \\ &= O\left(\frac{1}{\gamma}\right) + O\left(\frac{\log\gamma\exp\{-T_0(\gamma)\}}{\gamma}\right) \\ &= O\left(\frac{1}{\gamma}\right) + O\left(\frac{\log\gamma\exp\{-(1+\theta)\log(\gamma)\}}{\gamma}\right) \\ &= O\left(\frac{1}{\gamma}\right). \end{aligned}$$

Thus in both the cases $y'(T_0(\gamma)) = O(1/\gamma)$. Hence, noting that $g(\gamma) - \frac{1}{2}\gamma g'(\gamma) = O(\gamma^\beta)$ for some $\beta \in [0, 2)$, and $T_0(\gamma) \leqslant T_2(\gamma)$ we get

$$y'(T_0(\gamma))\, T_0^{1/2}(\gamma) = O\left(\frac{1}{\gamma}[\max\{\gamma^{1/2}, \gamma^{\beta/2}\}]\right)$$
$$\to 0 \quad \text{as} \quad \gamma \to \infty.$$

This proves the lemma. ∎

Lemma 3.2.

$$\| J'(\tilde{u}_R) \| \to 0 \quad \textit{as} \quad R \to 0.$$

Proof. We assume without loss of generality that $x_0 = 0$ and $\overline{B(0,2)} \subset \Omega$. Also we consider R to be much smaller than 1. Let $\chi \in C_0^\infty(\Omega)$ be such that

$$\text{Support}(\chi) \subset B(0,2),$$
$$\chi \equiv 1 \qquad \text{on} \quad B(0,1),$$
$$0 \leqslant \chi \leqslant 1.$$

For $\phi \in H_0^1(\Omega)$ set

$$\phi_1 = \chi\phi,$$
$$\psi(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi_1(r\cos\theta, r\sin\theta)\, d\theta.$$

We note that for some $c > 0$ (independent of ϕ), it holds that

$$\| \phi_1 \| \leqslant c \| \phi \|.$$

Now,

$$|\psi(R)| = \left| \int_R^2 \psi'(r)\, dr \right|$$
$$\leqslant \left(\int_R^2 |\psi'(r)|^2 r\, dr \right)^{1/2} \left(\int_R^2 \frac{dr}{r} \right)^{1/2}$$
$$\leqslant \frac{1}{\sqrt{2\pi}} \| \phi_1 \| \left(\log\left(\frac{2}{R}\right) \right)^{1/2}$$
$$\leqslant \frac{c}{\sqrt{2\pi}} \| \phi \| \left(\log\left(\frac{2}{R}\right) \right)^{1/2}.$$

Letting $d\sigma$ denote the surface measure on $\partial B(R)$, we have

$$\langle J'(\tilde{u}_R), \phi \rangle = \langle J'(\tilde{u}_R), \phi_1 \rangle$$
$$= \int_\Omega \nabla \tilde{u}_R \cdot \nabla \phi_1 dx - \int_\Omega f(\tilde{u}_R)\phi_1 dx$$
$$= \int_{B(R)} \nabla u_R \cdot \nabla \phi_1 dx - \int_{B(R)} f(u_R)\phi_1 dx$$
$$= \int_{\partial B(R)} \frac{\partial u_R}{\partial r}\phi_1 d\sigma - \int_{B(R)} [\Delta u_R + f(u_R)]\phi_1\, dx$$

$$= \int_{\partial B(R)} \frac{\partial u_R}{\partial r} \phi_1 \, d\sigma$$

$$= u_R'(R) \int_{\partial B(R)} \phi_1 \, d\sigma$$

$$= R u_R'(R) \int_0^{2\pi} \phi_1(R\cos\theta, R\sin\theta) \, d\theta$$

$$= 2\pi R u_R'(R) \psi(R).$$

Hence, for some $c > 0$,

$$|\langle J'(\tilde{u}_R), \phi \rangle| \leqslant c \| \phi \| R |u_R'(R)| \left(\log\left(\frac{2}{R}\right) \right)^{1/2}.$$

Therefore $\| J'(\tilde{u}_R) \|$ will tend to zero as $R \to 0$ if we can show that $R|u_R'(R)|$ $(\log(2/R))^{1/2} \to 0$ as $R \to 0$. Since $u_R(R) = 0$, if we define T_0 by $R = 2\exp\{-T_0/2\}$ and set $y(t) = u_R(2\exp\{-t/2\})$, we see that y solves (P_γ) with $\gamma = u_R(0)$ and T_0 is the first zero of y. Hence by lemma 3.1,

$$|u_R'(R)| R \left(\log\left(\frac{2}{R}\right) \right)^{1/2} = \sqrt{2} y'(T_0) T_0^{1/2}$$

$$\to 0$$

as $u_R(0) = \gamma \to \infty$. Since $u_R(0) \to \infty$ if and only if $R \to 0$, it follows that

$$\| J'(\tilde{u}_R) \| \to 0 \quad \text{as} \quad R \to 0.$$

This proves the lemma. ∎

Lemma 3.3.

$$J(\tilde{u}_R) \to \tfrac{1}{2} \quad \textit{as} \quad R \to 0.$$

Proof. Since

$$\int_{B(R)} |\nabla u_R|^2 \, dx \to 1 \quad \text{as} \quad R \to 0,$$

by (i) of theorem D in §4 it is sufficient to show that

$$\int_\Omega F(\tilde{u}_R) \, dx \to 0 \quad \text{as} \quad R \to 0.$$

Let N be a large positive number. We have, using the fact $F(t) \leqslant M f(t) t^{1-\eta}$ for some $M > 0$, $\eta \in (0, 1]$ and all large t,

$$\int_\Omega F(\tilde{u}_R) \, dx = \int_{B(R)} F(u_R) \, dx$$

$$= \int_{u_R \leqslant N} F(u_R) \, dx + \int_{u_R > N} F(u_R) \, dx.$$

$$\leqslant \int_{u_R \leqslant N} F(u_R) \, dx + \frac{M}{N^\eta} \int_{u_R > N} f(u_R) u_R \, dx.$$

Now, since u_R solves (P_R), we have

$$\int_{B(R)} f(u_R)u_R\,dx = \int_{B(R)} |\nabla u_R|^2\,dx.$$

Therefore, the second integral on the right in the last inequality is of the order $1/N^\eta$. By the dominated convergence theorem, it follows that the first integral in the same inequality tends to zero as R tends to 0. Therefore,

$$\int_\Omega F(\tilde{u}_R)\,dx \to 0 \quad \text{as} \quad R \to 0.$$

Hence $J(u_R) \to 1/2$ as $R \to 0$ which proves the lemma. ■

Proof of theorem B. Putting lemmas 3.2 and 3.3 together, we get that $\{\tilde{u}_R\}$ as $R \to 0$ is a Palais–Smale sequence for J. Since $\tilde{u}_R \to 0$ in $H_0^1(\Omega)$ and $J(\tilde{u}_R) \to \frac{1}{2}$ as $R \to 0$, it follows that $\{\tilde{u}_R\}$ has no strongly convergent subsequence. Thus J does not satisfy the Palais–Smale condition at the energy level $\frac{1}{2}$. By a standard construction, for each positive integer k, instead of u_R we may consider sum of k such solutions with mutually disjoint supports. Then, we will have that J does not satisfy the Palais–Smale condition at the energy levels $k/2$, $k = 1, 2, 3, \ldots$. This proves the theorem. ■

4(1). Proof of theorem C: Blow-up behaviour of the Moser functions $m_{l,1}(x)$

Define $A_{l,\rho}(x) = 2\pi(m_{l,1}(\rho x)^2 - m_{l,1}(\rho y)^2)$.

Case (i). $\rho \leqslant l$ *for all* $l \geqslant 0$: In this case $|\rho y| \leqslant l$. Therefore

$$m_{l,1}(\rho y)^2 = -\frac{1}{2\pi}\log l,$$

and hence

$$A_{l,\rho}(x) = \begin{cases} 0 & 0 \leqslant |x| \leqslant \dfrac{l}{\rho}, \\ \dfrac{-(\log\rho + \log|x|)^2 + (\log l)^2}{\log l} & \dfrac{l}{\rho} \leqslant |x| \leqslant \dfrac{1}{\rho}. \end{cases}$$

$$= \begin{cases} 0 & 0 \leqslant |x| \leqslant \dfrac{l}{\rho}, \\ \left(\dfrac{\log\rho}{\log l} + 1\right)\log\left(\dfrac{l}{\rho}\right), & \\ -2\left(\dfrac{\log\rho}{\log l}\right)\log|x| - \dfrac{(\log|x|)^2}{\log l} & \dfrac{l}{\rho} \leqslant |x| \leqslant \dfrac{1}{\rho}. \end{cases} \tag{4.1}$$

Also, since $\rho \leqslant l$,

$$a = \lim \frac{l}{\rho} \geqslant 1.$$

If $a=\infty$, it easily follows from (4.1) that $A_{l,\rho}(x)\to 0$ as $l\to 0$, the convergence being uniform on compact subsets of $\mathbb{R}^2$.

Suppose $a<\infty$. Then, (4.1) immediately implies (since $b=\lim_{l\to 0}(\log\rho/\log l)$) that $b=1$. Therefore,

$$\frac{\log\rho}{\log l}=1+o(l) \quad \text{as} \quad l\to 0.$$

Hence,

$$\left(\frac{\log\rho}{\log l}+1\right)\log\left(\frac{l}{\rho}\right)\to 2\log a \quad \text{as} \quad l\to 0.$$

Therefore, from (4.1), as $l\to 0$,

$$\lim_{l\to 0} A_{l,\rho}(x)=\begin{cases} 0 & 0\leqslant|x|\leqslant a, \\ 2\log\left(\dfrac{a}{|x|}\right) & a\leqslant|x|, \end{cases}$$

uniformly on compact subsets of $\mathbb{R}^2$.

Case (ii). $l<\rho<1$ *for all* $l>0$: In this case, $|\rho y|>l$. Therefore

$$m_{l,1}(\rho y)^2=-\frac{1}{2\pi}\frac{(\log\rho)^2}{\log l},$$

and hence,

$$A_{l,\rho}(x)=\begin{cases} \dfrac{(\log\rho)^2-(\log l)^2}{\log l} & 0\leqslant|x|\leqslant\dfrac{l}{\rho}, \\ \dfrac{(\log\rho)^2-(\log\rho+\log|x|)^2}{\log l} & \dfrac{l}{\rho}\leqslant|x|\leqslant\dfrac{1}{\rho}. \end{cases} \tag{4.2}$$

Also, we have

$$a\leqslant 1 \quad \text{and} \quad b\leqslant 1.$$

If $b<1$, by (4.1) we get $a=0$. Hence $l/\rho\to 0$ as $l\to 0$ and (4.2) gives

$$\lim_{l\to 0} A_{l,\rho}(x)=2b\log\left(\frac{1}{|x|}\right),$$

uniformly on compact subsets of $\mathbb{R}^2\backslash\{0\}$.

If $b=1$, there are two cases: $a=0$ and $a>0$. In the former case, again $l/\rho\to 0$ as $l\to 0$ and as before

$$\lim_{l\to 0} A_{l,\rho}(x)=2\log\left(\frac{1}{|x|}\right),$$

uniformly on compact subsets of $\mathbb{R}^2\backslash\{0\}$.

In the latter case, for $|x| > a$, from (4.2) it follows that uniformly on compact subsets of $\{x: |x| > a\}$,

$$\lim_{l \to 0} A_{l,\rho}(x) = 2 \log\left(\frac{1}{|x|}\right).$$

For $|x| \leqslant a$, we have from (4.2)

$$A_{l,\rho}(x) = \left(\frac{\log \rho}{\log l} + 1\right)\left(\log\left(\frac{\rho}{l}\right)\right)$$

$$\to 2 \log\left(\frac{1}{a}\right) \quad \text{as } l \to 0 \tag{4.3}$$

uniformly on $|x| \leqslant a$. That is

$$\lim_{l \to 0} A_{l,\rho}(x) = \begin{cases} 2 \log\left(\dfrac{1}{a}\right) & 0 \leqslant |x| \leqslant a, \\ 2 \log\left(\dfrac{1}{|x|}\right) & |x| > a, \end{cases}$$

uniformly on compact subsets of $\mathbb{R}^2$. This proves the theorem. ■

4(2). Proof of theorem D: Blow-up behaviour of the solutions u_R

Here we consider a more general situation where u_R is any solution of the problem (P) posed on $\Omega = B(R)$ and with $f(s) = h(s) \exp\{4\pi s^2\}$, $h \in \mathscr{B}$, satisfying the hypotheses listed in the statement of theorem B. The theorem shows that u_R concentrates to the Dirac mass at the origin as $R \to 0$. Also, the blow-up behaviour of u_R helps us to find the 'limiting equation' associated to the problem under consideration. The asymptotics developed in [6] will be crucial to the proof of the theorem.

We will find it convenient to prove the ODE version of theorem D. This ODE version is stated as theorem D′ below. First we indicate how the statements in theorem D transform to corresponding statements in theorem D′. We are considering the problem

$$\begin{aligned} -\Delta u &= h(u) \exp\{4\pi u^2\} && \text{in} \quad B(R), \\ u &> 0 && \text{in} \quad B(R), \\ u &= 0 && \text{on} \quad \partial B(R). \end{aligned} \tag{P_R}$$

By Gidas–Ni–Nirenberg [14], every solution u_R of (P_R) is radial about the origin. Define for $r \in [0, R]$, and any $x \in \partial B(r)$,

$$w_R(r) = u_R(x).$$

Clearly, w_R solves the following ODE:

$$\begin{aligned} -w'' - \frac{1}{r} w' &= h(w) \exp\{4\pi w^2\} && \text{in} \quad (0, R), \\ w &> 0 && \text{in} \quad (0, R), \\ w'(0) &= w(R) = 0. \end{aligned} \tag{$\tilde{P}_R$}$$

Let $\gamma = w_R(0)$. Define $y_\gamma(t) = w_R(2\exp\{-t/2\}), t \in \mathbb{R}$. Then, $T_0(\gamma) = 2\log(2/R)$ will be the first zero of y_γ. Also, y_γ solves the following ODE:

$$\begin{aligned} -y'' &= \exp\{-t\}h(y)\exp\{4\pi y^2\} \quad \text{in } (T_0(\gamma), \infty), \\ y &> 0 \qquad\qquad\qquad\qquad\quad \text{in } (T_0(\gamma), \infty), \\ y(\infty) &= \gamma,\ y'(\infty) = 0. \end{aligned} \tag{P_γ}$$

Assuming that a parameter ρ as in statement (ii) of the theorem exists, we let $\mu = 2\log(2/\rho)$. Now, $\gamma \to \infty$ if and only if $R \to 0$ and we have assumed that $\rho \to 0$ as $R \to 0$. Therefore, we get that $\mu \to \infty$ as $\gamma \to \infty$. Henceforth, we shall denote simply by y the solution y_γ of (P_γ). Due to the above reduction arguments, it is clear that statements (i) and (ii) in the theorem concerning the behaviour of u_R can be rewritten as the following statements concerning the behaviour of y.

Theorem D′.

(a) $$\int_{T_0(\gamma)}^{\infty} y'(t)^2\,dt \to \frac{1}{4\pi} \quad as \quad \gamma \to \infty,$$

(b) *There exists a parameter* μ *depending continuously on* γ, $\mu \to \infty$ *as* $\gamma \to \infty$, *such that,*

$$y^2(\mu + t) - y^2(\mu) \to \frac{1}{2\pi}\log\left(\frac{2}{1+\exp\{-t\}}\right)$$

uniformly (with respect to t*) on compact subsets of* $\mathbb{R}$.

Therefore, it is sufficient to prove theorem D′ in order to prove the theorem D. We prove few simple lemmas from which theorem D′ will follow readily. Before doing so we make some preliminary observations.

Let $T_0(\gamma)$ denote the first zero of the solution y of (P_γ). Let k be large, but fixed, positive integer. Define

$$\delta = k\log\gamma,$$

$$T_1(\gamma) = g(\gamma) + \log\left(\frac{g'(\gamma)}{2}\right) - \delta.$$

For easy notation, in the sequel we let $g = g(\gamma)$, $g' = g'(\gamma)$, $g'' = g''(\gamma)$, $g''' = g'''(\gamma)$, $T_0 = T_0(\gamma)$, $T_1 = T_1(\gamma)$, etc. Define

$$z(t) = \gamma - \frac{2}{g'}\log\left(1 + \frac{g'}{2}\exp\{g - t\}\right).$$

The analysis in [6] gives the following relations:

$$y(t) \leqslant z(t),\ t \geqslant T_1, \tag{4.4}$$

$$y'(t) \geqslant z'(t),\ t \geqslant T_1, \tag{4.5}$$

$$y(T_1) = \gamma - \frac{2\delta}{g'} + O\left(\frac{\delta^2}{g}\right), \tag{4.6}$$

$$g(y(T_1)) = g - 2\delta + O\left(\frac{\delta^2}{g}\right), \tag{4.7}$$

$$y'(T_1) = \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right]. \tag{4.8}$$

Equations (4.4) to (4.8) appear respectively as equations (4.4), (5.1), (7.8), (7.9) *and* (7.17) *in* [6].

Lemma 4.1.

$$\sup_{t\in[T_1,\infty)} |z(t)-y(t)| \to 0 \quad as \quad \gamma \to \infty.$$

Proof. Let

$$l(t) = \frac{2}{g'}\log\left(1+\frac{g'}{2}\exp\{g-t\}\right).$$

Then, from lemma 1 in [6], for $t > T_1$,

$$g(y(t)) \geqslant g - g'l(t).$$

Applying g^{-1} on either side of the above inequality and using Taylor's expansion of order two, we get, for some $\xi \in [g-g'l(t), g]$,

$$y(t) \geqslant g^{-1}(g) - (g^{-1})'(g)g'l(t) + \frac{(g^{-1})''(\xi)}{2}(g')^2 l^2(t). \tag{4.9}$$

Differentiating the identity $g^{-1}(g(s)) = s$ successively twice, we get

$$(g^{-1})'(g(s)) = \frac{1}{g'(s)},$$

$$(g^{-1})''(g(s)) = \frac{-g''(s)}{(g'(s))^2}.$$

The above equations together with the facts $g' \sim \gamma$ and $g'' = O(1)$ imply that

$$\frac{(g^{-1})''(\xi)(g')^2 l^2(t)}{2} = O\left(\frac{\log^2\gamma}{\gamma^3}\right).$$

Hence, from (4.9) we get

$$\begin{aligned} y(t) &\geqslant \gamma - \frac{2}{g'}\log\left(1+\left(\frac{g'}{2}\right)\exp\{g-t\}\right) + O\left(\frac{\log^2\gamma}{\gamma^3}\right) \\ &= z(t) + O\left(\frac{\log^2\gamma}{\gamma^3}\right) \end{aligned}$$

uniformly with respect to $t \geqslant T_1$. Thus the lemma is proved. ■

Lemma 4.2.

$$\sup_{t\in[T_1,\infty)} |(z')^2(t) - (y')^2(t)| \to 0 \quad as \quad \gamma \to \infty.$$

Proof. Clearly $z(\infty) = \gamma$. Also, it can be checked easily that $z(t)$ satisfies the differential equation

$$z''(t) = -\exp\{g + g'(z(t)-\gamma) - t\}. \tag{4.10}$$

Now we have

$$z'(t) = \frac{\exp\{g-t\}}{(1+(g'/2)\exp\{g-t\})}$$

which implies

$$z'(T_1) = \frac{2}{g'}\left[1+O\left(\frac{1}{\gamma^k}\right)\right]. \tag{4.11}$$

From (4.4) we have, for all $t \geqslant T_1$,

$$y(t) \leqslant z(t).$$

Let

$$l(t) = \frac{2}{g'}\log\left(1+\frac{g'}{2}\exp\{g-t\}\right). \tag{4.12}$$

Then for all large γ and $t \geqslant T_1$ we have

$$\begin{aligned} g(z(t)) &= g(\gamma - l(t)) \\ &= g(\gamma) - g'(\gamma)l(t) + \frac{g''(\xi)}{2}l^2(t), \end{aligned}$$

for some $\xi \in [\gamma - l(t), \gamma]$. Since g'' is bounded for large γ,

$$\begin{aligned} g(z(t)) = g - g'\left[\frac{2}{g'}\log\left(1+\frac{g'}{2}\exp\{g-t\}\right)\right] \\ + O\left(\frac{\log^2(1+(g'/2)\exp\{g-t\})}{(g')^2}\right). \end{aligned} \tag{4.13}$$

Since $g' \sim \gamma$ and for $t \geqslant T_1$, $\log(1+(g'/2)\exp\{g-t\}) = O(\log\gamma)$, the above equation becomes

$$g(z(t)) = g + g'(z(t)-\gamma) + O\left(\frac{\log^2\gamma}{\gamma^2}\right). \tag{4.14}$$

Since $y(t) \leqslant z(t)$ for $t \geqslant T_1$, and g is increasing in $[T_1, \infty)$, it follows from (4.14) and (4.10)

$$\begin{aligned} y'(t) &= \int_t^\infty \exp\{g(y)-s\}\,ds \\ &\leqslant \int_t^\infty \exp\{g(z)-s\}\,ds \\ &= \exp\left\{O\left(\frac{\log^2\gamma}{\gamma^2}\right)\right\}\int_t^\infty \exp\{g+g'(z(s)-\gamma)-s\}\,ds \\ &= z'(t)\left[1+O\left(\frac{\log^2\gamma}{\gamma^2}\right)\right]. \end{aligned}$$

Since from (4.5), $y'(t) \geqslant z'(t)$ for all $t \geqslant T_1$, we get

$$z'(t) \leqslant y'(t) \leqslant z'(t)\left[1+O\left(\frac{\log^2\gamma}{\gamma^2}\right)\right]$$

and hence for all $t \geq T_1$,

$$(z')^2(t) \leq (y')^2(t) \leq (z')^2(t)\left[1 + O\left(\frac{\log^2 \gamma}{\gamma^2}\right)\right]$$

which proves the lemma. ■

Lemma 4.3.

$$\int_{T_1}^{\infty} (y')^2(t)\,dt \to 0 \quad as \quad \gamma \to \infty.$$

Proof. By the previous lemma, it is enough to show

$$\int_{T_1}^{\infty} (z')^2(t)\,dt \to 0 \quad as \quad \gamma \to \infty.$$

We have

$$z'(t) = \frac{\exp\{g - t\}}{1 + (g'/2)\exp\{g - t\}}$$

which implies $z'(t) \leq 1$ for all $t \in [T_1, \infty)$, for all large γ. Hence

$$\begin{aligned}\int_{T_1}^{\infty} (z')^2(t)\,dt &\leq \int_{T_1}^{\infty} z'(t)\,dt \\ &= z(\infty) - z(T_1) \\ &= \frac{2}{g'}\log(1 + \gamma^k) \\ &\to 0 \text{ as } \gamma \to \infty.\end{aligned}$$

■

Define $T_2 < T_1$ by $y(T_2) = s_0$, where s_0 as in the hypotheses of the theorem is chosen as large as required.

Lemma 4.4.

$$\int_{T_2}^{T_1} (y')^2(t)\,dt \to \frac{1}{4\pi} \quad as \quad \gamma \to \infty.$$

Proof. From lemma 2 in [6], we have for all $t \in [T_2, T_1]$,

$$g(y(t)) - t \leq g(y(t)) - g + \frac{g'}{2}(\gamma - y(t)) - \log\left(\frac{g'}{2}\right) \equiv \psi(y).$$

Then $\psi''(s) = g''(s) > 0$ for all $s \in [y(T_2), y(T_1)]$. Thus, ψ is convex in $[y(T_2), y(T_1)]$ and hence for all $t \in [T_2, T_1]$,

$$g(y(t)) - t \leq \max\{\psi(y(T_1)), \psi(y(T_2))\}.$$

We have by the hypothesis (iv), for some $\eta > 0$ and all large γ,

$$\begin{aligned}\psi(y(T_2)) &= g(s_0) - g + \frac{1}{2}\gamma g' - \frac{1}{2}s_0 g' - \log\left(\frac{g'}{2}\right) \\ &\leq -\eta\gamma.\end{aligned}$$

Also, using the asymptotics (4.7) and (4.8) we get

$$\psi(y(T_1)) = g(y(T_1)) - g + \frac{1}{2}(\gamma - y(T_1))g' - \log\left(\frac{g'}{2}\right)$$
$$\leqslant -\delta + O(1). \tag{4.15}$$

Therefore, for all $t \in [T_2, T_1]$,

$$g(y(t)) - t \leqslant \max\{-\eta\gamma, -\delta + O(1)\} = -\delta + O(1).$$

Hence, for all $t \in [T_2, T_1]$ we have, if k is large enough

$$y'(t) = y'(T_1) + \int_t^{T_1} \exp\{g(y(s)) - s\}\,ds$$
$$\leqslant y'(T_1) + O\left(\frac{T_1 - t}{\gamma^k}\right)$$
$$= y'(T_1) + O\left(\frac{1}{\gamma^{k-2}}\right)$$
$$= \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right].$$

Since $y'(t) \geqslant y'(T_1)$ for any $t \in [T_2, T_1]$, we get, for any t in this range,

$$y'(t) = \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right].$$

Therefore, for some $\xi \in [T_2, T_1]$,

$$T_2 = T_1 - \frac{1}{y'(\xi)}(y(T_2) - y(T_1))$$
$$= g + \log\left(\frac{g'}{2}\right) - \delta$$
$$+ \left[\frac{2}{g'}\left(1 + O\left(\frac{\delta^2}{g}\right)\right)\right]^{-1}\left(s_0 - \gamma + \frac{2\delta}{g'} + O\left(\frac{\delta^2}{g}\right)\right)$$
$$= g - \frac{\gamma g'}{2} + \frac{s_0 g'}{2} + \log\left(\frac{g'}{2}\right) + O\left(\frac{\delta^2}{g'}\right). \tag{4.16}$$

Thus,

$$\int_{T_2}^{T_1} (y')^2(t)\,dt = \frac{4}{(g')^2}\left[1 + O\left(\frac{\delta^2}{g}\right)\right](T_1 - T_2)$$
$$= \frac{2\gamma}{g'} + O\left(\frac{1}{\gamma}\right)$$
$$\to \frac{1}{4\pi} \quad \text{as} \quad \gamma \to \infty.$$

This proves the lemma. ∎

Lemma 4.5.

$$\int_{T_0}^{T_2} (y')^2(t)\,dt \to 0 \quad as \quad \gamma \to \infty.$$

Proof. Let $T_3 = \max\{T_0, 4\log\gamma\}$. Note that T_3 depends on γ. Also, since

$$\sup\{f(y): 0 \leqslant y \leqslant y(T_2) = s_0\} = O(1),$$

we get, upon integrating the ODE $-y'' = \exp\{-t\} f(y)$ successively twice between the limits $t = T_3$ and $t = T_2$ and using the concavity of y together with the fact $T_3 \geqslant 4\log\gamma$,

$$y(T_3) = y(T_2) - y'(T_2)(T_2 - T_3) + O\left(\frac{1}{\gamma^3}\right), \tag{4.17}$$

$$y'(T_3) = y'(T_2) + O\left(\frac{1}{\gamma^4}\right). \tag{4.18}$$

We now distinguish two cases regarding the behaviour of $T_0(\gamma)$ as $\gamma \to \infty$.

Case (i). *For some subsequence* $\gamma_k \to \infty$, $T_0(\gamma_k) > 4\log\gamma_k$: Clearly in this case, $T_3 = T_0$ and hence from (4.18) and the fact that y' is decreasing on $[T_0, \infty)$ we get, $y'(t) = (2/g')[1 + O(\delta^2/g)]$ for all $t \in [T_0, T_2]$. Hence

$$\int_{T_0}^{T_2} (y')^2(t)\,dt = \frac{4}{(g')^2}\left[1 + O\left(\frac{\delta^2}{g}\right)\right](T_2 - T_0)$$
$$= O\left(\frac{T_2}{\gamma^2}\right).$$

Since $g - (\gamma g'/2) = O(\gamma^\beta)$ for some $\beta \in [0, 2)$, (4.16) implies that $T_2 = O(\gamma^\beta)$. Therefore, the last inequality implies that

$$\int_{T_0}^{T_2} (y')^2(t)\,dt \to 0 \quad \text{as} \quad \gamma_k \to \infty$$

and so the lemma is true in this case.

Case (ii). $T_0(\gamma) \leqslant 4\log\gamma$ *for all large* γ: In this case, $T_3 = 4\log\gamma$ for all large γ. Hence (4.17) gives

$$y(T_3) = s_0 - \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right]\left(g - \frac{\gamma g'}{2} + \frac{s_0 g'}{2} + \log\left(\frac{g'}{2}\right)\right.$$
$$\left. - 4\log\gamma + O\left(\frac{\delta^2}{g}\right)\right)$$
$$= -\frac{2}{g'}\left(g - \frac{g'\gamma}{2} + \log\left(\frac{g'}{2}\right)\right) + O\left(\frac{\log\gamma}{\gamma}\right).$$

Since by assumption (iv) in theorem B, $g - (g'\gamma/2) + \log(g'/2) \geqslant (1 + \theta)\log\gamma$ for some $\theta > -1$ and all large γ, the above equation gives

$$y(t) = O\left(\frac{\log\gamma}{\gamma}\right) \quad \text{for all} \quad t \in [T_0, T_3].$$

Now, for any $t \in [T_0, T_3]$,

$$y'(t) = y'(T_3) + \int_t^{T_3} f(y) \exp\{-s\}\, ds.$$

Since $f(0) = 0$, for any $s \in [T_0, T_3]$, $f(y(s)) = O(y(s)) = O(\log \gamma/\gamma)$. Therefore, the last equation implies that for all $t \in [T_0, T_3]$,

$$y'(t) = y'(T_3) + O\left(\frac{\log \gamma}{\gamma}\right)$$

$$= O\left(\frac{\log \gamma}{\gamma}\right).$$

Therefore, since $T_0 \geqslant 0$ for all large γ (in fact, $T_0 \to \infty$ as $\gamma \to \infty$),

$$\int_{T_0}^{T_2} (y')^2(t)\, dt \leqslant O\left(\frac{\log^2 \gamma}{\gamma^2}\right) T_2.$$

Since $T_2 = O(\gamma^\beta)$ for some $\beta \in [0, 2)$, the above inequaliy implies

$$\int_{T_0}^{T_2} (y')^2(t)\, dt \to 0 \quad \text{as} \quad \gamma \to \infty$$

and so the lemma is true in this case also. This proves the lemma. ∎

Proof of theorem D'. (a) follows from lemmas 4.3–4.5. In view of lemma 4.1, (b) is proved if we show that there exists a parameter μ depending continuously on γ, $\mu \to \infty$ as $\gamma \to \infty$, such that as $\mu \to \infty$

$$z^2(\mu + t) - z^2(\mu) \to \frac{1}{2\pi} \log\left(\frac{2}{1 + \exp\{-t\}}\right)$$

uniformly (with respect to t) on compact subsets of $\mathbb{R}$.

We take $\mu = g + \log(g'/2)$. Then,

$$z^2(\mu + t) - z^2(\mu) = \left(\gamma - \frac{2}{g'} \log(1 + \exp\{-t\})\right)^2 - \left(\gamma - \frac{2}{g'} \log 2\right)^2$$

$$= \frac{4\gamma}{g'} \log\left(\frac{2}{1 + \exp\{-t\}}\right) + O\left(\frac{1}{\gamma^2}\right)$$

$$\to \frac{1}{2\pi} \log\left(\frac{2}{1 + \exp\{-t\}}\right)$$

uniformly (with respect to t) on compact subsets of $\mathbb{R}$ as $\mu \to \infty$. This proves (b) and hence the theorem. ∎

5. Generalizations

In this section we generalize theorems A and B by replacing $-\Delta$ in problem (P) by the n-Laplacian $-\Delta_n$ in $\mathbb{R}^n$, $n \geqslant 2$. Let $\mathscr{B}_n$ denote the subset of $C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ consisting of

functions $h(s)$ which vanish only at $s=0$ and which satisfy the following growth conditions:

For every $\delta > 0$,

$$\lim_{s\to\infty} h(s)\exp\{\delta s^{n/(n-1)}\} = \infty,$$

$$\lim_{s\to\infty} h(s)\exp\{-\delta s^{n/(n-1)}\} = 0.$$

We call a function $f \in C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ a function of critical growth in $\mathbb{R}^n$ if $f(s) = h(s)s^{n-2}\exp\{\alpha_n s^{(n/(n-1))}\}$ with $h \in \mathscr{B}_n$ and $\alpha_n = n\omega_n^{1(n-1)}$, ω_n = volume (S^{n-1}). Consider the following problem with f having critical growth in $\mathbb{R}^n$ and Ω a smooth bounded domain in $\mathbb{R}^n$:

$$\begin{aligned} -\Delta_n u &= f(u) \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{P_n}$$

The associated energy functional J_n: $W_0^{1,n}(\Omega) \to \mathbb{R}$ is given by

$$J_n(u) = \frac{1}{n}\int_\Omega |\nabla u|^n \, dx - \int_\Omega F(u)\, dx,$$

where F denotes the primitive of f.

5.1 *Generalization of theorem A*

Let $0 < l < R$. Define the Moser functions

$$m_{l,R}(x) = \frac{1}{\omega_n^{1/n}} \begin{cases} [\log(R/l)]^{1-1/n} & 0 \leqslant |x| \leqslant l, \\ \dfrac{\log(R/|x|)}{[\log(R/l)]^{1/n}} & l \leqslant |x| \leqslant R, \\ 0 & |x| \geqslant R. \end{cases}$$

Let $x_0 \in \Omega$ and $R < d(x_0, \partial\Omega)$. Define $m_{l,R,x_0}(x) = m_{l,R}(x - x_0)$. We can now state the following.

Theorem A′. *Let $f(s) = h(s)s^{n-2}\exp\{\alpha_n s^{(n/(n-1))}\}$, $s \geqslant 0$, $h \in \mathscr{B}_n$ and h satisfies the following hypothesis: $|h'(t)| \leqslant ch(t)t^{(((1/(n-1))-\eta)}$ and $F(t) \leqslant cf(t)t^{(((1/(n-1))-\eta)}$ for $\eta \in (0, 1/(n-1))]$, for some $c > 0$ and all large t.*

Let R depend on l so that $\lim_{l\to 0}(\log(2/R)/\log(R/l)) = 0$. Then there exists a sequence $\rho_l \to 1$ as $l \to 0$ such that

(a) $\{\rho_l m_{l,R,x_0}\}_{l>0}$ *is a Palais–Smale sequence for J_n as $l \to 0$,*
(b) *J_n fails to satisfy the Palais–Smale condition along the above sequence at the energy levels k/n, k any positive integer.*

Proof. Except for some technical modifications, same as that of theorem A.

5.2 *Generalization of Theorem B*

Let $x_0 \in \Omega$ and $R > 0$ be small enough so that $B(x_0, R) = \{x : |x - x_0| < R\} \subset \Omega$. We now state the following generalization of theorem B.

Theorem B′. *Let* $h \in \mathscr{B}_n$ *and* $f(s) = h(s)s^{n-2}\exp\{\alpha_n s^{(n/(n-1))}\}$, $s \geqslant 0$, *satisfies the following hypothesis: There exists* $s_0 > 0$ *such that*

(i) $h(s)s^{n-1} \to \infty$ *as* $s \to \infty$,

(ii) $g(s) = \log f(s)$ *is* C^3 *and convex for* $s \geqslant s_0$,

(iii) $f(s)$ *is strictly increasing for* $s \geqslant s_0$,

(iv) $\left(g(s) - \left(\frac{n-1}{n}\right)sg'(s)\right) + (n-1)\log\left(\frac{(n-1)}{n}g'(s)\right) \geqslant (n-1+\theta)\log s$ *for all* $s \geqslant s_0$ *and some* $\theta > 1-n$,

(v) $\limsup_{s\to\infty}\left(g(s) - \left(\frac{n-1}{n}\right)sg'(s)\right)s^{-\beta} = O(1)$ *for some* $\beta \in [0, (n/(n-1)))$,

(vi) $\limsup_{s\to\infty}\frac{sg^{(k+1)}(s)}{g^{(k)}(s)}$, $k = 0, 1$, *exist and are different from* 0 (*here* $g^{(k)}$ *denotes the* k*-th derivative of* g).

Let u_R *denote the solution to problem* (P_n) *with the above choice of* f *which is obtained in* [1]. *Let* $\tilde{u}_R$ *denote the extension of* u_R *obtained by setting* $\tilde{u}_R \equiv 0$ *in* $\Omega \backslash B(x_0, R)$. *Then*

(a) $\{\tilde{u}_R\}_{R>0}$ *as* $R \to 0$ *are Palais–Smale sequences for* J_n,

(b) J_n *does not satisfy the Palais–Smale condition at the energy levels* k/n, k *any positive integer.*

Proof. Except for some technical modifications, same as that of theorem B.

6. The Neumann case

If we consider the problem (P_n) with the Dirichlet boundary condition replaced by the homogeneous Neumann condition, the associated energy functional J_n: $W^{1,n}(\Omega) \to \mathbb{R}$ will be given again by

$$J_n(u) = \frac{1}{n}\int_\Omega |\nabla u|^n \, dx - \int_\Omega F(u)\, dx.$$

If f satisfies the assumptions of either theorem A′ or theorem B′, it can be shown that J_n fails to satisfy the Palais–Smale condition at the energy levels $k/2n$, k any positive integer. The proof is done first for the case of half-space $\mathbb{R}^+_n$, where one uses the results from Dirichlet case to show that the associated energy functional fails to satisfy the Palais–Smale condition at the energy levels $k/2n$, k any positive integer. The proof for a general domain Ω with smooth boundary is accompolished by the standard localization argument involving partition of unity.

7. Concluding remarks

1. Analogues of theorems A and B can be shown to hold even when the Palais–Smale sequences considered in §2 and §3 approach a boundary point of Ω and/or there are multiple concentrations at a single point in Ω or boundary of Ω. In each of these cases Palais–Smale still fails at the energy levels $k/2$, k any positive integer.

2. If the non-linearity f grows like $\exp\{bs^2\}$ for some $b > 0$ as $s \to \infty$ similar arguments as in theorems A and B will show that the corresponding energy functional

will fail to satisfy the Palais–Smale condition at the energy levels $2k\pi/b$, k any positive integer.

3. Blow-up result similar to the one presented in part (ii) of theorem D were first obtained in [20] for the maximizing sequence for the Trudinger imbedding.

4. The blow-up result in part (ii) of theorem D extends the results of [17, 18], where h is assumed to have atmost polynomial growth. Also, we have greatly simplified the proof given in these references.

5. In fact, the blow-up analysis given here can be easily seen to hold even when the non-linearity f grows like $\exp\{s^\alpha\}$ as $s \to \infty$ for any $\alpha \geqslant 1$.

6. Lemmas 4.1 and 4.2 are generalizations of the corresponding results in Volkmer [23].

8. Appendix

Proof of Lemma 2.3. Let $\varepsilon \in (0,1)$ be fixed. For $\delta > 0$ denote $\hat{\delta} = \delta/4\pi$. Define $T(\varepsilon, n) = (1-\varepsilon)t_n + \varepsilon \bar{t}_n$. Then, for any $\delta > 0$, and all large n,

$$\begin{aligned}
&\int_0^{T(\varepsilon,n)} f(v_n) v_n \exp\{-t\}\, dt \\
&= \int_0^{T(\varepsilon,n)} h(v_n) v_n \exp\left\{\rho_n^2 \frac{(t-\bar{t}_n)^2}{t_n - \bar{t}_n} - t\right\} dt \\
&\leqslant \int_0^{T(\varepsilon,n)} h(v_n) v_n \exp\{\rho_n^2(1-\varepsilon)(t-\bar{t}_n) - t\}\, dt \\
&= \int_0^{T(\varepsilon,n)} h(v_n) \exp\{\rho_n^2(1-\varepsilon)(t-\bar{t}_n) - t - \delta v_n^2 + \delta v_n^2\}\, dt \\
&\leqslant \int_0^{T(\varepsilon,n)} h(v_n) \exp\{-\delta v_n^2\} v_n \exp\{\rho_n^2(1-\varepsilon)(1+\hat{\delta})(t-\bar{t}_n) - t\}\, dt \\
&\leqslant \int_0^{T(\varepsilon,n)} h(v_n) \exp\{-\delta v_n^2\} v_n \exp\{\rho_n^2(1-\varepsilon)(1+\hat{\delta}) - 1)t\}\, dt. \qquad (8.1)
\end{aligned}$$

Let M be an upper bound for $h(v_n)\exp\{-\delta v_n^2\}$.

We choose $\delta > 0$ small enough so that for some $\eta > 0$ and all large n,

$$(1-\varepsilon)(1+\hat{\delta})\rho_n^2 - 1 \leqslant -\eta.$$

Now (8.1) implies, with η and M chosen as above,

$$\begin{aligned}
&\int_0^{T(\varepsilon,n)} f(v_n) v_n \exp\{-t\}\, dt \\
&\leqslant \frac{\rho_n M}{\sqrt{4\pi}(t_n - \bar{t}_n)^{1/2}} \int_{\bar{t}_n}^{T(\varepsilon,n)} t \exp\{-\eta t\}\, dt \\
&= O(t_n^{-1/2}). \qquad (8.2)
\end{aligned}$$

Integrating by parts, we obtain for some $v>0$,

$$\begin{aligned}
&\int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n \exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\\
&=\int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n\left(\frac{2\rho_n^2(t-\bar t_n)}{t_n-\bar t_n}-1\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\right)\mathrm{d}t\\
&=\left[h(v_n)v_n\left(\frac{2\rho_n^2(t-\bar t_n)}{t_n-\bar t_n}-1\right)^{-1}\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\right]_{t=T(\varepsilon,n)}^{t=t_n}\\
&\quad-\int_{T(\varepsilon,n)}^{t_n}\left\{-h(v_n)v_n\left(\frac{2\rho_n^2(t-\bar t_n)}{t_n-\bar t_n}-1\right)^{-2}\frac{2\rho_n^2}{(t_n-\bar t_n)}\right.\\
&\quad\left.+(h(v_n)v_n)'\left(\frac{2\rho_n^2(t-\bar t_n)}{t_n-\bar t_n}-1\right)^{-1}\right\}\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\\
&=\frac{h(\bar v_n)\bar v_n}{2\rho_n^2-1}\exp\left\{\rho_n^2(t_n-\bar t_n)-t_n\right\}+O(\exp\{-vt_n\})\\
&\quad+O\left(\frac{1}{t_n}\int_{T(\varepsilon,n)}^{t_n}h(v_n)v_n\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\right)\\
&\quad+O\left(\int_{T(\varepsilon,n)}^{t_n}(h(v_n)v_n)'\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\right).
\end{aligned}\tag{8.3}$$

By the hypothesis on h we get, for some $\eta\in(0,1]$, for some $c>0$ and all large t,

$$\begin{aligned}
|h'(v_n)|v_n &\leqslant ch(v_n)v_n^{1-\eta}v_n\\
&\leqslant c\left(\frac{t_n-\bar t_n}{4\pi}\right)^{(1-\eta)/2}h(v_n)v_n.
\end{aligned}$$

Therefore, for some positive numbers c_1, c_2 and c_3,

$$\begin{aligned}
&\int_{T(\varepsilon,n)}^{t_n}|h(v_n)v_n)'|\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\\
&\quad\leqslant\int_{T(\varepsilon,n)}^{t_n}\{|h'(v_n)v_n'|v_n+h(v_n)|v_n'|\}\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\\
&\quad=\frac{c_1}{(t_n-\bar t_n)^{1/2}}\int_{T(\varepsilon,n)}^{t_n}\{|h'(v_n)|v_n+h(v_n)\}\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\\
&\quad\leqslant\frac{c_2}{(t_n-\bar t_n)^{\eta/2}}\int_{T(\varepsilon,n)}^{t_n}h(v_n)v_n\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t\\
&\qquad+\frac{c_3}{t_n-\bar t_n}\int_{T(\varepsilon,n)}^{t_n}h(v_n)v_n\exp\left\{\frac{\rho_n^2(t-\bar t_n)^2}{t_n-\bar t_n}-t\right\}\mathrm{d}t.
\end{aligned}\tag{8.4}$$

Since by lemma 2.2, $\rho_n^2\to 1$ as $n\to\infty$, (2.13) implies

$$\int_0^{t_n}f(v_n)\exp\{-t\}\,\mathrm{d}t=O(1).$$

Therefore (8.4) gives

$$\int_{T(\varepsilon,n)}^{t_n} |(h(v_n)v_n)'| \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt = O(t_n^{-\eta/2}).$$

Hence, (8.3) implies

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n \exp\{-t\}\, dt = \frac{f(\bar{v}_n)\bar{v}_n \exp\{-t_n\}}{2\rho_n^2 - 1} + O(t_n^{-\eta/2}). \tag{8.5}$$

Combining (8.2) with the last equation, we get

$$\int_{0}^{t_n} f(v_n)v_n \exp\{-t\}\, dt = \frac{f(\bar{v}_n)\bar{v}_n \exp\{-t_n\}}{2\rho_n^2 - 1} + O(t_n^{-\eta/2}). \tag{8.6}$$

The lemma now follows from (2.13) and the fact that $\rho_n^2 \to 1$ as $n \to \infty$. ■

Proof of Lemma 2.5. From (2.6), it is sufficient to show that

$$I_n = \int_0^\infty \left| \int_t^\infty f(v_n) \exp\{-s\}\, ds - \frac{dv_n}{dt} \right|^2 dt \to 0$$

as $n \to \infty$.

Using the explicit form of v_n, we find that

$$\frac{dv_n}{dt} = \frac{\rho_n}{\sqrt{4\pi}} \begin{cases} 0 & 0 \leqslant t < \bar{t}_n, \\ (t_n - \bar{t}_n)^{-1/2} & \bar{t}_n < t < t_n, \\ 0 & t_n < t. \end{cases}$$

We split

$$I_n = A_n + B_n + C_n$$

with

$$A_n = \int_0^{\bar{t}_n} \left| \int_t^\infty f(v_n) \exp\{-s\}\, ds \right|^2 dt,$$

$$B_n = \int_{\bar{t}_n}^{t_n} \left| \int_t^\infty f(v_n) \exp\{-s\}\, ds - \frac{\rho}{\sqrt{4\pi}} (t_n - \bar{t}_n)^{-1/2} \right|^2 dt,$$

$$C_n = \int_{t_n}^\infty \left| \int_t^\infty f(v_n) \exp\{-s\}\, ds \right|^2 dt.$$

We show that A_n, B_n and C_n tend to zero as $n \to \infty$. This we do in the following series of claims. Let $\varepsilon > 0$ be arbitrary. Define $T(\varepsilon, n) = (1-\varepsilon)t_n + \varepsilon \bar{t}_n$.

Claim 1. There exists an $\eta > 0$ such that for any $t \in [\bar{t}_n, T(\varepsilon, n)]$, as $n \to \infty$,

$$\int_t^{T(\varepsilon,n)} f(v_n) \exp\{-s\}\, ds = O((t_n - \bar{t}_n)^{-1/2}(1+t)\exp\{-t\}) + O(\exp\{-\eta(t_n - \bar{t}_n)^{1/2}\}). \tag{8.7}$$

Proof of claim.

Since $h(0)=0$ and $v_n(s)\leqslant 1$ in $[0,(t_n-\bar{t}_n)^{1/2}]$, we have $h(v_n(s))=O(v_n(s))$ as $n\to\infty$, uniformly for $s\in[0,(t_n-\bar{t}_n)^{1/2}]$. Let $\delta>0$ and $c_\delta>0$ be such that $h(t)\leqslant c_\delta\exp\{4\pi\delta t^2\}$ for all $t\geqslant 0$. Also, we may choose δ small enough so that the following inequality holds for all $s\leqslant T(\varepsilon,n)$ and some $\eta>0$:

$$(1+\delta)\rho_n^2(s-\bar{t}_n)^2(t_n-\bar{t}_n)^{-1}-s\leqslant -\eta s.$$

Therefore, for $t\in[\bar{t}_n, T(\varepsilon,n)]$, all large n and some positive constant c,

$$\begin{aligned}
\int_t^{T(\varepsilon,n)} f(v_n)\exp\{-s\}\,ds &= \int_t^{(t_n-\bar{t}_n)^{1/2}} f(v_n)\exp\{-s\}\,ds \\
&\quad + \int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} f(v_n)\exp\{-s\}\,ds \\
&\leqslant c\int_t^{(t_n-\bar{t}_n)^{1/2}} v_n(s)\exp\{-s\}\,ds \\
&\quad + c_\delta\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} \exp\left\{(1+\delta)\rho_n^2\frac{(s-\bar{t}_n)^2}{(t_n-\bar{t}_n)}-s\right\}ds \\
&\leqslant \frac{c\rho_n}{\sqrt{4\pi}(t_n-\bar{t}_n)^{1/2}}\int_t^{(t_n-\bar{t}_n)^{1/2}} s\exp\{-s\}\,ds \\
&\quad + c_\delta\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} \exp\{-\eta s\}\,ds \\
&= O((t_n-\bar{t}_n)^{-1/2}(1+t)\exp\{-t\}) \\
&\quad + O(\exp\{-\eta(t_n-\bar{t}_n)^{1/2}\}).
\end{aligned}$$

This proves the claim.

Claim 2. For all large enough n,

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n)\exp\{-s\}\,ds\in\left[\frac{(1-\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n},\frac{(1+\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{(1-\varepsilon)4\sqrt{\pi}\rho_n}\right].$$

Proof of claim. From lemma 2.3, we may obtain the following two relations for all large n:

$$\int_0^{t_n} f(v_n)v_n\exp\{-s\}\,ds\in\left[\frac{1-\varepsilon}{8\pi},\frac{1+\varepsilon}{8\pi}\right], \tag{8.8}$$

$$f(\bar{v}_n)\bar{v}_n\exp\{-t_n\}\in\left[\frac{1-\varepsilon}{8\pi},\frac{1+\varepsilon}{8\pi}\right]. \tag{8.9}$$

We have, for all large n,

$$\begin{aligned}
\int_{T(\varepsilon,n)}^{t_n} f(v_n)\exp\{-s\}\,ds &\leqslant \frac{1}{v_n(T(\varepsilon,n))}\int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n\exp\{-s\}\,ds \\
&\leqslant \frac{1+\varepsilon}{8\pi v_n(T(\varepsilon,n))} \\
&= \frac{(1+\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}(1-\varepsilon)\rho_n}.
\end{aligned} \tag{8.10}$$

Now,

$$\int_{\bar{t}_n}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,ds = \int_{\bar{t}_n}^{(t_n-\bar{t}_n)^{1/2}} f(v_n)v_n \exp\{-s\}\,ds + \int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,ds.$$

Since by (2.13) $f(v_n)v_n \exp\{-t\} \in L^1((0,\infty))$ and $f(v_n)v_n \to 0$ as $n \to \infty$ pointwise in $[\bar{t}_n, (t_n-\bar{t}_n)^{1/2}]$, by dominated convergence theorem

$$\int_{\bar{t}_n}^{(t_n-\bar{t}_n)^{1/2}} f(v_n)v_n \exp\{-s\}\,ds \to 0 \quad \text{as} \quad n \to \infty.$$

By the reasoning that led to the second 'order term' in claim 1, we can find an $\eta > 0$ such that

$$\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,ds = O(\exp\{-\eta(t_n-\bar{t}_n)^{1/2}\}).$$

Therefore,

$$\int_{\bar{t}_n}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,ds \to 0 \text{ as } n \to \infty.$$

Now, since

$$\int_{\bar{t}_n}^{t_n} f(v_n)v_n \exp\{-s\}\,ds \to \frac{1}{8\pi} \text{ as } n \to \infty,$$

we obtain for all large n,

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n \exp\{-s\}\,ds \geqslant \frac{1-\varepsilon}{8\pi}.$$

Therefore,

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n)\exp\{-s\}\,ds \geqslant \frac{\sqrt{4\pi}(t_n-\bar{t}_n)^{-1/2}}{\rho_n} \int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n \exp\{-s\}\,ds \geqslant \frac{(1-\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n}. \tag{8.11}$$

Combining (8.10) and (8.11) we prove the claim. ■

Claim 3.

$$\lim_{n\to\infty} A_n = 0.$$

Proof of Claim. Since $f(0) = 0$, it follows that

$$A_n \leqslant \bar{t}_n \left| \int_{\bar{t}_n}^{\infty} f(v_n)\exp\{-s\}\,ds \right|^2.$$

Claims 1 and 2 imply that

$$\int_{\bar{t}_n}^{t_n} f(v_n)\exp\{-s\}\,ds = O((t_n-\bar{t}_n)^{-1/2}).$$

Since by lemma 2.3,

$$\int_{t_n}^{\infty} f(v_n)\exp\{-s\}\,ds = f(\bar{v}_n)\exp\{-t_n\} = O((t_n-\bar{t}_n)^{-1/2}),$$

it follows that

$$A_n \leqslant O\left(\frac{\bar{t}_n}{t_n-\bar{t}_n}\right).$$

Since $(\bar{t}_n/(t_n-\bar{t}_n))\to 0$ by assumption, the claim follows. ∎

Claim 4.

$$\lim_{n\to\infty} B_n = 0.$$

Proof of claim. We split

$$B_n = B_n^1 + B_n^2$$

with

$$B_n^1 = \int_{\bar{t}_n}^{T(\varepsilon,n)} \left| \int_t^{\infty} f(v_n)\exp\{-s\}\,ds - \frac{\rho}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2} \right|^2 dt.$$

Now,

$$\int_t^{\infty} f(v_n)\exp\{-s\}\,ds - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}$$
$$= \int_t^{t_n} f(v_n)\exp\{-s\}\,ds + f(\bar{v}_n)\exp\{-t_n\} - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}. \quad (8.12)$$

From (8.9) we obtain,

$$f(\bar{v}_n)\exp\{-t_n\} - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}$$
$$\in \left[\frac{(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n}(1-\varepsilon-2\rho_n^2), \frac{(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n}(1+\varepsilon-2\rho_n^2)\right]. \quad (8.13)$$

Therefore, using claims 1 and 2 we get from (8.12)

$$\int_t^{\infty} f(v_n)\exp\{-s\}\,ds - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}$$
$$\in \left[\frac{(t_n-\bar{t}_n)^{-1/2}}{\sqrt{4\pi}\rho_n}(1-\varepsilon-\rho_n^2), \frac{(t_n-\bar{t}_n)^{-1/2}}{\sqrt{4\pi}\rho_n}\left(\frac{1+\varepsilon}{1-\varepsilon}-\rho_n^2\right)\right]$$
$$+ O((t_n-\bar{t}_n)^{-1/2}(1+t)\exp\{-t\})$$
$$+ O(\exp\{-\eta(t_n-\bar{t}_n)^{-1/2}\}).$$

Therefore,

$$B_n^1 \leqslant 2\int_{\bar{t}_n}^{T(\varepsilon,n)} \frac{(t_n-\bar{t}_n)^{-1}}{4\sqrt{\pi}\rho_n^2} \max\left\{(1-\varepsilon-\rho_n^2)^2, \left(\frac{1+\varepsilon}{1-\varepsilon}-\rho_n^2\right)^2\right\} dt$$
$$+ O\left(\int_{\bar{t}_n}^{T(\varepsilon,n)} (t_n-\bar{t}_n)^{-1}(1+t)^2 \exp\{-2t\}\, dt\right)$$
$$+ O\left(\int_{\bar{t}_n}^{T(\varepsilon,n)} \exp\{-2\eta(t_n-\bar{t}_n)^{1/2}\}\, dt\right)$$
$$= \frac{(1-\varepsilon)}{2\pi\rho_n^2} \max\left\{(1-\varepsilon-\rho_n^2)^2, \left(\frac{1+\varepsilon}{1-\varepsilon}-\rho_n^2\right)^2\right\} + O((t_n-\bar{t}_n)^{-1}).$$

Hence,

$$\limsup B_n^1 \leqslant \frac{(1-\varepsilon)}{2\pi} \max\left\{\varepsilon^2, \left(\frac{2\varepsilon}{1-\varepsilon}\right)^2\right\}$$
$$= \frac{2\varepsilon^2}{\pi(1-\varepsilon)}. \tag{8.14}$$

From claim 2 and the fact $f(\bar{v}_n)\exp\{-t_n\} = O((t_n-\bar{t}_n)^{-1/2})$, it follows that for $t \in [T(\varepsilon,n), t_n]$,

$$\left|\int_t^\infty f(v_n)\exp\{-s\}\, ds - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}\right|^2 = O((t_n-\bar{t}_n)^{-1}).$$

Therefore,

$$B_n^2 = O\left(\int_{T(\varepsilon,n)}^{t_n} (t_n-\bar{t}_n)^{-1}\, dt\right)$$
$$= O(\varepsilon). \tag{8.15}$$

Since $\varepsilon > 0$ was arbitrary, combining (8.14) and (8.15) we obtain the claim. ∎

Claim 5.

$$\lim_{n\to\infty} C_n = 0.$$

Proof of claim. We have

$$C_n = f(\bar{v}_n)^2 \int_{t_n}^\infty \exp\{-2t\}\, dt$$
$$= \frac{1}{2}(f(\bar{v}_n)\exp\{-t_n\})^2$$
$$= O((t_n-\bar{t}_n)^{-1})$$

which proves the claim. ∎

Claims 3, 4 and 5 imply that $\lim_{n\to\infty} I_n = 0$, which proves the lemma. ∎

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From Tanaka's formula to Ito's formula: The fundamental theorem of stochastic calculus

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Abstract. In this article we give a new proof of Ito's formula in $\mathbb{R}^n$ starting from the one-dimensional Tanaka formula. The proof is algebraic and does not use any limiting procedure. It uses the integration by parts formula, Fubini's theorem for stochastic integrals and essential properties of local times.

Keywords. Semi-martingales; Ito formula; Tanaka formula; local times.

1. Introduction

In this note we give a proof of Ito's formula starting from Tanaka's formula. Our approach was motivated by the proof of Ito's formula given in [1], and the results of [5], where we have proved and generalized the Tanaka formula, starting from first principles. In the case of dimension one, the basic method is well known, involving an integration in the space variable and then using the occupation density formula. We use a variation of this method, to go from Tanaka's formula to Ito's formula. In effect, we introduce an increasing process viz. the space integral of the local time and prove the occupation density formula and Ito's formula with this increasing process, which is then identified as the quadratic variation process by taking $f(x) = x^2$ in Ito's formula. The surprising fact is that the same method works in higher dimensions. Indeed, our formalism also yields a proof of the fundamental theorem of ordinary calculus, though only for twice differentiable functions: Let $x(t)$, $y(t)$ be continuous functions on $[0, \infty)$, of bounded variation on finite intervals and first let f be a C^2 function on $\mathbb{R}$ with compact support. Then

$$f(x_t) = f(x_0) + \int_0^t f'(x_s)\, dx_s.$$

To prove this we first note that the 'ordinary' Tanaka formula holds for (x_t): If $a \in \mathbb{R}$, then

$$(x_t - a)^+ = (x_0 - a)^+ + \int_0^t I_{\{x_s > a\}} dx_s.$$

This equation follows upon noting that the set $\{x_s > a\}$ is a union of disjoint open intervals of $[0, \infty)$. The integral in the RHS then reduces to a sum which together with the first term adds up to $(x_t - a)^+$. Integrating the ordinary Tanaka formula with respect to $f''(a)\, da$ and using Fubini's theorem we get the above expression for $f(x_t)$. We can then extend this equation for $f(x_t)$ to the arbitrary C^2 functions in the usual

manner. Taking $f(x) = x^2$ we get

$$x_t^2 = x_0^2 + 2\int_0^t x_s \, dx_s.$$

Polarization then yields

$$x_t y_t = x_0 y_0 + \int_0^t x_s \, dy_s + \int_0^t y_s \, dx_s.$$

Applying this equation to the functions $(x_t - a)^+$, $(y_t - b)^+$ we get

$$(x_t - a)^+ (y_t - b)^+ = (x_0 - a)^+ (y_0 - b)^+ + \int_0^t (x_s - a)^+ I_{\{y_s > b\}} \, dy_s + \int_0^t (y_s - b)^+ I_{\{x_s > a\}} \, dx_s.$$

Next if we let $f(x, y)$ be a C^2 function with compact support, then integrating the above equation with respect to $\partial_x^2 \partial_y^2 f(a, b) \, da db$ we arrive at

$$f(x_t, y_t) = f(x_0, y_0) + \int_0^t \partial_x f(x_s, y_s) \, dx_s + \int_0^t \partial_y f(x_s, y_s) \, dy_s.$$

Our proof of Ito's formula in higher dimensions is an exact analogue of the above computations. The local time terms in the stochastic analogue of the equation for $(x_t - a)^+ (y_t - b)^+$, integrated with $\partial_x^2 \partial_y^2 f(a, b) \, da db$, yield the quadratic variation terms of the Ito's formula, via the occupation density formula.

We remark on two other aspects of these techniques: Firstly, we wish to emphasize the role of local time techniques in a multidimensional context. It is well-known that the point local times need not exist in higher dimensions. The resuls of this paper show that this fact need not necessarily restrict the use of local times in higher dimensional situations. Secondly, we would like to point out that each term in the expression for $(x_t - a)^+ (y_t - b)^+$, for fixed t, is a locally integrable function in (a, b) and hence can be read as an equation for the corresponding Schwartz distribution. Applying the distributional derivative $\partial_x^2 \partial_y^2$ to both sides we get the fomal expression

$$\delta_{(x_t, y_t)} = \delta_{(x_0, y_0)} - \int_0^t \partial_x \delta_{(x_s, y_s)} \, dx_s - \int_0^t \partial_y \delta_{(x_s, y_s)} \, dy_s$$

which is like a distributional analogue of the fundamental theorem. The point here is that there is a duality relation embedded in the statement of the fundamental theorem (for test functions) in both the ordinary and stochastic cases. In a forthcoming article, we hope to follow up this aspect in the stochastic case.

The paper is organized as follows. After the preliminaries in § 2, we prove the formula for continuous semi-martingales in § 3, and discuss the necessary modifications for arbitrary (discontinuous) semi-martingales in § 4.

2. Preliminaries (Continuous case)

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space with the filtration $(\mathcal{F}_t)$ satisfying usual conditions. Let (X_t) be a continuous semi-martingale viz. $X_t = X_0 + M_t + V_t$ where (M_t) is a continuous $\mathcal{F}_t$ local martingale, (V_t) is an $\mathcal{F}_t$ - adapted process of finite variation and $M_0 = V_0 \equiv 0$.

We recall Tanaka's formulas: $\forall\, a \in \mathbb{R}$,

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_s > a\}} \mathrm{d}X_s + \frac{1}{2} L(t, a), \tag{1}$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t I_{\{X_s \leqslant a\}} \mathrm{d}X_s + \frac{1}{2} L(t, a),$$

where for each a, $(L(t,a))_{t \geqslant 0}$ is a continuous, increasing adapted process, $L(0,a) \equiv 0$ and satisfying

$$L(t,a) = \int_0^t I_{\{X_s = a\}} \mathrm{d}L(s,a). \tag{2}$$

We shall assume, following the results of [7], that the mapping $(t,a,w) \to L(t,a,w)$ is $\mathscr{B}[0,\infty) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{F}_t / \mathscr{B}(\mathbb{R})$ measurable where $\mathscr{B}(\cdot)$ is the Borel σ-algebra on $(\cdot)$. Recall that for two continuous semi-martingales (X_t), (Y_t), a unique, continuous, adapted process of finite variation $([X, Y]_t)_{t \geqslant 0}$, (the quadratic co-variation process of X and Y) is defined by the following equation:

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_s \mathrm{d}Y_s - \int_0^t Y_s \mathrm{d}X_s. \tag{3}$$

We will denote by $\mathbb{D}(\mathbb{R}^n)$ (or simply by $\mathbb{D}$ when the context is clear) the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, let

$$\alpha(x,y) = \prod_{k=1}^{n} (x_k - y_k)^+;$$

$$\alpha_i(x,y) = I_{\{x_i > y_i\}} \prod_{k \neq i} (x_k - y_k)^+$$

$$\alpha_{ij}(x,y) = I_{\{x_i > y_i, x_j > y_j\}} \prod_{k \neq i,j} (x_k - y_k)^+$$

and

$$\bar{\alpha}_i(x,y) = \prod_{k \neq i} (x_k - y_k)^+.$$

3. Ito's formula and occupation density formula: The continuous case

Let (X_t) be a continuous semi-martingale as in § 2 and $(L(t,x))_{t \geqslant 0}$ its local time process at $x \in \mathbb{R}$.

PROPOSITION 1

Let (X_t) be a continuous, bounded semi-martingale. Then a.s.,

$$\int_{\mathbb{R}} L(t,x) \mathrm{d}x < \infty \quad \forall\, t \geqslant 0.$$

Proof. Since (X_t) is a bounded process, $\exists M > 0 \ni |X_t(w)| \leqslant M\ \forall t, w$. Let $K = [-M, M]$. By (2) and our boundedness assumption, $\forall\, x \notin K$, $L(t,x) = 0$ a.s. Hence by Fubini's

theorem, a.s., $L(t,x)=0$ for almost all $x \in K^c$. Hence,

$$\int_{\mathbb{R}} L(t,x)\,dx = \int_K L(t,x)\,dx.$$

Since K is compact, $\int_K \{(X_t-x)^+ - (X_0-x)^+\}\,dx < \infty$. Further the process $f(s,w)=\int_K I_{\{X_s(w)>x\}}\,dx$ is bounded and previsible. Hence by Fubini's theorem for stochastic integrals (see [6], Chap. V, p. 165)

$$\int_K dx \int_0^t I_{\{X_s>x\}}\,dX_s = \int_0^t f(s)\,dX_s < \infty \quad \text{a.s.} \quad \forall t \geqslant 0.$$

Integrating eq. (1) over K with respect to dx, it follows that a.s., $\int_K L(t,x)\,dx < \infty$. This completes the proof. □

PROPOSITION 2

Let (X_t) be a bounded semi-martingale. Then $\Lambda_t := \int_{\mathbb{R}} L(t,x)\,dx$ is an adapted continuous and non-decreasing process. Moreover we have a.s., $\forall f \geqslant 0$ measurable

$$\int_0^t f(X_s)\,d\Lambda_s = \int_{\mathbb{R}} f(x)L(t,x)dx. \tag{4}$$

Proof. From Fubini's theorem it follows that Λ_t is an adapted process. Proposition (1) says that a.s., $\Lambda_t < \infty\, \forall t \geqslant 0$. Since for every x, $(L(t,x))$ is a non-decreasing and continuous process, the same is true of (Λ_t).

By Fubini's theorem applied to the measure $L(dt,x)\,dx$, it is immediate that

$$\int_0^t f(X_s)\,d\Lambda_s = \int_{\mathbb{R}} \int_0^t f(X_s)\,L(ds,x)dx$$

for $f \geqslant 0$, measurable. Now the result follows from eq. (2). □

Theorem 1. (*The occupation density formula*). *Let (X_t) be a continuous semi-martingale and $[X,X]_t$, the quadratic variation process defined via eq.* (3). *Let $f:[0,\infty)\times\mathbb{R}\times\Omega\to[0,\infty]$ be a measurable function. Then a.s.,*

$$\int_0^t f(s,X_s,w)\,d[X,X]_s = \int_{\mathbb{R}} dx \int_0^t f(s,x,w)\,dL(s,x) \quad \forall t \geqslant 0. \tag{5}$$

Proof. It is sufficient to prove the case when f is a function of x above, since the general case reduces to this case by standard monotone class arguments. Further, it is sufficient to consider the case when X_0 is bounded since replacement of X_t by $I_A(X_0)X_t$ merely multiples eq. (5) by $I_A(X_0)$. Again if eq. (5), is true when $[X,X]_t$, $L(t,x)$ are replaced by $[X,X]_{t\wedge T_n}$, $L(t\wedge T_n,x)$ respectively, for a sequence of stopping times T_n, increasing to ∞, then letting $n\to\infty$, eq. (5) is true for any $t \geqslant 0$. Thus, by stopping X if necessry we can assume that it is a bounded process. By Proposition 2, it therefore suffices to show, that if (X_t) is bounded then, $\Lambda_t \equiv [X,X]_t$. But if $f \in \mathbb{D}(\mathbb{R})$, then integrating eq. (1) with respect to the measure $f''(a)\,da$ we get, using proposition 2,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\Lambda_s.$$

It follows by standard localizing arguments that

$$\Lambda_t = X_t^2 - X_0^2 - 2\int_0^t X_s \, dX_s \equiv [X, X]_t.$$

This completes the proof. □

Let $X_t = (X_t^1, \ldots, X_t^n)$ be a continuous vector valued semi-martingale, where each X_t^i is a continuous real valued semi-martingale whose local time process at $x_i \in R$ will be denoted by $(L^i(t, x_i))_{t \geq 0}$.

Theorem 2. (*Ito's formula*). *Let (X_t) be a continuous n-dimensional semi-martingale. Then for all $f \in \mathbb{D}(\mathbb{R}^n)$ we have*

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_s) \, dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) \, d[X^i, X^j]_s. \tag{6}$$

Proof. Recall the functions $\alpha(x, y)$, $\alpha_i(x, y)$, $\alpha_{ij}(x, y)$ and $\bar{\alpha}_i(x, y)$ defined in § 2. Then using Tanaka's formula (eq. (1)) and the integration by parts formula (eq. (3)), we get by induction on n,

$$\alpha(X_t, y) = \alpha(X_o, y) + \sum_{i=1}^n \int_0^t \alpha_i(X_s, y) \, dX_s^i + \frac{1}{2}\sum_{i=1}^n \int_0^t \bar{\alpha}_i(X_s, y) \, dL^i(s, y_i)$$
$$+ \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^n \int_0^t \alpha_{ij}(X_s, y) \, d[X^i, X^j]_s. \tag{7}$$

Let D^2 denote the differential operator $D^2 = \partial^{2n}/(\partial y_1^2, \ldots, \partial y_n^2)$. The following identities are easily verified for $f \in \mathbb{D}(\mathbb{R}^n)$ by integration by parts:

$$f(x) = \int_{\mathbb{R}^n} dy D^2 f(y) \alpha(x, y), \tag{8}$$

$$\frac{\partial f}{\partial x_i}(x) = \int_{\mathbb{R}^n} dy D^2 f(y) \alpha_i(x, y), \tag{9}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} dy D^2 f(y) \alpha_{ij}(x, y). \tag{10}$$

Note that by Fubini's theorem and the occupation density formula (eq. (5)), we have

$$\int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \bar{\alpha}_i(X_s, y) \, dL^i(s, y_i) = \int_0^t \frac{\partial^2 f}{\partial x_i^2}(X_s) \, d[X^i, X^j]_s. \tag{11}$$

Integrating eq. (7) with respect to $D^2 f(y) \, dy$ and using eq. (8), we get

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \alpha_i(X_s, y) \, dX_s^i$$
$$+ \frac{1}{2}\sum_{i=1}^n \int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \bar{\alpha}_i(X_s, y) \, dL^i(s, y_i)$$
$$+ \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^n \int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \alpha_{ij}(X_s, y) \, d[X^i, X^i]_s.$$

Using Fubini's theorem for stochastic integrals (see [6], Chap. IV, p. 165) and eq. (9) for terms of the 1st summation, eq. (11) for terms of the 2nd summation, the Fubini's theorem for ordinary integrals and eq. (10) for terms of the 3rd summation, the theorem follows. □

4. Ito's formula and occupation density formula: The general case

We now consider the case of a general semi-martingale $X_t = X_0 + M_t + V_t$ where (M_t) is a local martingale and (V_t) an adapted process of finite variation with $M_0 \equiv V_0 \equiv 0$. The Tanaka formula for (X_t), for $a \in \mathbb{R}$, is given by

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_{s-} > a\}} \, dX_s + l(t,a) + \frac{1}{2} L(t,a), \tag{12}$$

where $l(t,a) = \Sigma_{s \le t} (I_{\{X_{s-} \le a\}} (X_s - a)^+ + I_{\{X_{s-} > a\}} (X_s - a)^-)$ and $(L(t,a))_{t \ge 0}$ is a continuous adapted non-decreasing process satisfying

$$L(t,a) = \int_0^t I_{\{X_s = a\}} \, dL(s,a). \tag{13}$$

Further by the results of [7], the map $(t,a,w) \to L(t,a,w)$ can be taken to be jointly measurable. Note that $(l(t,a))_{t \ge 0}$ is a non-negative and non-decreasing process. If (X_t), (Y_t) are two semi-martingales, then the quadratic co-variation process $[X, Y]_t$ defined by

$$[X,Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t Y_{s-} dX_{s-} - \int_0^t X_{s-} dY_s \tag{14}$$

is an adapted process of finite variation such that

$$[X,Y]_t - \sum_{s \le t} \Delta X_s \Delta Y_s \equiv [X,Y]_t^c \tag{15}$$

is a continuous adapted process of finite variation. If (X_t) is a semi-martingale and T a stopping time, then the pre-stopped process $X^{T-}(t)$ is also a semi-martingale where $X^{T-}(t) \equiv X_t I_{[0,T)}(t) + X_{T-} I_{[T,\infty]}(t)$ with $X_{0-} \equiv X_0$. Moreover, $[X^{T-}, X^{T-}]^c(t) = [X,X]^c(t \wedge T)$.

We refer to [2] for these and other propositions on general semi-martingales.

PROPOSITION 3

Let (X_t) be a bounded semi-martingale. Then a.s.,

$$\int_{\mathbb{R}} (L(t,x) + l(t,x)) \, dx < \infty \quad \forall \, t \ge 0.$$

Proof. Let K be a compact set such that a.s., $X_t \in K \, \forall \, t \ge 0$. Then it is clear from the definition of $l(t,x)$ that if $x \notin K$, $l(t,x) \equiv 0$. The rest of the argument is the same as in the continuous case (Proposition 1), with $L(t,x)$ replaced by $L(t,x) + l(t,x)$, where for the application of Fubini's theorem for stochastic integrals of general semi-martingales we refer to [3], Chap. V, p. 180 or [4], Chap. IV, p. 157. □

PROPOSITION 4

Let (X_t) be a bounded semi-martingale, then $\Lambda := \int_{\mathbb{R}} L(t,x)\,dx$ is an adapted process which is continuous and non-decreasing. Moreover we have a.s., for all f non-negative and measurable,

$$\int_0^t f(X_s)\,d\Lambda_s = \int_{\mathbb{R}} f(x)\,L(t,x)\,dx \quad \forall t \geqslant 0. \tag{16}$$

Proof. The first part of the proposition follows from Proposition 3 and the fact that for each $x \in \mathbb{R}$ $(L(t,x))_{t \geqslant 0}$ is a continuous non-decreasing adapted process. The proof of the second part is similar to that of the continuous case (Proposition 2). □

Theorem 3. (*The occupation density formula, general case*). *Let (X_t) be a semi-martingale and $[X,X]_t$ its quadratic variation process defined by eq.* (14). *Let $f:[0,\infty] \times \mathbb{R} \times \Omega \to [0,\infty]$ be a measurable function. Then a.s.,*

$$\int_0^t f(x, X_s, w)\,d[X,X]^c_s = \int_{\mathbb{R}} dx \int_0^t f(s,x,w)\,dL(s,x) \quad \forall t \geqslant 0. \tag{17}$$

Proof. It is sufficient to consider the case when f is a function of x alone, X_0 is bounded and $[X,X]^c_t$, $L(t,x)$ are replaced by $[X,X]^c_{t \wedge T_n}$, $L(t \wedge T_n, x)$ respectively for a sequence of stopping times T_n increasing to ∞. Note that $[X,X]^c_{t \wedge T_n}$, $L(t \wedge T_n, x)$ are respectively, the continuous part of the quadratic variation process, and local time process at x, of the semi-martingale $X_t^{T_n -}$. Thus, by pre-stopping X if necessary we can assume that it is a bounded process.

By Proposition (4), it therefore suffices to show that if (X_t) is a bounded semi-martingale, then $\Lambda_t \equiv [X,X]^c_t$. If $f \in \mathbb{D}(\mathbb{R})$, then integrating eq. (12) with respect to the measure $f''(a)da$, we get using Proposition 4 that

$$f(X_t) = f(X_0) + \int_0^t f(X_{s-})\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\Lambda_s$$

$$+ \sum_{s \leqslant t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

If $X_t \in K\ \forall\, t \geqslant 0$, where K is a compact set, then choosing $f \in \mathbb{D}(\mathbb{R})$, $f(x) = x^2$, $x \in K$ we get, from above

$$\Lambda_t = X_t^2 - X_0^2 - 2\int_0^t X_{s-}\,dX_s - \sum_{s \leqslant t} (\Delta X_s)^2$$

$$\equiv [X,X]^c_t.$$

This completes the proof. □

Theorem 4. *Let (X_t) be an $\mathbb{R}^n$-valued semi-martingale with components (X_t^i) $1 \leqslant i \leqslant n$. Then, $\forall f \in \mathbb{D}(\mathbb{R}^n)$ we have a.s.,*

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_{s-})\,dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\,d[X^i, X^j]^c_s$$

$$+ \sum_{s \leqslant t} \{f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\}. \tag{18}$$

Proof. For $x, y \in \mathbb{R}^n$, we recall the functions $\alpha(x,y)$, $\alpha_i(x,y)$, $\bar{\alpha}_i(x,y)$, $\alpha_{ij}(x,y)$ of § 2. Let $L^i(s,y_i)$ denote the local time of (X_s^i) at $y_i \in \mathbb{R}$, where $y=(y_1,\ldots,y_n)$. Then using Tanaka formula (12), and the integration by parts formula (14), we get by induction on n, that

$$\begin{aligned}\alpha(X_t,y) = \alpha(X_0,y) &+ \sum_{i=1}^{n} \int_0^t \alpha_i(X_{s-},y)\,\mathrm{d}\,X_s^i \\ &+ \frac{1}{2}\sum_{i=1}^{n}\int_0^t \bar{\alpha}_i(X_{s-},y)\,\mathrm{d}\,L^i(s,y_i) \\ &+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_0^t \alpha_{ij}(X_{s-},y)\,\mathrm{d}\,[X^i,X^j]_s^c \\ &+ \sum_{s\leqslant t}\left(\sum_{i=1}^{n}\bar{\alpha}_i(X_{s-},y)\Delta l^i(s,y_i)\right) \\ &+ \sum_{s\leqslant t}\left\{\alpha(X_s,y)-\alpha(X_{s-},y)-\sum_{i=1}^{n}\bar{\alpha}_i(X_{s-},y)\Delta(X_s^i-y_i)^+\right\}.\end{aligned}$$

It is easily verified that

$$\bar{\alpha}_i(X_{s-},y)\Delta l^i(s,y_i)-\bar{\alpha}_i(X_{s-},y)\Delta(X_{s-}^i-y_i)^+ = -\alpha_i(X_{s-},y)\,\Delta X_s^i$$

and hence,

$$\begin{aligned}\alpha(X_t,y) = \alpha(X_0,y) &+ \sum_{i=1}^{n}\int_0^t \alpha_i(X_{s-},y)\,\mathrm{d}\,X_s^i \\ &+ \frac{1}{2}\sum_{i=1}^{n}\int_0^t \bar{\alpha}_i(X_{s-},y)\,\mathrm{d}\,L^i(s,y_i) \\ &+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_0^t \alpha_{ij}(X_{s-},y)\,\mathrm{d}\,[X^i,X^j]_s^c \\ &+ \sum_{s\leqslant t}\left\{\alpha(X_s,y)-\alpha(X_{s-},y)-\sum_{i=1}^{n}\alpha_i(X_{s-},y)\Delta X_s^i\right\}. \qquad (19)\end{aligned}$$

Let $D^2 = \partial^{2n}/(\partial y_1^2 \ldots \partial y_n^2)$. Integrating eq. (19) with respect to the measure $D^2 f(y)\mathrm{d}y$ and using Fubini's theorem for ordinary and stochastic integrals (for the latter case, see [3] or [4]), the proof is completed as in the continuous case (Theorem 2) with the aid of eqs (8), (9), (10) and the analog of eq. (11) for the discontinuous case. □

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Positive values of non-homogeneous indefinite quadratic forms of type (1, 4)

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Abstract. Let $\Gamma_{r,n-r}$ denote the infimum of all number $\Gamma > 0$ such that for any real indefinite quadratic form in n variables of type $(r, n-r)$, determinant $D \neq 0$ and real numbers $c_1, c_2, \ldots, c_n$, there exist integers $x_1, x_2, \ldots, x_n$ satisfying

$$0 < Q(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n) \leqslant (\Gamma |D|)^{1/n}.$$

All the values of $\Gamma_{r,n-r}$ are known except for $\Gamma_{1,4}$. Earlier it was shown that $8 \leq \Gamma_{1,4} \leq 16$. Here we improve the upper bound to get $\Gamma_{1,4} < 12$.

Keywords. Equivalent forms; reduction; soluble.

1. Introduction

Let $Q(x_1, x_2, \ldots, x_n)$ be a real indefinite quadratic form in n variables of type $(r, n-r)$ and of determinant $D \neq 0$. Let $\Gamma_{r,n-r}$ denote the infimum of all numbers $\Gamma > 0$ for which there exist integers $x_1, x_2, \ldots, x_n$ satisfying

$$0 < Q(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n) \leqslant (\Gamma |D|)^{1/n}$$

for any real numbers $c_1, c_2, \ldots, c_n$. The values of $\Gamma_{r,n-r}$ are known for all n and r except for $\Gamma_{1,4}$ (see reference [1]). From the work of Jackson [2] it follows that $\Gamma_{1,4} \leqslant 32$. It is conjectured that $\Gamma_{1,4} = 8$. Dumir and Sehmi [1] proved that $8 \leqslant \Gamma_{1,4} \leqslant 16$. In this paper we improve the upper bound to get $\Gamma_{1,4} < 12$.

We modify the method of Dumir and Sehmi [1]. This method can yield at the most $\Gamma_{1,4} \leqslant 32/3$, but then the proof becomes very complicated and we do not get any form which attains the minimum 32/3. So we content ourselves to give a proof of $\Gamma_{1,4} < 12$. For any improvement of 32/3, one will need an improvement of Macbeath's result (stated as Lemma 5 in [1]) or an analogous result in three variables. More precisely we prove the following:

Theorem. *Let $Q(x_1, x_2, x_3, x_4, x_5)$ be a real indefinite quadratic form of type (1, 4) and determinant $D \neq 0$. Then given any real numbers $c_j, 1 \leqslant j \leqslant 5$, there exist integers $x_j, 1 \leqslant j \leqslant 5$, such that*

$$0 < Q(x_1 + c_1, x_2 + c_2, \ldots, x_5 + c_5) < (12|D|)^{1/5}.$$

We start as in Dumir and Sehmi [1] and will use many of their lemmas. Henceforth, we will refer to this paper as DS.

2. Reduction

As in the reduction of DS, we can assume that Q is a zero rational quadratic form and that

$$Q = (x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5)x_2 - \phi(x_3, x_4, x_5),$$

where ϕ is a positive definite ternary quadratic form with determinant $\Delta = 4|D|$. Let $d^5 = 12|D|$.

Lemma 1. *If* $\phi(x_3, x_4, x_5)$ *represents a number with* $0 < a < d/4$, *then the following inequality*

$$0 < (x_1 + a_2x_2 + \cdots + a_5x_5)x_2 - \phi(x_3, x_4, x_5) < d$$

is soluble.

The proof is similar to that of Lemma 6 of DS. Here as our constant is smaller, a smaller range of a is covered.

Working as in § 3 of DS we can assume that

$$d/4 \leqslant a \leqslant (2\Delta)^{1/3} = (2d^5/3)^{1/3}. \tag{2.1}$$

This gives

$$d \geqslant \sqrt{6}/16. \tag{2.2}$$

Further we can suppose that

$$\phi(x_3, x_4, x_5) = a(x_3 + h_4x_4 + h_5x_5)^2 + A(x_4 + \lambda x_5)^2 + tx_5^2,$$

where

$$0 \leqslant \lambda \leqslant 1/2, 0 < A \leqslant \sqrt{4\delta/3} = \sqrt{4d^5/9a}, \mathrm{t} = \delta/A = d^5/3aA, \tag{2.3}$$

$$A \leqslant C = t + A\lambda^2 \text{ which gives } 3A/4 \leqslant t, \tag{2.4}$$

$$-1/2 < a_i, h_j \leqslant 1/2, \text{ for } i = 2, 3, 4, 5, \quad j = 4, 5, \tag{2.5}$$

and

$$A \geqslant 3a/4 \text{ and if } h_4 = 0 \text{ then } A \geqslant a. \tag{2.6}$$

We need to show that

$$0 < Q = (x_1 + a_2x_2 + \cdots + a_5x_5)x_2 - a(x_3 + \cdots)^2 - A(x_4 + \lambda x_5)^2 + tx_5^2 < d \tag{2.7}$$

is soluble. Without loss of generality, we can suppose that

$$-1/2 < c_i \leqslant 1/2 \quad \text{for} \quad 1 \leqslant i \leqslant 5.$$

Lemma 2. *Let* α, β *and* δ *be any real numbers, then*

$$0 < (x_1 + \alpha x_2 + v)x_2 + \beta < \delta$$

is soluble for

$$\delta > 1/2 \quad \text{if} \quad c_2 \neq 0,$$

$$\delta > 1 \quad \text{if} \quad c_2 = 0.$$

Proof: is trivial.

By Lemma 2, we can assume that

$$d \leqslant 1/2 \quad \text{if} \quad c_2 \neq 0; \quad d \leqslant 1 \quad \text{if} \quad c_2 = 0. \tag{2.8}$$

Let m be the integer defined by $m < d/a \leqslant m + 1$. From (2.1) we have

$$4 \geqslant d/a \geqslant d/(2d^5/3)^{1/3} > 1, \quad \text{so that } m = 1, 2 \text{ or } 3.$$

We can rewrite Q as

$$Q(x_1,\dots,x_5) = -a(x_3+a_3x_2/2a+\cdots)^2+(x_1+a_2'x_2+a_4'x_4+a_5'x_5)x_2 \\ -A(x_4+\lambda x_5)^2-tx_5^2,$$

where a_2', a_4', a_5' are some real numbers satisfying

$$\left.\begin{aligned} &a_2' \equiv a_2+a_3^2/4a(\text{mod } 1),\\ &a_4' \equiv a_4-a_3b_4(\text{mod } 1), \qquad -1/2<a_2',\ a_4',\ a_5'\leqslant 1\\ &a_5' \equiv a_5-a_3b_5(\text{mod } 1). \end{aligned}\right\}. \tag{2.9}$$

Now (2.7) is soluble by Lemma 3 of DS if we can solve

$$0<F(x_1,x_2,x_4,x_5)=(x_1+a_2'x_2+a_4'x_4+a_5'x_5)x_2-A(x_4+\lambda x_5)^2 \\ -tx_5^2-a/4<d+(m^2-1)a/4=\delta_m \quad \text{(say)}. \tag{2.10}$$

Let K be the integer defined by $K<\delta_m/A\leqslant K+1$. Since $a\geqslant d/(m+1)$ and $\delta_m=d+(m^2-1)a/4\geqslant (m+3)\,d/4$, we have from (2.3) that

$$\delta_m/A\geqslant\frac{(m+3)d}{4}\sqrt{\frac{9a}{4d^5}}\geqslant\frac{(m+3)}{\sqrt{m+1}}\cdot\frac{3}{8}\cdot\frac{1}{d}. \tag{2.11}$$

Therefore we find because of (2.8), that

$$K\geqslant 2 \text{ if } c_2\neq 0 \text{ and } K\geqslant 1 \text{ if } c_2=0. \tag{2.12}$$

We can rewrite F as

$$F=-A(x_4+a_4'x_2/2A+\lambda x_5)^2+(x_1+a_2''x_2+a_5''x_5)x_2-tx_5^2-a/4,$$

where

$$a_2''\equiv a_2'+a_4'^2/4A\ (\text{mod } 1),\ a_5''\equiv a_5'-\lambda a_4'\ (\text{mod } 1),\ -1/2<a_2'',a_5''\leqslant 1/2. \tag{2.13}$$

Now (2.10) and hence (2.7) is soluble, by Lemma 3 of DS, if we can solve

$$0<G(x_1,x_2,x_5)=(x_1+a_2''x_2+a_5''x_5)\,x_2-tx_5^2-(a+A)/4 \\ <\delta_m+(K^2-1)A/4=\delta_{m,K}(\text{say}). \tag{2.14}$$

By definition of m and K, we have

$$a\geqslant d/(m+1),A\geqslant(m+3)\,d/4(K+1),\quad \delta_{m,K}\geqslant(m+3)(K+3)d/16. \tag{2.15}$$

Now since $(x_1+a_2''x_2+a_5''x_5)x_2-tx_5^2$ is a zero form, (2.14) is soluble, by Lemma 4 of DS, if

$$\delta_{m,K}\geqslant 2(t/4)^{1/3}=(2d^5/3aA)^{1/3}$$

i.e. using (2.15) we have

$$\left(\frac{m+3}{4}\right)^4\left(\frac{K+3}{4}\right)^3\left(\frac{1}{m+1}\right)\left(\frac{1}{K+1}\right)>\frac{2}{3}. \tag{2.16}$$

As left hand side of (2.16) is an increasing function of K, we find that (2.16) is satisfied for $m=3$, $K\geqslant 2$ and for $m=2$, $K\geqslant 3$.

For $m=1$, using (2.6) we have

$$\delta_m/A=d/A\leqslant 4d/3a\leqslant 8/3 \text{ (as } a\geqslant d/2),$$

so that we have $K \leqslant 2$. Thus, we consider the following cases.

I. $m=3$, $K=1$
II. $m=2$, $K=1$
III. $m=2$, $K=2$
IV. $m=1$, $K=2$
V. $m=1$, $K=1$

In each of these cases we will prove that either (2.7) or (2.10) or (2.14) is soluble; proving thereby the Theorem.

Lemma 3. *If* $c_2 \neq 0$, *then* (2.14) *is soluble.*

Proof. Since from (2.12) we have $K \geqslant 2$, we need to consider cases III and IV only.

Now (2.14) is soluble by Lemma 2, if $\delta_{m,K} > 1/2$ i.e. using (2.15) we have

$$(m+3)(K+3)\,d/16 > 1/2. \tag{2.17}$$

For $m=1$, since $d/2 \leqslant a \leqslant (2d^5/3)^{1/3}$ from (2.1), we get $d \geqslant \sqrt{3}/4$. For $m=2$, $k=2$, we have from (2.15) and (2.3) that $5d/12 \leqslant A \leqslant \sqrt{4d^5/9a} \leqslant \sqrt{4/3}\,d^2$, which gives $d \geqslant 5\sqrt{3}/24$. In both these cases we find that (2.17) is satisfied, proving thereby (2.14).

Lemma 4. *If* $c_2 = 0$, *then* (2.7) *is soluble unless*

(i) $a < 1/2$, $a+d \leqslant 1$ (2.18)
(ii) $Q=(x_1+a_2x_2+\cdots)x_2 - x_3^2/2 - x_4^2/2 - tx_5^2$, (2.19)

where $t=1/2$ *or* 1; (a_3,c_3) *and* (a_4,c_4) *are either* (0, 1/2) *or* (1/2, 0). *For* $t=1/2$, $(a_5c_5)=(0,1/2)$ *or* $(1/2,0)$; *for* $t=1$, $(a_5,c_5)=(0,0)$ *or* $(1/2,\pm 1/4)$ *or* (0,1/2).

The proof is similar to that of Lemmas 7 and 10 of DS and so is left.

Notation. For $n=1,2,3,\ldots,$ denote

$$f_n = n|c_1| + n^2a_2,\ f'_n = n|c_1| + n^2a'_2,\ f''_n = n|c_1| + n^2a''_2,$$
$$g_n = -n|c_1| + n^2a_2,\ g'_n = -n|c_1| + n^2a'_2,\ g''_n = -n|c_1| + n^2a''_2,$$
$$p'_n = nc_1 + n^2a'_2,\ q'_n = -nc_1 + n^2a'_2,\ p''_n = nc_1 + n^2a''_2,\ q''_n = -nc_1 + n^2a''_2.$$

One finds that $-1/2 < f_1, f'_1, f''_1 \leqslant 1$, $-1 < g_1, g'_1, g''_1 \leqslant 1/2$ and $-1 < p'_1, p''_1, q'_1, q''_1 \leqslant 1$.

Lemma 5. *Let* Q *be as in case* (ii) *of Lemma* 4 *with* $t=1/2$, *then* (2.7) *is soluble.*

Proof. Here we have

$$Q=(x_1+a_2x_2+\cdots)x_2 - (x_3^2+x_4^2+x_5^2)/2, \quad d=(3/8)^{1/5}=0{\cdot}821\cdots$$

and $(a_i,c_i)=(0,1/2)$ or $(1/2,0)$ for $i=3,4,5$. We work exactly as in Lemma 11 of DS and find that (2.7) is soluble by choosing the same values of $x_1, x_2, \ldots, x_5$.

emma 6. *Let* Q *be as in case* (ii) *of Lemma* 4 *with* $t=1$, *then* (2.7) *is soluble.*

roof. Here

$$Q=(x_1+\cdots)x_2 - (x_3^2+x_4^2)/2 - x_5^2, \quad d=(3/4)^{1/5}=0{\cdot}944\cdots$$

d $(a_5,c_5)=(0,0)$, $(1/2,\pm 1/4)$ or $(0,1/2)$. From (2.9) and (2.13) we get $a''_5=a'_5=a_5$, $=a_4$. Since $d/a=2d=1{\cdot}88\ldots$, we have $m=1$, $\delta_m/A=2d=1{\cdot}88\ldots$, we have $K=1$, that $\delta_{m,K}=d$.

Case (i) $(a_5, c_5) = (0, 0)$. Here (2.14) reduces to

$$0 < G = (x_1 + a_2'' x_2)\, x_2 - x_5^2 - 1/4 < d.$$

The following table gives a solution of (2.14).

Range	x_1	$x_1 x_2$	x_5	G
$f_1'' > \frac{1}{4}$	c_1	$\lvert c_1\rvert$	0	$f_1'' - \frac{1}{4}$
$f_1'' < d - \frac{3}{4}$	c_1	$\lvert c_1\rvert$	1	$f''^1 + \frac{3}{4}$
$d - \frac{3}{4} \leqslant f_1'' \leqslant \frac{1}{4}$, $-\frac{3}{4} < g_1'' < d - \frac{3}{4}$	c_1	$-\lvert c_1\rvert$	1	$g_1'' + \frac{3}{4}$
$d - \frac{3}{4} \leqslant f_1'' \leqslant \frac{1}{4}$, $g_1'' \geqslant d - \frac{3}{4}$	c_1	$2\lvert c_1\rvert$	0	$f_2'' - \frac{1}{4}$
$d - \frac{3}{4} \leqslant f_1'' \leqslant \frac{1}{4}$, $g_1'' \leqslant -\frac{3}{4}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	1	$f_2'' + \frac{3}{4}$

Case (ii) $(a_5, c_5) = (1/2, \pm 1/4)$. Here

$$G = (x_1 + a_2''\, x_2 + x_5/2)x_2 - x_5^2 - 1/4.$$

Changing $x_1 \to -x_1, x_2 \to -x_2, x_5 \to -x_5$, if necessary, we can assume that $c_5 = 1/4$.

Range	x_1	x_2	x_5	G
$p_1'' > \frac{3}{16}$	c_1	1	$\frac{1}{4}$	$p_1'' - \frac{3}{16}$
$-\frac{13}{16} < p_1'' < d - \frac{13}{16}$	$1 + c_1$	1	$\frac{1}{4}$	$p_1'' + \frac{13}{16}$
$p_1'' < d - \frac{29}{16}$	$2 + c_1$	1	$\frac{1}{4}$	$p_1'' + \frac{29}{16}$
$d - \frac{13}{16} \leqslant p_1'' \leqslant \frac{3}{16}$, $q_1'' > \frac{7}{16}$	c_1	-1	$\frac{1}{4}$	$q_1'' - \frac{7}{16}$
$d - \frac{13}{16} \leqslant p_1'' \leqslant \frac{3}{16}$, $d - \frac{9}{16} \leqslant q_1'' \leqslant \frac{7}{16}$	c_1	2	$\frac{1}{4}$	$p_2'' - \frac{1}{16}$
$d - \frac{13}{16} \leqslant p_1'' \leqslant \frac{3}{16}$, $-\frac{9}{16} < q_1'' < d - \frac{9}{16}$	$1 + c_1$	-1	$\frac{1}{4}$	$q_1'' + \frac{9}{16}$
$d - \frac{13}{16} \leqslant p_1'' \leqslant \frac{3}{16}$, $d - \frac{25}{16} \leqslant q_1'' \leqslant -\frac{9}{16}$	$1 + c_1$	2	$-\frac{3}{4}$	$p_2'' + \frac{7}{16}$
$d - \frac{13}{16} \leqslant p_1'' \leqslant \frac{3}{16}$, $q_1'' < d - \frac{25}{16}$	$2 + c_1$	-1	$\frac{1}{4}$	$q_1'' + \frac{25}{16}$
$d - \frac{29}{16} \leqslant p_1'' \leqslant -\frac{13}{16}$, $q_1'' > \frac{7}{16}$	c_1	-1	$\frac{1}{4}$	$q_1'' - \frac{7}{16}$
$d - \frac{29}{16} \leqslant p_1'' \leqslant -\frac{13}{16}$, $d - \frac{9}{16} \leqslant q_1'' \leqslant \frac{7}{16}$	c_1	-2	$-\frac{3}{4}$	$q_2'' - \frac{1}{16}$
$d - \frac{29}{16} \leqslant p_1'' \leqslant -\frac{13}{16}$, $q_1'' < d - \frac{9}{16}$	$1 + c_1$	-1	$\frac{1}{4}$	$q_1'' + \frac{9}{16}$

Case (iii) $(a_5, c_5) = (0, 1/2)$. Already we have (a_3, c_3) and $(a_4, c_4) = (0, 1/2)$ or $(1/2, 0)$ (see Case (ii) of Lemma 4). Because of symmetry we need to consider (i) $(a_4, c_4) = (0, 1/2)$, (a_3, c_3) may be $(0, 1/2)$ or $(1/2, 0)$, (ii) $(a_4, c_4) = (1/2, 0), (a_3, c_3) = (1/2, 0)$.

Subcase (i) $(a_4, c_4) = (0, 1/2)$, (a_3, c_3) may be $(0, 1/2)$ or $(1/2, 0)$.

The following table gives a solution of (2.10) i.e.

$$0 < F(x_1, x_2, x_4, x_5) = (x_1 + a_2' x_2) x_2 - x_4^2/2 - x_5^2 - 1/8 < d.$$

Range		x_1	x_1x_2	x_4	x_5	F
$f_1' > \frac{1}{2}$		c_1	$\lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$f_1' - \frac{1}{2}$
$f_1' < d - \frac{1}{2}$		$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$f_1' + \frac{1}{2}$
$d - \frac{1}{2} \leqslant f_1' \leqslant \frac{1}{2}$	$g_1' \geqslant d - \frac{1}{2}$	c_1	$-2\lvert c_1\rvert$	$\frac{3}{2}$	$\frac{1}{2}$	$g_2' - \frac{3}{2}$
	$-\frac{1}{2} < g_1' < d - \frac{1}{2}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$g_1' + \frac{1}{2}$
	$g_1' \leqslant -\frac{1}{2}$	c_1	$2\lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$f_2' - \frac{1}{2}$

Subcase (ii) $a_3 = a_4 = 1/2, c_3 = c_4 = 0$.
Here $Q = (x_1 + a_2 x_2 + x_3/2 + x_4/2) x_2 - (x_3^2 + x_4^2)/2 - x_5^2$.
The following table gives a solution of (2.7).

Range		x_1	x_1x_2	x_3x_2	x_3	x_4	x_5	Q
$f_1 > \frac{1}{4}$		c_1	$\lvert c_1\rvert$	0	0	0	$\frac{1}{2}$	$f_1 - \frac{1}{4}$
$f_1 < d - \frac{3}{4}$		$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	0	0	0	$\frac{1}{2}$	$f_1 + \frac{3}{4}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}$	$g_1 \geqslant d - \frac{3}{4}$	c_1	$2\lvert c_1\rvert$	0	0	0	$\frac{1}{2}$	$f_2 - \frac{1}{4}$
	$-\frac{3}{4} < g_1 < d - \frac{3}{4}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	0	0	0	$\frac{1}{2}$	$g_1 + \frac{3}{4}$
	$d - \frac{7}{4} \leqslant g_1 \leqslant -\frac{3}{4}$	c_1	$2\lvert c_1\rvert$	2	1	0	$\frac{1}{2}$	$f_2 + \frac{1}{4}$
	$g_1 < d - \frac{7}{4}$	$\pm 2 + c_1$	$2 - \lvert c_1\rvert$	0	0	0	$\frac{1}{2}$	$g_1 + \frac{7}{4}$

This completes the proof of Lemma 6. We are left now with the case (i) of Lemma 4 i.e.

$$c_2 = 0, a < 1/2, a + d \leqslant 1, d \leqslant 1. \tag{2.20}$$

By Lemma 2, (2.14) is soluble if $\delta_{m,K} > 1$. So we can suppose that $\delta_{m,K} \leqslant 1$, or by using (2.15) we can take

$$d \leqslant \frac{4}{m+3} \cdot \frac{4}{K+3}. \tag{2.21}$$

From (2.15) and (2.3) we have

$$\frac{\delta_{m,K}}{t} = \frac{3aA}{d^5} \delta_{m,K} \geqslant \frac{3}{d^5} \cdot \frac{d}{m+1} \frac{(m+3)d}{4} \frac{(m+3)}{4} \frac{(K+3)}{4} d,$$

$$= \frac{3}{64} \frac{(m+3)^2 (K+3)}{(m+1)(K+1)} \frac{1}{d^2}. \tag{2.22}$$

If for some m and K, $\delta_{m,K}/t$ turns out to be greater than 1, we will define an integer

$L \geqslant 1$ by $L < \delta_{m,K}/t \leqslant L+1$. Then applying Lemma 3 of DS, (2.14) will be soluble if we can solve

$$0 < H(x_1, x_2) = (x_1 + a_2'' x_2)x_2 - (a + A + t)/4 < \delta_{m,K} + (L^2 - 1)t/4$$
$$= \delta_{m,K,L} \text{ (say)}. \tag{2.23}$$

Since $\delta_{m,K,L} \geqslant \frac{(m+3)}{4}\cdot\frac{(K+3)}{4}\cdot\frac{(L+3)}{4}\, d$, (2.23) will be soluble by Lemma 2, if we have

$$\delta_{m,K,L} > 1 \text{ or if } \frac{(m+3)}{4}\cdot\frac{(K+3)}{4}\cdot\frac{(L+3)}{4} d > 1. \tag{2.24}$$

Otherwise we can assume that

$$\delta_{m,K,L} \leqslant 1 \text{ and } d \leqslant 64/(m+3)(K+3)(L+3). \tag{2.25}$$

Then from definition of L and (2.22) we have

$$d^2 \geqslant \frac{3}{64}\frac{(m+3)^2(K+3)}{(m+1)(K+1)(L+1)}. \tag{2.26}$$

Further, F can be rewritten as

$$F = -C(x_5 + A\lambda x_4/C - a_5' x_2/2C)^2 + (x_1 + a_2^* x_2 + a_4^* x_4)x_2 - Atx_4^2/C - a/4,$$

where

$$\left.\begin{aligned} &C = A\lambda^2 + t, a_2^* \equiv a_2' + a_5'^2/4C \pmod 1, \\ &a_4^* \equiv a_4' - \lambda A a_5'/C \pmod 1, -1/2 < a_2^*, a_4^* \leqslant 1/2 \end{aligned}\right\}. \tag{2.27}$$

If M is an integer $\geqslant 1$ such that $M < \delta_m/C \leqslant M+1$, then by Lemma 3 of DS, (2.10) will be soluble if

$$0 < G^* = (x_1 + a_2^* x_2 + a_4^* x_4)x_2 - Atx_4^2/C - a/4 - C/4$$
$$< \delta_m + (M^2 - 1)C/4 = \delta_{m,M}^* \text{ (say)} \tag{2.28}$$

is soluble.

When (2.20) holds true, we will apply Macbeath's lemma (stated as Lemma 5 in DS) repeatedly.

Choose $x_5 \equiv c_5 (\text{mod } 1)$ arbitrarily, take $x_2 = \pm 1$ and $x_1 = x + c_1$, $x_4 = y + c_4$, so that $F = \pm x + \beta_F y - Ay^2 + v$, where $\beta_F = \pm a_4' - 2A(c_4 + \lambda x_5)$ and v is some real number. We will choose suitable integers $2h$ and k such that

$$|h - Ak^2| + 1/2 < \delta_m. \tag{2.29}$$

Then by Macbeath's lemma, (2.10) will be soluble unless $A = h/k^2$ and $\beta_F \equiv h/k (\text{mod } 1/k, 2A)$. This will fix the values of A and a_4', c_4.

Similarly,

$$G(x + c_1, \pm 1, y + c_5) = \pm x + \beta_G y - ty^2 + v,$$

where $\beta_G = \pm a_5'' - 2tc_5$.

If we can find integers $2h$ and k such that

$$|h - tk^2| + 1/2 < \delta_{m,K}, \tag{2.30}$$

then (2.14) is soluble unless $t = h/k^2$ and $\beta_G \equiv h/k \pmod{1/k, 2t}$.
This will fix t and (a_5'', c_5). Sometimes for $x_4 \equiv c_4 \pmod 1$ chosen arbitrarily, we work with

$$F(x + c_1, \pm 1, x_4, y + c_5) = \pm x + \beta_F' y - Cy^2 + v,$$

where

$$\beta_F' = \pm a_5' - 2A\lambda x_4 - 2Cc_5.$$

Then we need to find integers $2h$ and k such that

$$|h - Ck^2| + 1/2 < \delta_m. \tag{2.31}$$

To fix a and (a_3, c_3), we will take

$$Q(x + c_1, \pm 1, y + c_3, x_4, x_5) = \pm x + \beta_Q y - ay^2 + v,$$

where $\beta_Q = \pm a_3 - 2a(c_3 + h_4 x_4 + h_5 x_5)$ and find integers $2h$ and k such that

$$|h - ak^2| + 1/2 < d. \tag{2.32}$$

Also

$$G^*(x + c_1, \pm 1, y + c_4) = \pm x + \beta_G^* y - Aty^2/C + v,$$

where $\beta_G^* = \pm a_4^* - 2Atc_4/C$.

Then (2.28), and hence (2.10) will be soluble if there exists integers $2h$ and k such that

$$|h - Atk^2/C| + 1/2 < \delta_{m,M}^*. \tag{2.33}$$

We distinguish cases I to V in the following sections.

3. Case I: m = 3, K = 1

From (2.3) and (2.15) we have

$$3d/4 \leqslant A \leqslant \sqrt{4d^5/9a} \leqslant 4d^2/3. \tag{3.1}$$

So we have from (3.1) and (2.21),

$$9/16 \leqslant d \leqslant 2/3. \tag{3.2}$$

Let $h = 1/2$, $k = 1$. Since $A + d + 2a \geqslant 3d/4 + d + d/2 = 9d/4 > 1$ and $A < \delta_m$ (because $K = 1$), we find that (2.29) is satisfied. Therefore by Macbeath's lemma, (2.10) is soluble unless $A = 1/2$ and $\beta_F \equiv 1/2 \pmod 1$. Taking $x_5 = c_5$ and $1 + c_5$ in β_F we get $\lambda = 0$. Equation (2.4) gives $A \leqslant t$. Now

$$1/2 = A \leqslant t = 2d^5/3a \leqslant 8d^4/3. \tag{3.3}$$

so that

$$2/3 \geqslant d \geqslant (3/16)^{1/4} = 0{\cdot}656\cdots. \tag{3.4}$$

Also $\pm a_4' - c_4 \equiv 1/2 \pmod 1$ gives $(a_4', c_4) = (0, 1/2)$ or $(1/2, 0)$. Taking $h = 1/2$, $k = 1$, and using (3.3) and (3.4) we find that (2.31) is satisfied. Therefore (2.10) is soluble unless $A = 1/2$ and $\beta_F = \pm a_5' - c_5 \equiv 1/2 \pmod 1$ which gives $(a_5', c_5) = (0, 1/2)$ or $(1/2, 0)$.

Because of symmetry in x_4 and x_5 we consider

(i) $a'_4 = a'_5 = 0, c_4 = c_5 = 1/2,$ (3.5)

(ii) $a'_4 = 0, a'_5 = 1/2, c_4 = 1/2, c_5 = 0,$ (3.6)

(iii) $a'_4 = a'_5 = 1/2, c_4 = c_5 = 0.$ (3.7)

Also $1/2 = t = d^5/3aA$ gives $a = 4d^5/3$. Thus

$$Q = (x_1 + a_2 x_2 + \cdots) x_2 - a(x_3 + \cdots)^2 - x_4^2/2 - x_5^2/2.$$

Taking $h = 3/2, k = 3$, and using (3.4) we find that (2.32) is true. Therefore (2.7) is soluble unless $a = 1/6$, so that $d = (1/8)^{1/5}$.

Case (i) $a'_4 = a'_5 = 0, c_4 = c_5 = 1/2$. Here

$$F = (x_1 + a'_2 x_2) x_2 - (x_4^2 + x_5^2)/2 - 1/24.$$

The following table gives a solution of (2.10).

Range	x_1	$x_1 x_2$	x_4	x_5	F
$f'_1 > \frac{7}{24}$	c_1	$\lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$f'_1 - \frac{7}{24}$
$f'_1 < d - \frac{9}{24}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$f'_1 + \frac{17}{24}$
$d - \frac{9}{24} \leqslant f'_1 \leqslant \frac{7}{24}$: $d - \frac{9}{24} \leqslant g'_1 \leqslant \frac{7}{24}$	c_1	$2\lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$f'_2 - \frac{7}{24}$
$d - \frac{9}{24} \leqslant f'_1 \leqslant \frac{7}{24}$: $-\frac{17}{24} < g'_1 < d - \frac{9}{24}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$g'_1 + \frac{17}{24}$
$d - \frac{9}{24} \leqslant f'_1 \leqslant \frac{7}{24}$: $d - \frac{33}{24} \leqslant g'_1 \leqslant -\frac{17}{24}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	$\frac{3}{2}$	$\frac{1}{2}$	$f'_2 + \frac{17}{24}$
$d - \frac{9}{24} \leqslant f'_1 \leqslant \frac{7}{24}$: $g'_1 < d - \frac{33}{24}$	$\pm 2 + c_1$	$2 - \lvert c_1\rvert$	$\frac{1}{2}$	$\frac{1}{2}$	$g'_1 + \frac{41}{24}$

Case (ii) $a'_4 = 0, a'_5 = 1/2, c_4 = 1/2, c_5 = 0$. Here

$$F = (x_1 + a'_2 x_2 + x_5/2) x_2 - (x_4^2 + x_5^2)/2 - 1/24.$$

Range	x_1	$x_1 x_2$	x_4	x_5	F
$f'_1 > \frac{1}{6}$	c_1	$\lvert c_1\rvert$	$\frac{1}{2}$	0	$f'_1 - \frac{1}{6}$
$f'_1 < d - \frac{1}{2}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\frac{1}{2}$	0	$f'_1 + \frac{5}{6}$
$d - \frac{1}{2} \leqslant f'_1 \leqslant \frac{1}{6}$: $d - \frac{1}{2} \leqslant g'_1 \leqslant \frac{1}{6}$	c_1	$2\lvert c_1\rvert$	$\frac{1}{2}$	0	$f'_2 - \frac{1}{6}$
$d - \frac{1}{2} \leqslant f'_1 \leqslant \frac{1}{6}$: $-\frac{5}{6} < g'_1 < d - \frac{1}{6}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\frac{1}{2}$	0	$g'_1 + \frac{5}{6}$
$d - \frac{1}{2} \leqslant f'_1 \leqslant \frac{1}{6}$: $d - \frac{3}{2} \leqslant g'_1 \leqslant -\frac{5}{6}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	$\frac{3}{2}$	0	$f'_2 + \frac{5}{6}$
$d - \frac{1}{2} \leqslant f'_1 \leqslant \frac{1}{6}$: $g'_1 < d - \frac{3}{2}$	$\pm 2 + c_1$	$2 - \lvert c_1\rvert$	$\frac{1}{2}$	0	$g'_1 + \frac{11}{6}$

Case (iii) $a'_4 = a'_5 = 1/2, c_4 = c_5 = 0$. Here

$$F = (x_1 + a'_2 x_2 + x_4/2 + x_5/2) x_2 - (x_4^2 + x_5^2)/2 - 1/24.$$

Range		x_1	x_1x_2	x_2x_4	x_2x_5	x_4	x_5	F
$f_1' > \frac{1}{24}$		c_1	$\lvert c_1\rvert$	0	0	0	0	$f_1' - \frac{7}{24}$
$f_1' < d - \frac{5}{8}$		$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	0	0	0	0	$f_1' + \frac{23}{24}$
$d - \frac{5}{8} \leqslant f_1' \leqslant \frac{1}{24}$	$d - \frac{5}{8} \leqslant g_1'$	c_1	$2\lvert c_1\rvert$	0	0	0	0	$f_2' - \frac{1}{24}$
	$-\frac{23}{24} < g_1' < d - \frac{5}{8}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	0	0	0	0	$g_1' + \frac{23}{24}$
	$d - \frac{13}{8} \leqslant g_1' \leqslant -\frac{23}{24}$	c_1	$2\lvert c_1\rvert$	2	2	± 1	± 1	$f_2' + \frac{23}{24}$
	$g_1' < d - \frac{13}{8}$	$\pm 2 + c_1$	$2 - \lvert c_1\rvert$	0	0	0	0	$g_1' + \frac{47}{24}$

4. Case II: m = 2, K = 1

From (2.3) and (2.15) we have

$$3d/8 \leqslant A \leqslant \sqrt{4d^5/9a} \leqslant \sqrt{4/3}\, d^2. \tag{4.1}$$

Also since $d/3 + d \leqslant a + d \leqslant 1$ (from (2.20)), we have

$$0{\cdot}5412\cdots = 5\sqrt{3}/16 \leqslant d \leqslant 3/4. \tag{4.2}$$

Working as in §3 and using (4.1), (4.2) instead of (3.1), we get that (2.10) is soluble unless $A = 1/2$, $\lambda = 0$ and $(a_4', c_4) = (0, 1/2)$ or $(1/2, 0)$.

From (2.4), we have

$$1/2 = A \leqslant t = 2d^5/3a \leqslant 2d^4 \tag{4.3}$$

so that

$$3/4 \geqslant d \geqslant (1/4)^{1/4} = 0{\cdot}7071\cdots. \tag{4.4}$$

Again working as in §3 and using (4.3), (4.4) we get (2.10) is soluble unless $t = 1/2$, and (a_4', c_4), (a_5', c_5) satisfy either (3.5) or (3.6) or (3.7).

Now $Q = (x_1 + a_2x_2 + \cdots)x_2 - a(x_3 + h_4x_4 + \cdots)^2 - x_4^2/2 - x_5^2/2$. Taking $h = 1$, $k = 2$, and using (4.4) we find that (2.32) is satisfied, as $d/3 \leqslant a \leqslant 1 - d$. Therefore (2.7) is soluble unless $a = 1/4$, so that $d = (3/16)^{1/5} = 0{\cdot}7154\cdots$ and $\beta_Q \equiv 1/2 \pmod{1/2}$ i.e.

$$\pm 2a_3 - (c_3 + h_4x_4 + h_5x_5) \equiv 0 \pmod 1. \tag{4.5}$$

When (3.5) or (3.6) holds, we give as in §3, certain values to x_1, x_2, x_4, x_5 and get a solution of (2.10) of

$$0 < F = (x_1 + a_2'x_2 + a_4'x_4 + a_5'x_5)x_2 - x_5^2/2 - x_4^2/2 - 1/16 < \delta_m = d + 3/16.$$

When (3.7) holds, i.e. when $a_4' = a_5' = 1/2, c_4 = c_5 = 0$, we take $x_4 = c_4$ and $1 + c_4$ in (4.5) and get $h_4 = 0$; similarly $h_5 = 0$; and then $\pm 2a_3 - c_3 \equiv 0 \pmod 1$. By Lemma 7 (proved in the end of this section) with $(b, c) = (a_3, c_3)$ we consider $(a_3, c_3) = (0, 0)$ and $(1/4, 1/2)$. Also from (2.9), $a_4' = a_4, a_5' = a_5$.

Subcase (i) $(a_3, c_3) = (0, 0)$. Here

$$Q = (x_1 + a_2x_2 + x_4/2 + x_5/2)x_2 - x_3^2/4 - x_4^2/2 - x_5^2/2.$$

The following table gives a solution of (2.7).

Range	x_1	x_1x_2	x_3	x_4	x_2x_4	x_5	Q
$f_1 \geqslant d + \frac{1}{4}$	c_1	$2\lvert c_1\rvert$	2	± 1	-2	0	$f_2 - \frac{5}{2}$
$\frac{1}{4} < f_1 < d + \frac{1}{4}$	c_1	$\lvert c_1\rvert$	1	0	0	0	$f_1 - \frac{1}{4}$
$f_1 < d - \frac{3}{4}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	1	0	0	0	$f_1 + \frac{3}{4}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}$, $f_2 < d - \frac{1}{4}$	c_1	$2\lvert c_1\rvert$	1	± 1	2	0	$f_2 + \frac{1}{4}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}$, $d - \frac{1}{4} \leqslant f_2 < d + \frac{1}{4}$	c_1	$2\lvert c_1\rvert$	1	0	0	0	$f_2 - \frac{1}{4}$
$g_1 \geqslant d - \frac{3}{4}$, $f_2 \geqslant d + \frac{1}{4}$	c_1	$2\lvert c_1\rvert$	2	± 1	2	0	$f_2 - \frac{1}{2}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}, -\frac{3}{4} \leqslant g_1 \leqslant d - \frac{3}{4}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	1	0	0	0	$g_1 + \frac{3}{4}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}$, $f_2 < d - \frac{3}{2}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	2	± 1	2	0	$f_2 + \frac{3}{2}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}$, $d - \frac{3}{2} \leqslant f_2 < d - 1$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	2	0	0	0	$f_2 + 1$
$d - \frac{7}{4} \leqslant f_1 \leqslant -\frac{3}{4}$, $f_2 \geqslant d - 1$	c_1	$2\lvert c_1\rvert$	0	± 1	2	0	$f_2 + \frac{1}{2}$
$d - \frac{3}{4} \leqslant f_1 \leqslant \frac{1}{4}, g_1 < d - \frac{7}{4}$	$\pm 2 + c_1$	$2 - \lvert c_1\rvert$	1	0	0	0	$g_1 + \frac{7}{4}$

Subcase (ii) $(a_3, c_3) = (1/4, 1/2)$. Here

$$Q = (x_1 + a_2x_2 + x_3/4 + x_4/2 + x_5/2)x_2 - x_3^2/4 - x_4^2/2 - x_5^2/2.$$

Range	x_1	x_1x_2	x_3	x_2x_3	x_4	x_2x_4	x_5	x_2x_5	Q
$f_1 \geqslant d + \frac{3}{16}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	$\pm\frac{1}{2}$	-1	0	0	0	0	$f_2 - \frac{37}{16}$
$\frac{3}{16} < f_1 < d + \frac{3}{16}$	c_1	$\lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	$f_1 - \frac{3}{16}$
$-\frac{1}{16} < f_1 \leqslant +\frac{3}{16}$	c_1	$\lvert c_1\rvert$	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$f_1 + \frac{1}{16}$
$f_1 < d - \frac{13}{16}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	$f_1 + \frac{13}{16}$
$d - \frac{13}{16} \leqslant f_1 \leqslant \frac{-1}{16}$, $g_1 \geqslant d - \frac{13}{16}$	c_1	$2\lvert c_1\rvert$	$\pm\frac{1}{2}$	-1	± 1	2	± 1	2	$f_2 + \frac{11}{16}$
$d - \frac{13}{16} \leqslant f_1 \leqslant \frac{-1}{16}$, $-\frac{13}{16} \leqslant g_1 < d - \frac{13}{16}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	$g_1 + \frac{13}{16}$
$f_1 \leqslant \frac{-1}{16}$, $g_1 \leqslant -\frac{13}{16}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$g_2 + \frac{17}{16}$

The following lemma is helpful in discarding many values of $(b, c) = (a_3, c_3)$ or (a_4', c_4) or (a_5'', c_5).

Lemma 7. Let Q or F or $G = f(x_1, x_2, y)$ + terms not containing y where $f(x_1, x_2, y) = (x_1 + a_2 x_2 + by)x_2 - \alpha y^2$, $y \equiv c \pmod 1$; $\alpha = r/2s$, r and s are positive integers with $(r, s) = 1$; $\pm b - 2ac \equiv h/k \pmod{1/k, 2a}$ where $2h$ is an integer and k is an integral factor of s. Then we just need to consider

$$\text{(i) } b = 0,\ c = \begin{cases} 0, 1/r, 2/r, \ldots & \text{if } sh/k \text{ is an integer,} \\ 1/2r, 3/2r, \ldots & \text{if } sh/k \text{ is half an integer,} \end{cases} \tag{4.6}$$

$$\text{(ii) } b = 1/2s,\ c = \begin{cases} 0, 1/r, 2/r, \ldots & \text{if } sh/k \text{ is half an integer,} \\ 1/2r, 3/2r, \ldots & \text{if } sh/k \text{ is an integer.} \end{cases} \tag{4.7}$$

Proof. We have

$$\pm b - rc/s \equiv h/k \pmod{1/k, r/s},$$

i.e.

$$\pm sb - rc \equiv sh/k \pmod 1. \tag{4.8}$$

Taking both the signs separately and subtracting, we get $2sb \equiv 0 \pmod 1$, which gives $b = l/2s$, $l \in \mathbb{Z}$. Taking $y \to -y$, if necessary, we can suppose $b \geqslant 0$, i.e. $l \geqslant 0$. Find integers p and q such that $rq - sp = [l/2]$ and consider the unimodular transformation $x_1 \to x_1 + py, x_2 \to x_2, y \to y + qx_2$. We find that

$$f(x_1, x_2, y) \to (x_1 + a_2^* x_2 + b^* y)x_2 - \alpha y^2,$$

where a_2^* is some real number and

$$b^* = \frac{2ps + l - 2rq}{2s} \begin{cases} 0 & \text{if } l \text{ is even} \\ 1/2s & \text{if } l \text{ is odd.} \end{cases}$$

Renaming a_2^* and b^* as a_2 and b respectively, we see that we need to consider $b = 0$ and $1/2s$ only.

When $b = 0$, we have from (4.8) that $rc \equiv sh/k \pmod 1$, and when $b = 1/2s$, we have $rc \equiv sh/k + 1/2 \pmod 1$.

Further changing $x_1 \to -x_1, x_2 \to -x_2, y \to -y$, if necessary, we can suppose that $c \geqslant 0$. This gives the required values of c.

5. Case III: m = 2, K = 2

From (2.3) and (2.15) we have

$$5d/12 \leqslant A \leqslant \sqrt{4d^5/9a} \leqslant \sqrt{4/3}\, d^2. \tag{5.1}$$

This together with (2.21), gives

$$0{\cdot}3608 \cdots = 5\sqrt{3}/24 \leqslant d \leqslant 16/25. \tag{5.2}$$

Further from (2.22), $\delta_{m,K}/t \geqslant 125/192\, d^2 \geqslant 125(25)^2/192(16)^2 > 1$; therefore if L is the integer defined by $L < \delta_{m,K}/t \leqslant L + 1$, we have $L \geqslant 1$. For $L \geqslant 5$ and d as in (5.2), we find that (2.24) is satisfied. We can assume, therefore, that $L \leqslant 4$. From (2.25) and (2.26) we get

$$\delta_{2,2,L} \leqslant 1, \sqrt{125/192(L+1)} \leqslant d \leqslant 64/25(L+3). \tag{5.3}$$

We distinguish the cases $L = 1, 2, 3$ and 4.

5.1 $L = 1, m = 2, K = 2$

From (5.3) we have

$$0{\cdot}5705\cdots = \sqrt{125/384} \leqslant d \leqslant 16/25. \tag{5.4}$$

Taking $h = 1/2$ and $k = 1$ and using (5.4) we find that (2.30) is satisfied, as $\delta_{m,K}/2 \leqslant t < \delta_{m,K}$ and $\delta_{m,K} \geqslant (m+3)(K+3)d/16 = 25d/16 > 2/3$. Therefore (2.14) is soluble unless $t = 1/2$ and $\beta_G = \pm a_5'' - c_5 \equiv 1/2 (\mathrm{mod}\ 1)$. This gives $(a_5'', c_5) = (0, 1/2)$ or $(1/2, 0)$.

Case (i) $(a_5'', c_5) = (0, 1/2)$. Here (2.14) reduces to solve

$$0 < G = (x_1 + a_2'' x_2) x_2 - x_5^2/2 - (a + A)/4 < d + 3a/4 + 3A/4 = \delta_{2,2}.$$

The following table gives a solution of (2.14).

Range		x_1	$x_1 x_2$	x_5	G
$f_1'' > \frac{1}{8} + \frac{a+A}{4}$		c_1	$\lvert c_1\rvert$	$\frac{1}{2}$	$f_1'' - \frac{1}{8} - \frac{a+A}{4}$
$f_1'' < d + a + A - \frac{7}{8}$		$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\frac{1}{2}$	$f_1'' + \frac{7}{8} - \frac{a+A}{4}$
$d + a + A - \frac{7}{8}$ $\leqslant f_1'' \leqslant \frac{1}{8} + \frac{a+A}{4}$	$g_1'' \geqslant d + a + A - \frac{7}{8}$	c_1	$2\lvert c_1\rvert$	$\frac{1}{2}$	$f_2'' - \frac{1}{8} - \frac{a+A}{4}$
	$\frac{-7}{8} + \frac{a+A}{4} < g_1'' < d + a + A - \frac{7}{8}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\frac{1}{2}$	$g_1'' + \frac{7}{8} - \frac{a+A}{4}$
	$g_1'' \leqslant \frac{-7}{8} + \frac{a+A}{4}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	$\frac{3}{2}$	$f_2'' + \frac{7}{8} - \frac{a+A}{4}$

Case (ii) $(a_5'', c_5) = (1/2, 0)$. Here since $a \geqslant d/3$, $A \geqslant 5d/12$ and $t = 1/2$ we have $5d/12 \leqslant A = 2d^5/3a \leqslant 2d^4$; this gives

$$d \geqslant (5/24)^{1/3} = 0{\cdot}592\cdots. \tag{5.5}$$

Also since $\delta_{2,2,1} \leqslant 1$, from (2.25) we have

$$A \leqslant 4[1 - (d + 3a/4)]/3 \leqslant 4(1 - 5d/4)/3. \tag{5.6}$$

Taking $h = 1$ and $k = 2$; using the bound (5.5) for $A > 1/4$, $d < 0{\cdot}624$ and the bound (5.6) for $A > 1/4$, $d \geqslant 0{\cdot}624$, one can easily see that (2.29) holds true. Therefore (2.10) is soluble unless $A = 1/4$ and $\beta_F \equiv 1/2 (\mathrm{mod}\ 1/2)$ i.e. $\pm a_4' - (c_4 + \lambda x_5) \equiv 0 (\mathrm{mod}\ 1)$. This gives as usual $\lambda = 0$. Then from Lemma 7, with $(b, c) = (a_4', c_4)$, we just consider $(a_4', c_4) = (0, 0)$ and $(1/4, 1/2)$ only. Also from (2.13) we have $a_5'' = a_5'$.

Subcase (i) $(a_4', c_4) = (0, 0)$, $(a_5', c_5) = (1/2, 0)$. Here (2.10) reduces to solving

$$0 < F = (x_1 + a_2' x_2 + x_5/2) x_2 - x_4^2/4 - x_5^2/2 - a/4 < d + 3a/4. \tag{5.7}$$

The following table gives a solution.

Range	x_1	x_1x_2	x_4	x_5	F
$f_1' > \frac{a+1}{4}$	c_1	$\lvert c_1\rvert$	1	0	$f_1' - \frac{a+1}{4}$
$\frac{a}{4} < f_1' \leqslant \frac{a+1}{4}$	c_1	$\lvert c_1\rvert$	0	0	$f_1' - \frac{a}{4}$
$f_1' < d + a - \frac{3}{4}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	1	0	$f_1' + \frac{3}{4} - \frac{a}{4}$
$d + a - \frac{3}{4} \leqslant f_1' \leqslant \frac{a}{4}$, $g_1' \geqslant d + a - \frac{3}{4}$	c_1	$2\lvert c_1\rvert$	0	0	$f_2' - \frac{a}{4}$
$d + a - \frac{3}{4} \leqslant f_1' \leqslant \frac{a}{4}$, $\frac{-3}{4} + \frac{a}{4} < g_1' < d + a - \frac{3}{4}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	1	0	$g_1' + \frac{3}{4} - \frac{a}{4}$
$d + a - \frac{3}{4} \leqslant f_1' \leqslant \frac{a}{4}$, $g_1' \leqslant \frac{-3}{4} + \frac{a}{4}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	2	0	$f_1' + 1 - \frac{a}{4}$

Subcase (ii) $(a_4', c_4) = (1/4, 1/2), (a_5', c_5) = (1/2, 0)$. Here

$$F = (x_1 + a_2'x_2 + x_4/4 + x_5/2)x_2 - x_4^2/4 - x_5^2/2 - a/4.$$

Range	x_1	x_1x_2	x_4	x_2x_4	x_5	F
$f_1' \geqslant d + a + \frac{3}{16}$	$\pm 1 + c_1$	$-2 + 2\lvert c_1\rvert$	$\pm\frac{1}{2}$	-1	0	$f_2' - \frac{37}{16}\frac{-a}{4}$
$\frac{3}{16} + \frac{a}{4} < f_1' < d + a + \frac{3}{16}$	c_1	$\lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	$f_1' - \frac{3}{16}\frac{-a}{4}$
$\frac{-1}{16} + \frac{a}{4} < f_1' \leqslant \frac{3}{16} + \frac{a}{4}$	c_1	$\lvert c_1\rvert$	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	$f_1' + \frac{1}{16}\frac{-a}{4}$
$f_1' \leqslant \frac{-1}{16} + \frac{a}{4}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	$f_1' + \frac{13}{16}\frac{-a}{4}$

5.2 $L = 2, m = 2, K = 2$

From (5.3) we have

$$0{\cdot}4658\cdots = 5\sqrt{5}/24 \leqslant d \leqslant 64/125 = 0{\cdot}512. \tag{5.8}$$

Taking $h = 1$ and $k = 2$ and using (5.8) we find that (2.30) is satisfied as $t \geqslant \delta_{2,2}/3$, $\delta_{2,2} \geqslant 25d/16$ and $t = d^5/3aA \leqslant 12d^3/5$. Therefore (2.14) is soluble unless $t = 1/4$ and $\beta_G \equiv 1/2 \pmod{1/2}$ i.e. $\pm 2a_5'' - c_5 \equiv 0 \pmod 1$. By Lemma 7 we need to consider $(a_5'', c_5) = (0, 0)$ or $(1/4, 1/2)$ only. In both these cases, it can be easily seen that (2.14) i.e.

$$0 < G = (x_1 + a_2''x_2 + a_5''x_5)x_2 - x_5^2/4 - (a + A)/4 < d + 3a/4 + 3A/4 = \delta_{2,2}$$

is soluble by giving certain values to x_1, x_2 and x_5 and noting that $1/4 = t = d^5/3aA \leqslant 12d^3/5$ gives $d \geqslant (5/48)^{1/3} = 0{\cdot}4705\cdots$.

5.3 $L=3, m=2, K=2$

From (5.3) we have

$$0{\cdot}4034\cdots = 5\sqrt{15}/48 \leqslant d \leqslant 32/75 = 0{\cdot}4266\cdots. \tag{5.9}$$

Further since $\delta_{2,2,L} = \delta_{2,2} + 2t \leqslant 1$, we have

$$t \leqslant (1-\delta_{2,2})/2 \leqslant (1-25d/16)/2, \tag{5.10}$$

as $\delta_{2,2} \geqslant 25d/16$. Also since $a \geqslant d/3$, $A \geqslant 5d/12$, we have

$$t = d^5/3aA \leqslant 12d^3/5. \tag{5.11}$$

Taking $h=3/2$ and $k=3$; using the bound (5.10) for $t \geqslant 1/6$, $d \geqslant 0{\cdot}42$ and the bound (5.11) for $t > 1/6$, $d < 0{\cdot}42$, one can easily see that (2.30) is satisfied. Therefore (2.14) is soluble unless $t = 1/6$ and $\beta_G \equiv 1/2 (\mathrm{mod}\ 1/3)$ i.e. $\pm 3a_5'' - c_5 \equiv 1/2 (\mathrm{mod}\ 1)$. Then by Lemma 7, we need consider $(a_5'', c_5) = (0, 1/2)$ or $(1/6, 0)$. If $a_5'' = 1/6$, considering the transformation $x_5 \to x_5 - x_2$, we may assume that $a_5'' = 1/2$. Further

$$1/6 = t \leqslant 12d^3/5 \text{ gives } d \geqslant (5/72)^{1/3} = 0{\cdot}411\cdots \tag{5.12}$$

and

$$1/6 = t = d^5/3aA \text{ gives } A = 2d^5/a \leqslant 6d^4. \tag{5.13}$$

Also $\delta_{2,2,3} = d + 3a/4 + 3A/4 + 1/3 \leqslant 1$ gives

$$a + A \leqslant 4(2/3 - d)/3 < 0{\cdot}3409\cdots. \tag{5.14}$$

Lemma 8. *For* $t = 1/6$, $(a_5'', c_5) = (0, 1/2)$, (2.14) *is soluble.*

Proof. Here (2.14) can be written as

$$0 < G = (x_1 + a_2'' x_2)\, x_2 - x_5^2/6 - (a+A)/4 < d + 3a/4 + 3A/4 = \delta_{2,2}.$$

Range		x_1	$x_1 x_2$	x_5	G
$f_1'' > \frac{3}{8} + \frac{a+A}{4}$		c_1	$\lvert c_1\rvert$	$\frac{3}{2}$	$f_1'' - \frac{3}{8} - \frac{a+A}{4}$
$\frac{1}{24} + \frac{a+A}{4} < f_1'' \leqslant \frac{3}{8} + \frac{a+A}{4}$		c_1	$\lvert c_1\rvert$	$\frac{1}{2}$	$f_1'' - \frac{1}{24} - \frac{a+A}{4}$
$f_1'' < d + a + A - \frac{5}{8}$		$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\frac{3}{2}$	$f_1'' + \frac{5}{8} - \frac{a+A}{4}$
$d + a + A - \frac{5}{8} \leqslant f_1'' \leqslant \frac{1}{24} + \frac{a+A}{4}$	$g_1'' \geqslant d + a + A - \frac{5}{8}$	c_1	$2\lvert c_1\rvert$	$\frac{1}{2}$	$f_2'' - \frac{1}{24} - \frac{a+A}{4}$
	$\frac{-5}{8} + \frac{a+A}{4} < g_1'' < d + a + A - \frac{5}{8}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\frac{3}{2}$	$g_1'' + \frac{5}{8} - \frac{a+A}{4}$
	$\frac{-23}{24} + \frac{a+A}{4} < g_1'' \leqslant \frac{-5}{8} + \frac{a+A}{4}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\frac{1}{2}$	$g_1'' + \frac{23}{24} - \frac{a+A}{4}$
	$g_1'' \leqslant \frac{-23}{24} + \frac{a+A}{4}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	$\frac{5}{2}$	$f_2'' + \frac{23}{24} - \frac{a+A}{4}$

Lemma 9. *For* $t=1/6$, $(a_5'', c_5)=(1/2,0)$, (2.14) *is soluble unless* $c_1=1/2$, $a_2''=1/4$, $d\leqslant 5/12$.

Proof. In this case (2.14) can be written as

$$0<G=(x_1+a_2''x_2+x_5/2)x_2-x_5^2/6-(a+A)/4<d+3a/4+3A/4=\delta_{2,2}.$$

Range	x_1	x_1x_2	x_5	x_2x_5	G
$\frac{a+A}{4}<f_1''<d+a+A$	c_1	$\|c_1\|$	0	0	$f_1''-\frac{a+A}{4}$
$\frac{-1}{3}+\frac{a+A}{4}<f_1''\leqslant\frac{a+A}{4}$	c_1	$\|c_1\|$	±1	1	$f_1''+\frac{1}{3}-\frac{a+A}{4}$
$f_1''\leqslant\frac{-1}{3}+\frac{a+A}{4}$ $g_1''\geqslant -1+d+a+A$	c_1	$-2\|c_1\|$	±2	4	$g_2''+\frac{4}{3}-\frac{a+A}{4}$
$g_1''<-1+d+a+A$	$\pm1+c_1$	$1-\|c_1\|$	0	0	$g_1''+1-\frac{a+A}{4}$
$f_1''\geqslant d+a+A$, $g_1''>\frac{-1}{3}+\frac{a+A}{4}$	c_1	$-\|c_1\|$	±1	1	$g_1''+1-\frac{a+A}{4}$

The above table shows that (2.14) is soluble unless

$$f_1''\geqslant d+a+A \text{ and } g_1''\leqslant -\frac{1}{3}+\frac{a+A}{4}. \tag{5.15}$$

Case (i) $a_2''\leqslant 1/4$.

Let

$$a_2''=1/4-\varepsilon_2,\ |c_1|=1/2-\varepsilon_1,\ \varepsilon_1\geqslant 0,\ \varepsilon_2\geqslant 0. \tag{5.16}$$

From (5.15), $d+a+A\leqslant f_1''=|c_1|+a_2''$, we get

$$\varepsilon_1+\varepsilon_2\leqslant 3/4-(d+a+A). \tag{5.17}$$

Suppose $(\varepsilon_1,\varepsilon_2)\neq(0,0)$. Choose an integer $l(\geqslant 1)$ such that

$$\varepsilon_1 3^{(l-1)}+\varepsilon_2(3^{(l-1)})^2\leqslant 3/4-(d+a+A)<\varepsilon_1 3^l+\varepsilon_2(3^l)^2. \tag{5.18}$$

Let p be an integer such that

$$2p^2+9\equiv 0(\text{mod } 3^{l+1}).$$

Clearly

$$f(p,l)=9+2p^2+2.3^{l+1}p-3^{2l+1}-2.3^{l+1}\equiv 0(\text{mod } 3^{l+1}).$$

Also

$$f(p,l)=2p(p+3^{l+1})-3(3^l+1)^2+12\equiv 0(\text{mod } 4).$$

Let q be an integer such that $f(p,l)=4.3^{l+1}.q$, so that

$$q.3^l+3^l/2+3^{2l}/4-3^lp/2-p^2/6=3/4. \tag{5.19}$$

Take $x_2 = \pm 3^l$, $x_5 = \pm p, x_1 = \pm q + c_1$, such that $x_1x_2 = q.3^l + 3^l|c_1|$, $x_2x_5 = -p.3^l$, so that

$$\begin{aligned} G &= q3^l + 3^l|c_1| + 3^{2l}a_2'' - p3^l/2 - p^2/6 - (a+A)/4 \\ &= 3/4 - 3^l\varepsilon_1 - 3^{2l}\varepsilon_2 - (a+A)/4 \quad \text{(from (5.16) and (5.19)} \\ &< d + 3a/4 + 3A/4 = \delta_{2,2} \quad \text{(from (5.18)).} \end{aligned}$$

Also

$$\begin{aligned} G &= 3/4 - (a+A)/4 - \left(\frac{3^l}{3^{l-1}}\right)^2\left[\varepsilon_1\frac{(3^{l-1})^2}{3^l} + \varepsilon_2(3^{l-1})^2\right] \\ &> 3/4 - (a+A)/4 - 9[\varepsilon_1 3^{(l-1)} + \varepsilon_2(3^{(l-1)})^2] \\ &\geqslant 3/4 - (a+A)/4 - 9[3/4 - (d+a+A)] \quad \text{(from (5.18))} \\ &= 9d + 35(a+A)/4 - 6 > 0. \end{aligned}$$

Case (ii) $a_2'' > 1/4$. Let

$$a_2'' = 1/4 + \varepsilon_2, |c_1| = 1/2 - \varepsilon_1, \varepsilon_1 \geqslant 0, \varepsilon_2 > 0. \tag{5.20}$$

From (5.15), we get

$$\varepsilon_1 + \varepsilon_2 \leqslant (a+A)/4 - 1/12. \tag{5.21}$$

Choose an integer $l(\geqslant 1$ because of (5.21)) such that

$$\varepsilon_1 3^{(l-1)} + \varepsilon_2(3^{(l-1)})^2 \leqslant (a+A)/4 - 1/12 < \varepsilon_1 3^l + \varepsilon_2(3^l)^2. \tag{5.22}$$

Let p be an integer such that $2p^2 + 1 \equiv 0 \pmod{3^{l+1}}$. Clearly

$$f(p,l) = 1 + 2p^2 + 6.3^l - 3.3^{2l} + 6.3^l p \equiv 0 \pmod{3^{l+1}}.$$

Also $f(p,l) = 2p(p+3^{l+1}) - 3(3^l-1)^2 + 4 \equiv 0 \pmod 4$. Let q be an integer such that $f(p,l) = 4.3^{l+1}.q$, so that

$$q.3^l - 3^l/2 + 3^{2l}/4 - 3^l p/2 - p^2/6 = 1/12. \tag{5.23}$$

Now take $x_2 = \pm 3^l, x_5 = \pm p, x_1 = \pm q + c_1$, such that $x_1x_2 = q3^l - 3^l|c_1|, x_2x_5 = -p3^l$, so that we have

$$\begin{aligned} G &= q3^l - 3^l|c_1| + 3^{2l}a_2'' - p3^l/2 - p^2/6 - (a+A)/4 \\ &= 1/12 + 3^l\varepsilon_1 + 3^{2l}\varepsilon_2 - (a+A)/4 \qquad \text{(using (5.20) and (5.23))} \\ &> 0. \qquad \text{(using (5.22))} \end{aligned}$$

Also

$$\begin{aligned} G &= \left(\frac{3^l}{3^{l-1}}\right)^2\left[\varepsilon_1\frac{(3^{l-1})^2}{3^l} + \varepsilon_2(3^{l-1})^2\right] - [(a+A)/4 - 1/12] \\ &< 9[\varepsilon_1 3^{(l-1)} + \varepsilon_2(3^{(l-1)})^2] - [(a+A)/4 - 1/12] \\ &\leqslant 9[(a+A)/4 - 1/12] - [(a+A)/4 - 1/12] \quad \text{(from (5.22))} \\ &= 2(a+A) - 2/3 < d + 3a/4 + 3A/4. \end{aligned}$$

Thus we see from case (i) and case (ii) that (2.14) is soluble unless $(\varepsilon_1, \varepsilon_2) = (0, 0)$ i.e. unless $a_2'' = 1/4$, $c_1 = 1/2$. Further from (5.19) we get $d + a + A \leqslant 3/4$ and $a + A \geqslant 1/3$, this gives $d \leqslant 5/12$.

Lemma 10. *When* $t = 1/6$, $(a_5'', c_5) = (1/2, 0)$, $c_1 = 1/2$, $a_2'' = 1/4$ *and* $0{\cdot}411\cdots \leqslant d \leqslant 5/12$, *then* (2.10) *is soluble.*

Proof. Since from (3.13) we have

$$1/4 = a_2'' \equiv a_2' + a_4'^2/4A (\text{mod } 1), \quad 1/2 = a_5'' \equiv a_5' - \lambda a_4' (\text{mod } 1),$$

and

$$-1/2 < a_2' \leqslant a_2' + a_4'^2/4A \leqslant a_2' + 1/16A < 1, \ -3/4 \leqslant a_5' - \lambda a_4' \leqslant 3/4,$$

so we have

$$a_2' + a_4'^2/4A = 1/4 \text{ and } a_5' - \lambda a_4' = 1/2 \text{ or } -1/2. \tag{5.24}$$

Since here

$$2 < 45d/8 \leqslant 9\delta_2/2 \leqslant \delta_2/C = (d + 3a/4)/C \leqslant \delta_2/A \leqslant 3,$$

we have $M = 2$, and

$$1/6 \geqslant At/C = A/6C = A/(6A\lambda^2 + 1) \geqslant A/(3A/2 + 1) > 1/8.$$

(as $A \geqslant 5d/12$ and $d > 0{\cdot}411\cdots$).

Taking $h = 3/2$, $k = 3$ for $A/6C > (2 - \delta_{2,2}^*)/9$ and $h = 1/2$, $k = 2$ for $A/6C \leqslant (2 - \delta_{2,2}^*)/9$ and noting that $\delta_{2,2}^* \geqslant \delta_{2,2} \geqslant 25d/16 > 0{\cdot}64$, we find that (2.33) is satisfied. Therefore (2.28) and hence (2.10) is soluble unless $A/6C = 1/6$ i.e. $A = C$ and $\beta_G^* \equiv 1/2$ (mod 1/3) i.e. $\pm 3a_4^* - c_4 \equiv 1/2$ (mod 1). From Lemma 7, we need to consider $(a_4^*, c_4) = (1/2, 0)$ or $(0, 1/2)$. Proceeding as in Lemmas 8 and 9, we find that (2.28) has a solution unless $(a_4^*, c_4) = (1/2, 0)$ and $a_2^* = 1/4$.

Working as before, we get from (2.27) that

$$a_2' + a_5'^2/4A = 1/4 \text{ and } a_4' - \lambda a_5' = 1/2 \text{ or } -1/2. \tag{5.25}$$

From (5.24) and (5.25), we get $a_4'^2 = a_5'^2$, so $a_4' = \pm a_5'$. As $A = C = A\lambda^2 + 1/6$, we get $A = 1/6(1 - \lambda^2)$. Further $5d/12 \leqslant A \leqslant 6d^4$, $0{\cdot}411\cdots \leqslant d \leqslant 5/12$ gives $0{\cdot}1635\cdots < \lambda \leqslant 0{\cdot}28$. If $a_4' = a_5'$, we have from (5.27) that $a_5' = \pm 1/2(1 - \lambda)$ giving thereby $|a_5'| > 1/2$, a contradiction. Therefore, we must have $a_4' = -a_5' = \pm 1/2(1 + \lambda)$. So $a_2' = 1/4 - a_5'^2/4A = (5\lambda - 1)/8(1 + \lambda)$. But then $x_1 = 1/2$, $x_2 = 1$, $x_4 = x_5 = 0$ gives a solution of (2.10). This completes the proof of the lemma.

5.4 $L = 4$, $m = 2$, $K = 2$

From (5.3) we have

$$0{\cdot}3608\cdots = 5\sqrt{3}/24 \leqslant d \leqslant 64/175 = 0{\cdot}3657\cdots. \tag{5.26}$$

Since $t \leqslant 12d^3/5 < 1/8$ and $t \geqslant \delta_{2,2}/5$, $\delta_{2,2} \geqslant 25d/16$, taking $h = 1/2$ and $k = 2$ and using (5.26) we find that (2.30) is satisfied. Therefore (2.14) is soluble by Macbeath's Lemma.

6. Case IV: m = 1, K = 2

Here since $a \geqslant d/2$, (2.1) and (2.20) give $d/2 \leqslant a \leqslant (2d^5/3)^{1/3}\, d/2 + d \leqslant a + d \leqslant 1$, which further gives

$$0{\cdot}433\cdots = \sqrt{3}/4 \leqslant d \leqslant 2/3. \tag{6.1}$$

Also from (2.6) we have

$$A \geqslant 3a/4 \geqslant 3d/8. \tag{6.2}$$

Further from (2.22) and (6.1) we have $\delta_{m,K}/t \geqslant 5/8d^2 > 1$; therefore the integer L defined by $L < \delta_{1,2}/t \leqslant L+1$ is at least 1. For $L \geqslant 5$ and d as in (6.1) we find that (2.24) is satisfied. We can assume therefore that $L \geqslant 4$. From (6.2) we get

$$\delta_{1,2} = d + 3A/4 \geqslant 41d/32. \tag{6.3}$$

Further since from (2.25) and (6.3)

$$\frac{L+3}{4}\cdot\frac{41}{32}\,d \leqslant \frac{L+3}{4}\cdot\delta_{1,2} \leqslant \delta_{1,2} + \frac{L-1}{4}\cdot t = \delta_{1,2,L} \leqslant 1,$$

we get

$$d \leqslant 128//41(L+3) \tag{6.4}$$

and for $L \neq 1$,

$$t \leqslant \frac{4}{L^2-1}(1-\delta_{1,2}) \leqslant \frac{4}{L^2-1}(1-41d/32). \tag{6.5}$$

Also by definition of L, (6.2) and (6.3)

$$\frac{41}{32}\,\frac{d}{(L+1)} \leqslant \frac{\delta_{1,2}}{(L+1)} \leqslant t = d^5/3aA \leqslant 16d^3/9, \tag{6.6}$$

which gives

$$d^2 \geqslant 369/512(L+1). \tag{6.7}$$

We distinguish the cases $L = 1$, 2, 3 and 4.

6.1 $L = 1, m = 1, K = 2$

From (6.1) and (6.7) we have

$$0{\cdot}6 < \sqrt{369/1024} \leqslant d \leqslant 2/3. \tag{6.8}$$

Working as in §5.1 and using (6.6), (6.3) and (6.8) we find that (2.14) is soluble unless $t = 1/2$ and $(a_5'', c_5) = (0, 1/2)$ or $(1/2, 0)$.

When $(a_5'', c_5) = (0, 1/2)$, we get as in §5.1, a solution of (2.14) directly. When $(a_5'', c_5) = (1/2, 0)$, we apply Macbeath's lemma to F and get that (2.10) is soluble unless $A = 1/4$ and $(a_4', c_4) = (0, 0)$ or $(1/4, 1/2)$. In each of these subcases a table similar to that of §5.1 gives a solution of (2.10).

6.2 $L = 2, m = 1, K = 2$

From (6.7) and (6.4) we have

$$0{\cdot}4901\cdots = \sqrt{123/512} \leqslant d \leqslant 128/205 = 0{\cdot}62439\cdots. \tag{6.9}$$

Taking

$$h=1/2,\ k=1 \text{ for } t>1-\delta_{1,2},$$
$$h=1,\ k=2 \text{ for } (3/2-\delta_{1,2})/4<t\leqslant \min(16d^3/9, 1-\delta_{1,2}),$$
$$h=2,\ k=3 \text{ for } t\leqslant (3/2-\delta_{1,2})/4<2/9,$$

and using (6.6) and (6.9), we find that (2.30) is satisfied. Therefore (2.14) is soluble unless $t=1/4$ and $\beta_G\equiv 1/2 (\mathrm{mod}\ 1/2)$ i.e. $\pm 2a_5''-c_5\equiv 0 \pmod 1$. From Lemma 7, we just need to consider $(a_5'', c_5)=(0,0)$ or $(1/4, 1/2)$.

Also $1/4=t\leqslant 16d^3/9$ gives

$$d\geqslant (9/64)^{1/3}=0{\cdot}52\cdots. \tag{6.10}$$

Tables similar to those of § 5.2 give a solution of (2.14).

6.3 $L=3, m=1, K=2$

From (6.1), (6.4) and (6.3) we get

$$\sqrt{3}/4\leqslant d\leqslant 64/123,\ \delta_{1,2}\geqslant 41d/32>0{\cdot}5547\cdots. \tag{6.11}$$

Also from (6.6) and (6.5) we have

$$\begin{aligned} 0{\cdot}1386<41d/128\leqslant t &\leqslant \min\{16d^3/9, (1-41d/32)/2\} \\ &= f_0(d), \text{ say} \\ &= \begin{cases} 16d^3/9 & \text{for } d\leqslant \gamma_0 \\ (1-41d/32)/2 & \text{for } d\geqslant \gamma_0, \end{cases} \end{aligned} \tag{6.12}$$

where γ_0 is positive real root of the cubic equation $16d^3/9+41d/64-1/2=0$. One finds that $0{\cdot}477<\gamma_0<0{\cdot}478$; and then $t\leqslant f(\gamma_0)<0{\cdot}19417=\lambda_0$, say.

We divide the range of t i.e. $0{\cdot}1386<t<0{\cdot}19417=\lambda_0$ into 19 subintervals $[\lambda_n, \lambda_{n-1}]$, $n=1, 2, \ldots, 19$ and in each subinterval choose suitable integers $2h_n$ and k_n satisfying (2.30) i.e.

$$|h-tk^2|+1/2<41d/32\leqslant\delta_{1,2}. \tag{6.13}$$

We take

n	h_n	k_n	λ_n	n	h_n	k_n	λ_n
1	3	4	0·18223	2	4·5	5	0·17645
3	1·5	3	0·15879	4	2·5	4	0·15227
5	5·5	6	0·15103	6	25·5	13	0·15062
7	15	10	0·149393	8	9·5	8	0·14752
9	65	21	0·147261	10	42·5	17	0·14689
11	33	15	0·14642	12	37·5	16	0·146284
13	58·5	20	0·14612	14	21	12	0·14553
15	14·5	10	0·14453	16	17·5	11	0·14418
17	52	19	0·1439	18	7	7	0·14174
19	3·5	5	0·1386				

ıe choice of (h_n, k_n) is done in the following way:

For a fixed value of the pair (h_n, k_n), $2h_n, k_n \in \mathbb{Z}$, (6.13) is satisfied if t lies in the interval $I_n = (a_n, b_n)$, where $a_n = a_n(d) = (h_n + 1/2 - 41d/32)/k_n^2, b_n = b_n(d) = (h_n - 1/2 + 41d/32)/k_n^2$ and h_n/k_n^2 is the centre of the interval I_n.

We take $(h_1, k_1) = (3, 4)$ so that the centre of I_1 namely $h_1/k_1^2 = 3/16$ lies to the left of λ_0 (or is very close to it) and also $f_0(d) < b_1(d)$. So for $t \in I_1 = (a_1, b_1)$, (6.13) is true.

Let now $t \leqslant a_1(d)$, also $t \leqslant 16d^3/9$, so that $t \leqslant \min\{16d^3/9, a_1(d)\} = f_1(d)$, say. If γ_1 is the positive real root of the cubic equation $16d^3/9 - a_1(d) = 0$; we compute some good enough bounds of γ_1 namely $0{\cdot}467 < \gamma_1 < 0{\cdot}468$ and so $t \leqslant f_1(d) \leqslant f_1(\gamma_1) < \lambda_1 = 0{\cdot}18223$. This means that (6.13) is satisfied for the region $\lambda_1 \leqslant t \leqslant \lambda_0$ by the pair (h_1, k_1).

We take $(h_2, k_2) = (4{\cdot}5, 5)$ so that $h_2/k_2^2 < \lambda_1$ and $f_1(d) < b_2(d)$. Repeating this process nineteen times, we find that (2.14) is soluble unless $t = h_n/k_n^2$, $1 \leqslant n \leqslant 19$.

For some of the central values of $t = h_n/k_n^2$ namely for $n = 2, 5, 6, 8, 9, 11$–$13, 15$–$17, 19$, we find that (6.13) is satisfied for some other choice of the pair (h, k). They are listed below.

We note that for $t = h_n/k_n^2 \leqslant 16d^3/9$, we have

$$d \geqslant (9t/16)^{1/3} = (9h_n/16k_n^2)^{1/3}. \tag{6.14}$$

n	h	k	t	n	h	k	t
2	6·5	6	4·5/25	5	2·5	4	5·5/36
6	34	15	25·5/169	8	12	9	9·5/64
9	59	20	65/441	11	37·5	16	33/225
12	33	15	37·5/256	13	64·5	21	58·5/400
15	17·5	11	14·5/100	16	14·5	10	17·5/121
17	63·5	21	52/361	19	5	6	3·5/25

Therefore by Macbeath's lemma, (2.14) is soluble unless $t = h_n/k_n^2$, $n = 1, 3, 4, 7, 10, 14, 18$ i.e. unless $t = 3/16, 1/6, 5/32, 3/20, 5/34, 7/48$ or $1/7$ and $\beta_G \equiv h_n/k_n (\mathrm{mod}\ 1/k_n, 2t)$. In each of these cases we will give solutions of (2.14) or (2.10) depending upon values of a_2'' and c_1.

From (6.14), (2.6) and (2.4) we have

$$d + A + a/4 \geqslant 3d/2 \geqslant 3(9t/16)^{1/3}/2,\ (a + A)/4 \leqslant 7A/12 \leqslant 7t/9. \tag{6.15}$$

Remark 1. We have $t = r/2s$, $(r, s) = 1$, $a_5'' - 2tc_5 \equiv h/k (\mathrm{mod}\ 1/k, 2t)$, where k is an integral factor of s. Therefore by Lemma 7, with $(b, c) = (a_5'', c_5'')$, we just need to consider $a_5'' = 0$ and $a_5'' = 1/2s$, c_5 as given in (4.6) or (4.7).

If we take $x_2 = \pm 1$ only (which we will do except in case when $t = 1/6$) then, for $a_5'' = 1/2s$, we, in fact need to consider the following values of c_5 only.

$$c_5 = \begin{cases} 1/2r,\ 5/2r,\ 9/2r, \ldots & \text{if } sh/k \text{ is an integer.} \\ 1/r,\ 3/r,\ 5/r, \ldots & \text{if } sh/k \text{ is half an integer.} \end{cases}$$

This is because, for $c_5 = l/2r$, $l \geqslant 0$, we have

$$G(-x+c_1, -1, y+c_5) = x+(-c_1+a_2'')-\frac{r}{2s}\left(y+\frac{l}{2+s}\right)\left(y+\frac{l+2}{2s}\right)-\frac{a+A}{4}.$$

Also for $c_5 = (l+2)/2r$, we have

$$G(x+c_1, 1, y+c_5) = x+(c_1+a_2'')-\frac{r}{2s}\left(y+\frac{l}{2+s}\right)\left(y+\frac{l+2}{2s}\right)-\frac{a+A}{4}.$$

Since $p_1'' = c_1 + a_2''$ and $q_1'' = -c_1 + a_2''$ both lie in the same interval $[-1,1]$, the same values of integral pairs (x, y) will give a solution of (2.14) for both $c_5 = l/2r$ and $c_5 = (l+2)/2r$.

Lemma 11. *If* $t = 3/16$, *then* (2.14) *is soluble.*

Proof. By Lemma 7 and Remark 1 we need to consider $(a_5'', c_5) = (0, 0)$, $(0, 1/3)$ and $(1/16, 1/6)$. From (6.14) and (6.15) we get

$$d+A+a/4 \geqslant 0{\cdot}7, \quad d+3A/4 > 0{\cdot}6, \quad (a+A)/4 \leqslant 7/48.$$

Case (i) $(a_5'', c_5) = (0, 0)$.

$$0 < G = (x_1 + a_2'' x_2)x_2 - 3x_5^2/16 - (a+A)/4 < d+3A/4.$$

Range	x_1	x_1x_2	x_5	G
$f_1'' > \frac{3}{4}+\frac{a+A}{4}$	c_1	$\lvert c_1\rvert$	2	$f_1''-\frac{3}{4}-\frac{a+A}{4}$
$\frac{3}{16}+\frac{a+A}{4} < f_1'' \leqslant \frac{3}{4}+\frac{a+A}{4}$	c_1	$\lvert c_1\rvert$	1	$f_1''-\frac{3}{16}-\frac{a+A}{4}$
$\frac{-1}{4}+\frac{a+A}{4} < f_1'' \leqslant \frac{3}{16}+\frac{a+A}{4}$	$\pm 1+c_1$	$1+\lvert c_1\rvert$	2	$f_1''+\frac{1}{4}-\frac{a+A}{4}$
$f_1'' \leqslant \frac{-1}{4}+\frac{a+A}{4}$	$\pm 1+c_1$	$1+\lvert c_1\rvert$	1	$f_1''+\frac{13}{16}-\frac{a+A}{4}$

Case (ii) $(a_5'', c_5) = (0, 1/3)$.

Range	x_1	x_1x_2	x_5	G
$f_1'' > \frac{25}{48}+\frac{a+A}{4}$	c_1	$\lvert c_1\rvert$	$\frac{-5}{3}$	$f_1''-\frac{25}{48}-\frac{a+A}{4}$
$\frac{1}{48}+\frac{a+A}{4} < f_1'' \leqslant \frac{25}{48}+\frac{a+A}{4}$	c_1	$\lvert c_1\rvert$	$\frac{1}{3}$	$f_1''-\frac{1}{48}-\frac{a+A}{4}$
$\frac{-23}{48}+\frac{a+A}{4} < f_1'' \leqslant \frac{1}{48}+\frac{a+A}{4}$	$\pm 1+c_1$	$1+\lvert c_1\rvert$	$\frac{-5}{3}$	$f_1''+\frac{23}{48}-\frac{a+A}{4}$
$f_1'' \leqslant \frac{-23}{48}+\frac{a+A}{4}$	$\pm 1+c_1$	$1+\lvert c_1\rvert$	$\frac{1}{3}$	$f_1''+\frac{47}{48}-\frac{a+A}{4}$

Case (iii) $(a_5'', c_5) = (1/16, 1/6)$. Let $x_2 = -1, x_1 = x + c_1, x_5 = y + 1/6$, so that we can find integers x and y such that

$$0 < G = x + q_1'' - \frac{3}{16}\left(y + \frac{1}{6}\right)\left(y + \frac{1}{2}\right) - \frac{a + A}{4} < d + 3A/4.$$

Range	x	y	G
$q_1'' > \frac{21}{64} + \frac{a+A}{4}$	0	1	$q_1'' - \frac{21}{64} - \frac{a+A}{4}$
$\frac{1}{64} + \frac{a+A}{4} < q_1'' \leqslant \frac{21}{64} + \frac{a+A}{4}$	0	0	$q_1'' - \frac{1}{64} - \frac{a+A}{4}$
$\frac{-31}{64} + \frac{a+A}{4} < q_1'' \leqslant \frac{1}{64} + \frac{a+A}{4}$	1	-2	$q_1'' - \frac{31}{64} - \frac{a+A}{4}$
$\frac{-63}{64} + \frac{a+A}{4} < q_1'' \leqslant \frac{-31}{64} + \frac{a+A}{4}$	1	0	$q_1'' + \frac{63}{64} - \frac{a+A}{4}$
$q_1'' \leqslant \frac{-63}{64} + \frac{a+A}{4}$	2	-2	$q_1'' + \frac{95}{64} - \frac{a+A}{4}$

Lemma 12. *If* $t = 5/32$ *or* $3/20$ *or* $5/34$ *or* $7/48$ *or* $1/7$, *then* (2.14) *is soluble.*

Proof. By Lemma 7 and Remark 1 for $t = 5/32$ we need to consider $(a_5'', c_5) = (0,0)$, $(0,1/5)$, $(0,2/5)$, $(1/32, 1/10)$ and $(1/32, 1/2)$; for $t = 3/20$ we need to consider $(a_5'', c_5) = (0, 0)$, $(0, 1/3)$ and $(1/20, 1/6)$; for $t = 5/34$ we need to consider $(a_5'', c_5) = (0, 1/10)$, $(0, 3/10)$, $(0, 1/2)$ and $(1/34, 1/5)$; for $t = 7/48$ we need to consider $(a_5'', c_5) = (0, 0), (0, 1/7)$, $(0, 2/7), (0, 3/7), (1/48, 1/14), (1/48, 5/14)$ and for $t = 1/7$ we need to consider $(a_5'', c_5) = (0, 0), (0, 1/2), (1/14, 1/4)$ only. Now taking different values of x_1, x_2 and x_5 depending upon the ranges of f_1'', p_1'' or q_1'' it can be easily shown as in previous lemma that (2.14) is soluble.

Lemma 13. *If* $t = 1/6$, *then* (2.14) *is soluble unless* $(a_5'', c_5) = (1/2, 0)$, $a_2'' = 1/4$, $c_1 = 1/2$.

Proof. We work as in § 5.3 and in Lemmas 8 and 9. Here we have from (6.14) and (6.15) that

$$d > 0{\cdot}45428, \delta_{1,2} > 0{\cdot}582, d + A + a/4 > 0{\cdot}6814 \text{ and } (a + A)/4 \leqslant 0{\cdot}13.$$

When $(a_5'', c_5) = (0, 1/2)$, a table similar to that in Lemma 8 gives a solution of (2.14).

When $(a_5'', c_5) = (1/2, 0)$ we find that (2.14) is soluble unless

$$f_1'' \geqslant d + A + \frac{a}{4} \quad \text{and} \quad g_1'' \leqslant \frac{-1}{3} + \frac{a+A}{4}.$$

Now working exactly as in Lemma 9 with d replaced by $d - 3a/4$, we find that (2.14) is soluble unless $a_2'' = 1/4$ and $c_1 = 1/2$.

Lemma 14. *If* $t = 1/6$, $(a_5'', c_5) = (1/2, 0)$, $a_2'' = 1/4$, $c_1 = 1/2$, $d \geqslant 0{\cdot}45428$, *then* (2.10) *is soluble.*

Proof. We work as in Lemma 10 and find that (5.24) holds here also. Since $2 < \delta_m/C = d/C < 3$, we get $M = 2$. Also we note, for $A \geqslant 3d/8$, $d \geqslant 0{\cdot}45428$, that

$$1/6 \geqslant A/6C \geqslant A/(3A/2+1) > 1/8,\ \delta^*_{1,2} \geqslant \delta_{1,2} = d + 3A/4 > 0{\cdot}582. \tag{6.16}$$

We take

$h = 1{\cdot}5,\ k = 3$ for $(2-\delta^*_{1,2})/9 < A/6C \leqslant 1/6$

$h = 2{\cdot}5,\ k = 4$ for $(3-\delta^*_{1,2})/16 < A/6C \leqslant (2-\delta^*_{1,2})/9$

$h = 5{\cdot}5,\ k = 6$ for $(6-\delta^*_{1,2})/36 < A/6C \leqslant (3-\delta^*_{1,2})/16$

$h = 15,\ k = 10$ for $(15{\cdot}5-\delta^*_{1,2})/100 < A/6C \leqslant (6-\delta^*_{1,2})/36$

$h = 9{\cdot}5,\ k = 8$ for $(10-\delta^*_{1,2})/64 < A/6C \leqslant (15{\cdot}5-\delta^*_{1,2})/100$

$h = 0{\cdot}5,\ k = 2$ for $A/6C = 9{\cdot}5/64$ $\left(\text{here } \delta^*_{1,2} = d + \frac{3}{4}\frac{64}{9{\cdot}5} > 0{\cdot}597\right)$

$h = 42{\cdot}5,\ k = 17$ for $(43-\delta^*_{1,2})/289 < A/6C \leqslant (10-\delta^*_{1,2})/64$

$h = 0{\cdot}5,\ k = 2$ for $A/6C = \dfrac{42{\cdot}5}{289} = \dfrac{5}{34}$ (here $\delta^*_{1,2} > 0{\cdot}59$)

$h = 33,\ k = 15$ for $(33{\cdot}5-\delta^*_{1,2})/225 < A/6C \leqslant (43-\delta^*_{1,2})/289$

$h = 0{\cdot}5,\ k = 2$ for $A/6C = \dfrac{33}{225}$(here $\delta^*_{1,2} > 0{\cdot}59$)

$h = 21,\ k = 12$ for $(33{\cdot}5-\delta^*_{1,2})/225 < A/6C \leqslant (43-\delta^*_{1,2})/289$

$h = 0{\cdot}5,\ k = 2$ for $A/6C = \dfrac{21}{144}$(here $\delta^*_{1,2} > 0{\cdot}6$)

$h = 0{\cdot}5,\ k = 2$ for $1/8 < A/6C \leqslant (21{\cdot}5 - \delta^*_{1,2})/144$

and find that (2.33) is satisfied. Therefore by Macbeath's lemma (2.28) and hence (2.10) is soluble unless $A/6C = 1/6$, $2{\cdot}5/16$ or $15/100$. When $A/6C = 2{\cdot}5/16$ or $15/100$, working as in Lemmas 11 and 12, it can be easily shown that (2.28) is soluble.

When $A/6C = 1/6$ i.e. $A = C = 1/6(1-\lambda^2)$, (2.28) is soluble unless $(a^*_4, c_4) = (0, 1/2)$ or $(1/2, 0)$. Working exactly as in Lemma 10 we find that $a'_4 = -a'_5 = \pm 1/2(1+\lambda)$, $a'_2 = 1/4 - a'^2_4/4A = (5\lambda-1)/8(1+\lambda)$. Since here $1/6(1-\lambda^2) = A \geqslant 3d/8$ and $d \geqslant 0{\cdot}45428$, we get $\lambda > 0{\cdot}147$. Now $F(1/2, 1, \pm 1, \mp 1)$ provides a solution of (2.10).

7. Case V: m = 1, k = 1

Here since $a \geqslant d/2$, $A \geqslant d/2$, we have from (2.3) and (2.20) that

$$d/2 \leqslant A \leqslant \sqrt{4d^5/9a} = \sqrt{8/9}\, d^2,\ d/2 + d \leqslant a + d \leqslant 1.$$

This gives

$$0{\cdot}53033\cdots < 3/4\sqrt{2} \leqslant d \leqslant 2/3. \tag{7.1}$$

Also from (2.3) and (2.4) we have

$$3d/8 \leqslant 3A/4 \leqslant t = d^5/3aA \leqslant 4d^3/3. \tag{7.2}$$

Using (7.1) and (7.2) we find that $0{\cdot}1988 < t < 0{\cdot}3951 = \lambda_0$ (say). We divide the range of

t into 72 subintervals $[\lambda_n, \lambda_{n-1}]$, $n = 1, 2, \ldots, 72$, and in each subinterval choose suitable integers $2h_n$ and k_n (listed below) satisfying (2.30) i.e.

$$|h - tk^2| + 1/2 < d = \delta_{1,1}. \tag{7.3}$$

The pair (h_n, k_n) $2h_n$, $k_n \in \mathbb{Z}$, is chosen in the same way as described in § 6.3.

n	h_n	k_n	λ_n	n	h_n	k_n	λ_n
1	1·5	2	0·3414	2	3	3	0·32009
3	5	4	0·3057	4	19·5	8	0·3031
5	7·5	5	0·2959	6	14·5	7	0·2938
7	10·5	6	0·28888	8	41·5	12	0·2876
9	14	7	0·2839	10	4·5	4	0·2757
11	13·5	7	0·2737	12	17·5	8	0·27218
13	1	2	0·2350	14	15	8	0·23353
15	52·5	15	0·2331	16	75·5	18	0·2329
17	33·5	12	0·2323	18	45·5	14	0·23186
19	28	11	0·23104	20	39	13	0·23054
21	23	10	0·22955	22	33	12	0·22881
23	18·5	9	0·22782	24	2	3	0·21748
25	3·5	4	0·21596	26	17·5	9	0·21555
27	31	12	0·21498	28	10·5	7	0·21348
29	48	15	0·21318	30	36	13	0·21279
31	61·5	17	0·21266	32	112·5	23	0·21259
33	85	20	0·21240	34	166·5	28	0·21233
35	143·5	26	0·21223	36	191	30	0·21218
37	275	36	0·212159	38	231	33	0·212084
39	30·5	12	0·21151	40	13·5	8	0·21032
41	17	9	0·20939	42	7·5	6	0·2073
43	35	13	0·20689	44	25	11	0·20632
45	82·5	20	0·206161	46	224·5	33	0·206127
47	185·5	30	0·206081	48	109	23	0·20599
49	59·5	17	0·20576	50	118·5	24	0·205667
51	139	26	0·205574	52	99·5	22	0·205505
53	197·5	31	0·20547	54	273·5	34	0·20542
55	577	53	0·205402	56	161	28	0·205321
57	20·5	10	0·20475	58	10	7	0·20349
59	13	8	0·20269	60	24·5	11	0·20224
61	45·5	15	0·20212	62	65·5	18	0·202064
63	247·5	35	0·202018	64	373·5	43	0·201984
65	276·5	37	0·20195	66	136·5	26	0·201882
67	89	21	0·201745	68	373	43	0·201717
69	390·5	44	0·201689	70	39·5	14	0·201439
71	29	12	0·20122				
72	5	5	$3d/8 \leqslant t < 0{\cdot}20122$				

Thus by Macbeath's lemma, (2.14) is soluble unless $t = h_n/k_n^2$, $1 \leqslant n \leqslant 72$. For $n = 6, 25, 26, 31, 32, 37, 52, 62$, we find that $t = h_n/k_n^2 > \lambda_{n-1}$, these values of t are therefore covered by the previous choice h_{n-1}, k_{n-1}. Since $t \leqslant 4d^3/3$, we have

$$d \geqslant (3t/4)^{1/3}. \tag{7.4}$$

Making use of (7.4), we find that (7.3) is satisfied for $t = h_n/k_n^2$, $n = 1, 4, 8, 9, 10, 11, 12, 14, 16, 17, 19, 21, 23, 27, 29, 30, 34, 35, 36, 39, 40, 41, 43, 44, 46, 47, 48, 50, 51, 53, 54, 55, 56, 57, 58, 60, 61, 63, 64, 65, 67, 68, 70$ and 71 for the choice $(h, k) = (0{\cdot}5, 1), (11, 6), (2{\cdot}5, 3), (4{\cdot}5, 4), (7, 5), (2{\cdot}5, 3)$ $(2{\cdot}5, 3)$ $(11{\cdot}5, 7)$ $(33{\cdot}5, 12), (59{\cdot}5, 16), (75, 18), (66{\cdot}5, 17), (58{\cdot}5, 16), (21{\cdot}5, 10), (77, 19), (54{\cdot}5, 16), (217{\cdot}5, 32), (178{\cdot}5, 29), (143{\cdot}5, 26), (14{\cdot}5, 14), (47{\cdot}5, 15), (21, 10), (53, 16), (40{\cdot}5, 14), (475, 48), (252{\cdot}5, 35), (198, 31), (150, 27), (99{\cdot}5, 22), (312{\cdot}5, 39), (513{\cdot}5, 50), (66{\cdot}5, 18), (29{\cdot}5, 12), (16{\cdot}5, 9), (58{\cdot}5, 17), (24{\cdot}5, 11), (220, 33), (220, 33), (233{\cdot}5, 34), (465, 48), (408{\cdot}5, 45), (29, 12)$ and $(39{\cdot}5, 14)$ respectively.

Thus (2.14) is soluble unless $t = h_n/k_n^2$, $n = 2, 3, 5, 7, 13, 15, 18, 20, 22, 24, 28, 33, 38, 42, 45, 49, 59, 66, 69$ and 72 i.e. unless $t = 1/3, 5/16, 3/10, 7/24, 1/4, 7/30, 13/56, 3/13, 11/48, 2/9, 3/14, 17/80, 7/33, 5/24, 33/160, 7/34, 13/64, 21/104, 71/352$ and $1/5$. For each of these values of t we will show that either (2.14) or (2.10) is soluble. Also we have

$$d/4 \leqslant (a + A)/4 \leqslant 7t/9. \tag{7.5}$$

We adopt either of the following two methods.

Method I

For $t = h_n/k_n^2$, (2.14) is soluble unless $\beta_G \equiv h_n/k_n (\text{mod } 1/k_n, 2t)$. This then by Lemma 7 and Remark 1, fixes the values of (a_5'', c_5) to be considered. For each value of (a_5'', c_5) we give solutions of (2.14) depending upon values of a_2'' and c_1. We follow this method for $t = 5/16, 3/10, 7/24, 7/30, 13/56, 3/13, 11/48, 2/9, 3/14, 5/24$ and $7/34$. Proof being similar to Lemma 11 is left.

Method II

For a fixed t, from (7.4) and (2.4) we have

$$(3t/4)^{1/3}/2 \leqslant d/2 \leqslant A \leqslant 4t/3. \tag{7.6}$$

Applying Macbeath's lemma to F, we fix the values of A and (a_4', c_4). This will fix the values of λ also as we have, from (2.3) and (2.4), that

$$1/2 \geqslant \lambda \geqslant \sqrt{1 - t/A}. \tag{7.7}$$

We note that $1 < d/C \leqslant d/A \leqslant 2$, so that $M = 1$. If $C \neq A$, we will choose integers $2h$ and k satisfying either (2.31) or (2.33). This will give a solution of (2.10). We apply this method for $t = 17/80, 7/33, 33/160, 13/64, 21/104$ and $71/352$. We illustrate this method for $t = 17/80$ in Lemma 15, others can be similarly done.

For $t = 1/3, 1/4$ and $1/5$, method I does not give solution of (2.14) for every choice of (a_5'', c_5). Also following method II, we find that $C = A\lambda^2 + t = A$ for some values of A, λ and t; so method II does not work. Here we apply method I and II simultaneously (see Lemmas 16–19).

Lemma 15. *If* $t = 17/80$, *then* (2.10) *is soluble.*

Proof. From (7.4) and (7.6) we have $d \geqslant 0{\cdot}542175$ and $0{\cdot}2710875 < A \leqslant 0{\cdot}28333\cdots$. We take $(h, k) = (h_n, k_n)$ for $A \in I_n$, where

n	h_n	k_n	I_n
1	22	9	(0·271084, 0·2721)
2	46	13	(0·2720, 0·2724)
3	33	11	(0·2723, 0·2730)
4	17·5	8	(0·2728, 0·2734)
5	2·5	3	(0·2731, 0·2824)
6	4·5	4	(0·2823, 0·2838)

and find that (2.29) is satisfied. Therefore (2.10) is soluble unless $A = h_n/k_n^2$, $1 \leqslant n \leqslant 6$. For $n = 1, 2, 4, 6$ and $A = h_n/k_n^2$, (2.29) is satisfied on taking $(h, k) = (69{\cdot}5, 16), (120, 21), (33, 11)$ and $(2{\cdot}5, 3)$ respectively. Thus (2.10) is soluble unless $n = 3$ or 5 i.e. unless $A = 3/11$ or $5/18$ and $\beta_F \equiv h_n/k_n \pmod{1/k_n, 2A}$. On taking $x_5 = c_5$ and $1 + c_5$ simultaneously in β_F we find that

$$6\lambda \equiv 0 \pmod 1 \text{ for } A = 3/11, \quad \text{and}$$

$$5\lambda \equiv 0 \pmod 1 \text{ for } A = 5/18. \tag{7.8}$$

From (7.8) and (7.7) we get that (2.10) is soluble unless $A = 3/11$ and $\lambda = 1/2$. But then $C = A\lambda^2 + t = 247/880$ and $|2{\cdot}5 - (247/880)9| + 1/2 < d$. Therefore (2.10) is soluble by method II.

Lemma 16. *If* $t = 1/3$, *then* (2.10) *is soluble.*

Proof. Here since $h = 3$, $k = 3$, $B_G \equiv 1 \pmod{1/3, 2/3}$; by Lemma 7 we need to consider $(a_5'', c_5) = (0, 0), (0, 1/2)$ or $(1/6, 1/4)$. From (7.4) and (7.1) we have

$$0{\cdot}6299 < d \leqslant 2/3, \; d + a/4 \geqslant 9d/8 > 0{\cdot}7. \tag{7.9}$$

Also

$$0{\cdot}31495 < d/2 \leqslant A = d^5/3at = d^5/a \leqslant 2d^4 < 0{\cdot}3951. \tag{7.10}$$

We take

$$h = 3/2, \; k = 2 \quad \text{for} \quad (2-d)/4 < d \leqslant 2d^4$$

$$h = 3, \; k = 3 \quad \text{for} \quad (3{\cdot}5 + d)/9 < A \leqslant (2-d)/4$$

$$h = 5, \; k = 4 \quad \text{for} \quad 0{\cdot}3149 < A \leqslant (3{\cdot}5 - d)/9,$$

and find using (7.9) that (2.29) is satisfied. Also for $A = 3/8$, (2.29) is satisfied with $(h, k) = (1/2, 1)$. Therefore by Macbeath's lemma (2.10) is soluble unless $A = 3/9 = 1/3$ and $\beta_F \equiv 1/3 \pmod{1/3, 2/3}$. On taking $x_5 = c_5$ and $1 + c_5$ simultaneously in β_F we get

$$2\lambda \equiv 0 \pmod 1 \quad \text{i.e. } \lambda = 0 \text{ or } 1/2.$$

If $\lambda = 1/2$, we have $C = A\lambda^2 + t = 5/12$ and $|1/2 - C.1| + 1/2 < d$; so that (2.31) is satisfied.

If $\lambda = 0$, we have $C = A = t = 1/3$, then method II fails and we have to go to method I. When $\lambda = 0$, $3\beta_F = \pm 3a_4' - 2c_4 \equiv 0 \pmod 1$; therefore by Lemma 7 we need consider $(a_4', c_4) = (0, 0), (0, 1/2)$ or $(1/6, 1/4)$ only. Also from (2.13) we have $a_5'' = a_5'$.

$$F = (x_1 + a_2'x_2 + a_4'x_4 + a_5'x_5)x_2 - x_4^2/3 - x_5^2/3 - a/4.$$

Because of symmetry in x_4 and x_5 we need to consider $(a'_4, c_4, a'_5, c_5) = (0, 0, 0, 0), (0, 0, 0, 1/2), (0, 1/2, 0, 1/2), (0, 0, 1/6, 1/4), (0, 1/2, 1/6, 1/4), (1/6, 1/4, 1/6, 1/4)$.

Case (i) $(a'_4, c_4, a'_5, c_5) = (0, 0, 0, 0)$. Here (2.10) reduces to solve

$$0 < F(\pm x + c_1, \pm 1, x_4, x_5) = x + f'_1 - x_4^2/3 - x_5^2/3 - a/4 < d.$$

Range	x	x_4	x_5	F
$f'_1 > \frac{1}{3} + \frac{a}{4}$	0	1	0	$f'_1 - \frac{1}{3} - \frac{a}{4}$
$\frac{a}{4} < f'_1 \leqslant \frac{1}{3} + \frac{a}{4}$	0	0	0	$f'_1 - \frac{a}{4}$
$-\frac{1}{3} + \frac{a}{4} < f'_1 \leqslant \frac{a}{4}$	1	1	1	$f'_1 + \frac{1}{3} - \frac{a}{4}$
$f'_1 \leqslant -\frac{1}{3} + \frac{a}{4}$	1	1	0	$f'_1 + \frac{2}{3} - \frac{a}{4}$

Case (vi) $(a'_4, c_4, a'_5, c_5) = (1/6, 1/4, 1/6, 1/4)$. Here we need to find integers x, y and z such that

$$0 < F(x + c_1, 1/4, y + 1/4, z + 1/4) = x + p'_1 - y^2/3 - z^2/3 + 2/48 - a/4 < d.$$

Range	x	y	z	F
$p'_1 > \frac{14}{48} + \frac{a}{4}$	0	0	1	$p'_1 - \frac{14}{48} - \frac{a}{4}$
$\frac{-2}{48} + \frac{a}{4} < p'_1 \leqslant \frac{14}{48} + \frac{a}{4}$	0	0	0	$p'_1 + \frac{2}{48} - \frac{a}{4}$
$-\frac{18}{48} + \frac{a}{4} < p'_1 \leqslant \frac{2}{48} + \frac{a}{4}$	1	1	1	$p'_1 + \frac{18}{48} - \frac{a}{4}$
$-\frac{34}{48} + \frac{a}{4} < p'_1 \leqslant \frac{18}{48} + \frac{a}{4}$	1	1	0	$p'_1 + \frac{34}{48} - \frac{a}{4}$
$-\frac{50}{48} + \frac{a}{4} < p'_1 \leqslant \frac{34}{48} + \frac{a}{4}$	1	0	0	$p'_1 + \frac{50}{48} - \frac{a}{4}$
$-1 < p'_1 \leqslant -\frac{50}{48} + \frac{a}{4}$	2	1	1	$p'_1 + \frac{66}{48} - \frac{a}{4}$

The other cases can be similarly dealt with.

Lemma 17. *If* $t = 1/4$, *then* (2.10) *is soluble unless* $A = 1/3, \lambda = 1/2$ *and* $(a'_4, c_4, a'_5, c_5) = (0, 0, 0, 0)$.

Proof. Here $h = 1, k = 2, \beta_G \equiv 1/2 \pmod{1/2}$; so we need to consider $(a''_5, c_5) = (0, 0)$ and $(1/4, 1/2)$ only. Further from (7.4) and (7.5) we have

$$d \geqslant 0{\cdot}572357, \quad 0{\cdot}1430 < (a + A)/4 \leqslant 0{\cdot}1944. \tag{7.11}$$

When $(a_5'', c_5) = (1/4, 1/2)$, the following table gives a solution of (2.14) i.e. of

$$0 < G = (x_1 + a_2'' x_2 + x_5/4)x_2 - x_5^2/4 - (a + A)/4 < d.$$

Range	x_1	x_1x_2	x_5	x_2x_5	G
$f_1'' \geqslant d + \frac{3}{16} + \frac{a+A}{4}$	$\pm 1 + c_1$	$-2 + 2\lvert c_1\rvert$	$\pm\frac{1}{2}$	-1	$f_1'' - \frac{37}{16} - \frac{a+A}{4}$
$\frac{3}{16} + \frac{a+A}{4} < f_1'' < d + \frac{3}{16} + \frac{a+A}{4}$	c_1	$\lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$f_1'' - \frac{3}{16} - \frac{a+A}{4}$
$\frac{-1}{16} + \frac{a+A}{4} < f_1'' < \frac{3}{16} + \frac{a+A}{4}$	c_1	$\lvert c_1\rvert$	$\pm\frac{1}{2}$	$\frac{1}{2}$	$f_1'' + \frac{1}{16} - \frac{a+A}{4}$
$f_1'' \leqslant \frac{-1}{16} + \frac{a+A}{4}$, $g_1'' \geqslant d + \frac{a+A}{4} - \frac{13}{16}$	$f_2'' > \frac{-3}{16} + \frac{a+A}{4}$	c_1	$2\lvert c_1\rvert$	$\pm\frac{1}{2}$	1	$f_2'' + \frac{3}{16} - \frac{a+A}{4}$
	$f_2'' \leqslant \frac{-3}{16} + \frac{a+A}{4}$	$\pm 1 + c_1$	$2 - 2\lvert c_1\rvert$	$\pm\frac{3}{2}$	-3	$g_2'' + \frac{11}{16} - \frac{a+A}{4}$
$\frac{a+A}{4} - \frac{13}{16} < g_1'' < d + \frac{a+A}{4} - \frac{13}{16}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$g_1'' + \frac{13}{16} - \frac{a+A}{4}$
$\frac{a+A}{4} - \frac{17}{16} < g_1'' \leqslant \frac{a+A}{4} - \frac{13}{16}$	$\pm 1 + c_1$	$1 - \lvert c_1\rvert$	$\pm\frac{1}{2}$	$\frac{1}{2}$	$g_1'' + \frac{17}{16} - \frac{a+A}{4}$
$f_1'' \leqslant \frac{-1}{16} + \frac{a+A}{4}$, $g_1'' \leqslant -\frac{17}{16} + \frac{a+A}{4}$	$f_2'' > \frac{-19}{16} + \frac{a+A}{4}$	$\pm 2 + c_1$	$4 + 2\lvert c_1\rvert$	$\pm\frac{5}{2}$	-5	$f_2'' + \frac{19}{16} - \frac{a+A}{4}$
	$f_2'' \leqslant \frac{-19}{16} + \frac{a+A}{4}$, $f_1'' > \frac{-13}{16} + d + \frac{a+A}{4}$	$\pm 1 + c_1$	$2 + 2\lvert c_1\rvert$	$\pm\frac{1}{2}$	-1	$f_2'' + \frac{27}{16} - \frac{a+A}{4}$
$\frac{-1}{2} < f_1'' \leqslant \frac{-13}{16} + d + \frac{a+A}{4}$	$\pm 1 + c_1$	$1 + \lvert c_1\rvert$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$f_1'' + \frac{13}{16} - \frac{a+A}{4}$

When $(a_5'', c_5) = (0, 0)$, (2.14) has no solution for $(a_2'', c_1) = (0, 0)$; we have to apply method II. From (7.6) we have

$$0{\cdot}2861 < d/2\ A \leqslant 1/3. \tag{7.12}$$

Also

$$A = d^5/3at \leqslant 8d^4/3 \tag{7.13}$$

We divide the range of A in 17 subintervals $[\lambda_n, \lambda_{n-1}]$, $n = 1, 2, \ldots, 17$; and in each subinterval choose suitable integers $2h_n, k_n$ (listed below) satisfying (2.29). The choice of (h_n, k_n) is done in the same way as described in § 6.3.

n	h_n	k_n	λ_n	n	h_n	k_n	λ_n
1	3	3	0·3234	2	8	5	0·3167
3	5	4	0·3075	4	11	6	0·30345
5	24·5	9	0·3018	6	7·5	5	0·2971
7	14·5	7	0·2946	8	29·5	10	0·29436
9	85	17	0·2940	10	35·5	11	0·2930
11	10·5	6	0·2897	12	18·5	8	0·2881
13	41·5	12	0·2877	14	194·5	26	0·28767
15	115	20	0·28733	16	73·5	16	0·2869
17	14	7	0·2860				

Thus (2.10) is soluble unless $A = h_n/k_n^2$, $1 \leqslant n \leqslant 17$. For $n=8$ and 14, $h_n/k_n^2 > \lambda_{n-1}$, so for these values of A, (2.29) is satisfied by (h_{n-1}, k_{n-1}). For $n = 2, 4, 5, 7, 10, 12, 13, 16, 17$ and $A = h_n/k_n^2$, (2.29) is satisfied with $(h, k) = (11·5, 6), (15, 7), (43·5, 12), (19, 8), (10·5, 6), (35, 11), (18·5, 8), (14, 7)$ and $(2·5, 3)$ respectively. Thus (2.10) is soluble unless $n = 1, 3, 6, 9, 11, 15$ i.e. unless $A = 1/3, 5/16, 3/10, 5/17, 7/24, 23/80$ and $\beta_F \equiv h_n/k_n \pmod{1/k_n, 2A}$. On taking $x_5 = c_5$ and $1 + c_5$ simultaneously in β_F we get

$$2\lambda \equiv 0 \pmod 1 \text{ for } A = 1/3, \qquad 5\lambda \equiv 0 \pmod 1 \text{ for } A = 5/16,$$

$$3\lambda \equiv 0 \pmod 1 \text{ for } A = 3/10, \qquad 10\lambda \equiv 0 \pmod 1 \text{ for } A = 5/17,$$

$$7\lambda \equiv 0 \pmod 1 \text{ for } A = 7/24, \qquad 23\lambda \equiv 0 \pmod 1 \text{ for } A = 23/80.$$

Using (7.7) we find that (2.10) is soluble unless $A = 1/3$ and $\lambda = 1/2$; $A = 5/17$ and $\lambda = 2/5$ or $1/2$; $A = 7/24$ and $\lambda = 3/7$; $A = 23/80$ and $\lambda = 9/23$, $10/23$ or $11/23$. For $A = 5/17$, $\lambda = 2/5$; $A = 7/24$, $\lambda = 3/7$; $A = 23/80$, $\lambda = 9/23$, $10/23$ or $11/23$, we find that (2.31) is satisfied on taking $(h, k) = (7·5, 5), (11, 6)\ (85, 17), (11, 6), (5, 4)$ respectively and noting that $d \geqslant (3A/8)^{1/4}$. When $A = 5/17$ and $\lambda = 1/2$ we have $At/C = 5/22$ and $|2 - (5/22)9| + 1/2 < d$; proving thereby (2.33). When $A = 1/3$ and $\lambda = 1/2$, we have $C = A$, so method II fails. Here

$$F = (x_1 + a_2' x_2 + a_4' x_4 + a_5' x_5)x_2 - (x_4^2 + x_4 x_5 + x_5^2)/3 - a/4.$$

Because of symmetry in x_4 and x_5, we must have $c_4 = 0$. Also $\beta_F \equiv 3/3 \pmod{1/3, 2A}$. Taking $c_4 = 0$, $x_5 = 0$ in β_F, we must have $3a_4' \equiv 0 \pmod 1$ i.e. $a_4' = 0$ or $1/3$. By symmetry we have $a_5' = 0$ or $1/3$. But from (2.13) we have $0 = a_5'' \equiv a_5' - a_4'/2 \pmod 1$ which gives $a_5' - a_4'/2 = 0$, as $|a_5' - a_4'/2| \leqslant 3/4$. So we must have $a_4' = a_5' = 0$. This proves the Lemma.

Lemma 18. *If* $t = 1/4$, $A = 1/3$, $\lambda = 1/2$, $(a_4', c_4, a_5', c_5) = (0, 0, 0, 0)$, *then* (2.10) *is soluble.*

Proof. Since here $d/2 \leqslant a = 4d^5$, we have

$$0·5964 < (1/8)^{1/4} \leqslant d \leqslant 2/3,\ d + a/4 > 2/3. \tag{7.14}$$

The following table gives a solution of (2.10), i.e. of

$$0 < F = (x_1 + a_2' x_2)x_2 - (x_4^2 + x_4 x_5 + x_5^2)/3 - a/4 < d.$$

Range		x_2	x_1x_2	x_4	x_5	F
$f_1' > \frac{1}{3}+\frac{a}{4}$		± 1	$\lvert c_1\rvert$	1	0	$f_1'-\frac{1}{3}-\frac{a}{4}$
$\frac{a}{4} < f_1' \leqslant \frac{1}{3}+\frac{a}{4}$		± 1	$\lvert c_1\rvert$	0	0	$f_1'-\frac{a}{4}$
$f_1' < \frac{-2}{3}+d+\frac{a}{4}$		± 1	$1+\lvert c_1\rvert$	1	0	$f_1'+\frac{2}{3}-\frac{a}{4}$
$\frac{-2}{3}+d+\frac{a}{4} \leqslant f_1' \leqslant \frac{a}{4}$	$f_2' > \frac{a}{4}$	± 2	$2\lvert c_1\rvert$	0	0	$f_2'-\frac{a}{4}$
	$f_2' \leqslant \frac{a}{4}, f_3' > \frac{a}{4}$	± 3	$3\lvert c_1\rvert$	0	0	$f_3'-\frac{a}{4}$
	$f_3' \leqslant \frac{a}{4}, g_1' > \frac{-2}{3}+d+\frac{a}{4}$	± 7	$7\lvert c_1\rvert$	0	0	$f_7'-\frac{a}{4}$
	$\frac{-2}{3}+\frac{a}{4} < g_1' \leqslant \frac{-2}{3}+d+\frac{a}{4}$	± 1	$1-\lvert c_1\rvert$	1	0	$g_1'+\frac{2}{3}-\frac{a}{4}$
	$-1+\frac{a}{4} < g_1' \leqslant \frac{-2}{3}+\frac{a}{4}$	± 1	$1-\lvert c_1\rvert$	0	0	$g_1'+1-\frac{a}{4}$
$\frac{-2}{3}+d+\frac{a}{4} \leqslant f_1'' \leqslant \frac{a}{4}$, $g_1' \leqslant -1+\frac{a}{4}$	$f_2' > -1+\frac{a}{4}$	± 2	$2+2\lvert c_1\rvert$	1	1	$f_2'+1-\frac{a}{4}$
	$f_2' \leqslant -1+\frac{a}{4}, f_3' > -3+\frac{a}{4}$	± 3	$3+3\lvert c_1\rvert$	0	0	$f_3'+3-\frac{a}{4}$
	$f_3' \leqslant -3+\frac{a}{4}, f_5' > -10+\frac{a}{4}$	± 5	$10+5\lvert c_1\rvert$	0	0	$f_5'+10-\frac{a}{4}$
	$f_5' \leqslant -10+\frac{a}{4}$	± 7	$21+7\lvert c_1\rvert$	0	0	$f_7'+21-\frac{a}{4}$

Lemma 19. *If* $t = 1/5$, *then* (2.14) *is soluble*

Proof. Here $h = k = 5$, $\beta_G \equiv 1 \pmod{1/5, 2/5}$. By Lemma 12 we need to consider $(a_5'', c_5) = (0, 0), (0, 1/2)$ or $(1/10, 1/4)$. Also from (9.4) and (7.5) we have

$$d \geqslant 0{\cdot}531329 \text{ and } 0{\cdot}1328 < (a+A)/4 < 0{\cdot}1555\cdots. \tag{7.15}$$

When $(a_5'', c_5) = (0, 0)$ or $(0, 1/2)$, we get a solution of (2.14) directly depending upon the values of c_1 and a_2''. When $(a_5'', c_5) = (1/10, 1/4)$, we first fix the values of A. From (7.15) and (7.6) we have

$$d > 0{\cdot}531529, \quad 0{\cdot}2656 < A \leqslant 4/15 = 0{\cdot}2666\cdots. \tag{7.16}$$

We take $(h, k) = (h_n, k_n)$ for $A \in I_n$, where

n	h_n	k_n	I_n
1	17	8	(0·2656, 0·2661)
2	45	13	(0·26609, 0·2664)
3	77	17	(0·26639, 0·26654)
4	60	15	(0·26653, 0·26667)

and find that (2.29) is satisfied. Therefore (2.10) is soluble unless $A = h_n/k_n^2$, $1 \leqslant n \leqslant 4$. When $n = 1, 2, 3$ and $A = h_n/k_n^2$, (2.29) is satisfied with $(h, k) = (13, 7), (45, 13)$ and $(106{\cdot}5, 20)$ respectively. Thus (2.10) is soluble unless $n = 4$, i.e. $A = 4/15$ and $\beta_F \equiv 60/15 \pmod{1/15, 8/15}$. On taking $x_5 = c_5$ and $1 + c_5$ simultaneously in β_F we get $8\lambda \equiv 0 \pmod 1$. This together with (7.7) gives $\lambda = 1/2$. Now

$$F = (x_1 + a_2' x_2 + a_4' x_4 + a_5' x_5) x_2 - 4(x_4^2 + x_4 x_5 + x_5^2)/15 - a/4 .$$

Since $c_4 = 1/4$, we can assume because of symmetry in x_4 and x_5 that $c_5 = 1/4$. Also from (2.13) we have $1/10 = a_5'' \equiv a_5' - a_4'/2 \pmod 1$. As $|a_5' - a_4'/2| \leqslant 3/4$, we must have $a_5' - a_4'/2 = 1/10$. By symmetry $a_4' - a_5'/2 = 1/10$. On solving we get $a_4' = a_5' = 1/5$

Let $x_1 = -x + c_1$, $x_2 = -1$, $x_4 = y + 1/4$, $x_5 = z + 1/4$, so that

$$F = x + p_1' - \frac{4}{15}(y+1)\left(y+\frac{1}{4}\right) - \frac{4}{15}(z+1)\left(z+\frac{1}{4}\right) - \frac{4}{15}\left(y+\frac{1}{4}\right)\left(z+\frac{1}{4}\right) - \frac{a}{4}$$

The following table gives a solution of (2.10).

Range	x	y	z	F
$p_1' > \frac{5}{12} + \frac{a}{4}$	0	-1	1	$p_1' - \frac{5}{12} - \frac{a}{4}$
$\frac{1}{60} + \frac{a}{4} < p_1' \leqslant \frac{5}{12} + \frac{a}{4}$	0	-1	0	$p_1' - \frac{1}{60} - \frac{a}{4}$
$-\frac{11}{60} + \frac{a}{4} < p_1' \leqslant \frac{1}{60} + \frac{a}{4}$	1	1	0	$p_1' + \frac{11}{60} - \frac{a}{4}$
$-\frac{7}{12} + \frac{a}{4} < p_1' \leqslant -\frac{11}{60} + \frac{a}{4}$	1	-1	1	$p_1' + \frac{7}{12} - \frac{a}{4}$
$-\frac{59}{60} + \frac{a}{4} < p_1' \leqslant \frac{-7}{12} + \frac{a}{4}$	1	-1	0	$p_1' + \frac{59}{60} - \frac{a}{4}$
$-1 \leqslant p_1' \leqslant \frac{-59}{60} + \frac{a}{4}$	2	1	0	$p_1' + \frac{71}{60} - \frac{a}{4}$

This completes the proof of the Lemma. Lemmas 1–19 complete the proof of the Theorem.

Remark 2. If we take $d = (32|D|/3)^{1/5}$, we find from (2.3) in the case $m = 1$, $K = 1$ that $d/2 \leqslant A \leqslant \sqrt{d^5/2a} \leqslant d^2$. So we have $d \geqslant 1/2$ in place of (7.1). The proof then (especially of § 7) becomes very complicated and we do not get any critical form either. If we take any constant smaller than 32/3, then we will have d smaller than 1/2; but then Macbeath's lemma, which is the most significant tool in this method, is not applicable.

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On asymptotic distribution on the a-adic integers

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Abstract. We show that the values of a polynomial with **a**-adic coefficients at integer and rational prime arguments are asymptotically distributed on the **a**-adic integers and that the integer parts of certain sequences known to be uniformly distributed modulo one, are uniformly distributed on the **a**-adic integers.

Keywords. **a**-adic integers; asymptotic distribution.

1. Introduction

By construction the natural numbers are dense in the p-adic integers. In fact, they can by shown to be uniformly distributed there. On the other hand the squares of natural numbers are dense but not uniformly distributed. Motivated by this observation in this paper we study the asymptotic distribution of certain sequences on the **a**-adic integers. To fix ideas we begin with some basic information about the **a**-adic integers. Let $\mathbf{a}=(a_n)_{n=0}^{\infty}$ be a sequence of rational integers greater than one. Using the notation of [HR, § 10] we define the **a**-adic integers $\Delta_{\mathbf{a}}$ to be the set of infinite sequences $(x_n)_{n=0}^{\infty}$ in $\prod_{n=1}^{\infty}\{0,1,\ldots,a_n-1\}$.

For $\mathbf{x}=(x_n)_{n=0}^{\infty}$ and $\mathbf{y}=(y_n)_{n=0}^{\infty}$ let $\mathbf{z}=(z_n)_{n=0}^{\infty}$ be defined as follows. Write $x_0+y_0=t_0a_0+z_0$, where $z_0\in\{0,1,\ldots,a_0-1\}$ and t_0 is a rational integer. Suppose $z_0,\ldots,z_k$ and $t_0,\ldots,t_k$ have been defined. Then write $x_{k+1}+y_{k+1}+t_k=t_{k+1}a_{k+1}+z_{k+1}$, where $z_{k+1}\in\{0,1,\ldots,a_{k+1}-1\}$ and t_{k+1} is a rational integer. We have thus inductively defined the sequence $\mathbf{z}=(z_n)_{n=0}^{\infty}$, which we deem to be $\mathbf{x}+\mathbf{y}$. The binary operation $+$ which we call addition makes $\Delta_{\mathbf{a}}$ an abelian group.

For each non-negative integer k, let

$$\Lambda_k=\{x\in\Delta_{\mathbf{a}}: x_n=0 \quad \text{if} \quad n<k\}.$$

These sets form a basis at $\mathbf{0}=(0,0,\ldots)$ for a topology on $\Delta_{\mathbf{a}}$. With respect to this topology $\Delta_{\mathbf{a}}$ is compact and the group operations are continuous, making $\Delta_{\mathbf{a}}$ a compact Abelian topological group. A second binary operation called multiplication, denoted by $\times$ and compatible with addition is defined as follows. Let $\mathbf{u}=(1,0,0,\ldots)$. Note that $(n\mathbf{u})_{n=0}^{\infty}$ is dense in $\Delta_{\mathbf{a}}$. First on $(n\mathbf{u})_{n=0}^{\infty}$ define $k_1\mathbf{u}\times k_2\mathbf{u}$ to be $k_1k_2\mathbf{u}$. Deeming multiplication to be continuous on $\Delta_{\mathbf{a}}$ defines it off $(n\mathbf{u})_{n=0}^{\infty}$. The binary operations addition and multiplication makes $\Delta_{\mathbf{a}}$ a topological ring.

For each non-negative integer n, let $\lambda_n(A)$ denote the measure on $\{0,1,\ldots,a_n-1\}$ given by $\lambda_n(A)=\mathrm{card}(A)/a_n$. Haar measure is the corresponding product measure on $\Delta_{\mathbf{a}}$.

The dual group of $\Delta_{\mathbf{a}}$, which we denote $\mathbf{Z}(\mathbf{a}^{\infty})$ consists of all rationals $t=l/A_r$ where $A_r=a_0\ldots a_r$ and $0\leqslant l\leqslant A_r$ for some non-negative integer r. To evaluate a character χ_t at $\mathbf{x}$ in $\Delta_{\mathbf{a}}$ we write

$$\chi_t(\mathbf{x})=e\left(\frac{l}{A_r}(x_0+a_0x_1+\cdots+a_0\cdots a_{r-1}x_r)\right),$$

where as usual, for a real number x, $e(x)$ denotes $e^{2\pi ix}$.

For a sequence $(x_n)_{n=0}^{\infty}$ in Δ_a, let $A(E;N)$ denote the number of elements of the set $\{x_0,\ldots,x_{n-1}\}$ that belong to E. If for every set E belonging to the algebra generated by the sets $\Lambda_k (k=0,1,\ldots)$ and their translates, the limit

$$\mu(E) = \lim_{N\to\infty} \frac{A(E;N)}{N}$$

exists, we say $(x_n)_{n=0}^{\infty}$ is asymptotically distributed on Δ_a with distribution μ. If μ coincides with Haar measure λ, we say $(x_n)_{n=0}^{\infty}$ is uniformly distributed.

Let $\alpha_0,\ldots,\alpha_k$ be an arbitrary but fixed set of elements of Δ_a and let

$$\rho(x) = \alpha_k x^k + \cdots + \alpha_1 x + \alpha_0.$$

In § 2 we prove the following theorem.

Theorem 1. *Suppose $(p_n)_{n=0}^{\infty}$ denotes the sequence of rational primes. The sequences $(\rho(n))_{n=0}^{\infty}$ and $(\rho(p_n))_{n=0}^{\infty}$ are asymptotically distributed on Δ_a. Moreover their distributions are continuous with respect to Haar measure.*

It is known [Me] that except in special cases the sequences considered in Theorem 1 are not uniformly distributed.

Suppose the sequence of integers $(k_n)_{n=1}^{\infty}$ is described by any of the following constructions:

(i) Suppose $S=(n_k)_{n=1}^{\infty} \subseteq \mathbf{N}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function I_S we may view it as a point in $\Lambda = \{0,1\}^{\mathbf{N}}$ the set of maps from $\mathbf{N}$ to $\{0,1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$ where for each natural number n we have $X_n = \{0,1\}$ and specify the probability π_n on X_n by $\pi_n(\{1\}) = q_n$ with $0 \leqslant q_n \leqslant 1$ and $\pi_n(\{0\}) = 1 - q_n$ such that $\lim_{n\to\infty} q_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\pi = \prod_{n=1}^{\infty} \pi_n$. The underlying σ-algebra β is that generated by the 'cylinders'

$$\{\lambda = (\lambda_n)_{n=1}^{\infty} \in \Lambda : \lambda_{i_1} = \alpha_{i_1}, \ldots, \lambda_{i_r} = \alpha_{i_r}\}$$

for all possible choices of $i_1,\ldots,i_r$ and $\alpha_{i_1},\ldots,\alpha_{i_r}$. Let $(k_n)_{n=1}^{\infty}$ be almost any point in Λ with respect to the measure π.

For a real number x let $[x]$ denote the greatest rational integer not exceeding x.

(ii) $k_n = [P(n)] (n=1,2,\ldots)$ where

$$P(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \cdots + \alpha_0$$

for real numbers $\alpha_0,\ldots,\alpha_k$ such that the numbers $\alpha_1,\ldots,\alpha_k$ are not all rational multiples of a single real number;

(iii) $k_n = [P(p_n)]$ $(n=1,2,\ldots)$ where $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes and $P(z)$ is as in (ii);

(iv) $k_n = [f(n)] (n=1,2,\ldots)$ where $f(z)$ denotes a non-polynomial entire function which is real on the real numbers and such that $|f(z)| \ll e^{(\log z)^{\alpha}}$ with $\alpha < \frac{4}{3}$;

(v) $k_n = [f(p_n)] (n=1,2,\ldots)$ where $f(z)$ is as in (iv) and p_n denotes the nth rational prime;

(vi) $k_n = [n^{\alpha}] (n=1,2,\ldots)$ where α is any real number greater than one.

(vii) $k_n = [a_n \cos(a_n x)]$ $(n=1,2,\ldots)$ for a strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ and almost all x with respect to Lebesgue measure;

(viii) $k_n = [a_n \cos(a_n x)]$ $(n = 1, 2, \ldots)$ for a strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ such that $a_n = O(n^p)$ and $p > 1$ and all x outside a set of Hausdorff dimension not greater than $1 - (1/(4p + \frac{1}{2}))$;
(ix) $k_n = [g_n(x)]$ $(n = 1, 2, \ldots)$ for almost all x with respect to Lebesgue measure in $[a, b]$ where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the following hypothesis. For each pair of distinct natural number m and n we have

(a) $g_n'(x) - g_m'(x)$ is monotone on $[a, b]$ and
(b) there is an absolute constant λ such that

$$|g_n'(x) - g'_m(x)| \geqslant \lambda > 0.$$

(x) $k_n = [g_n(x)]$ $(n = 1, 2, \ldots)$ for all x lying outside a set of Hausdorff dimension at most $1 - (1/p)$ in $[a, b]$ where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the hypothesis (a), (b) of (ix) and in addition
(c) for all x in $[a, b]$ we have

$$\sup_{x \in [a,b]} |g_n'(x)| = O(n^p)$$

for some $p > 1$ and with an implied constant independent of x and
(d) for each pair of distinct positive integers m and n the function

$$\frac{g_n'(x) g_m'(x)}{g_m'(x) - g_n'(x)}$$

is monotonic on $[a, b]$.

The proof of the following theorem is spread over §§ 3 and 4.

Theorem 2. *In each of the instances* (i)–(x) *the sequence of integers* $(k_n)_{n=1}^{\infty}$ *is uniformly distributed on* $\Delta_{\mathbf{a}}$.

2. Proof of Theorem 1

To prove Theorem 1 we need some lemmas. The next two lemmas generalize to the **a**-adic integer lemmas already known in the case where for each positive integer i, we have $a_i = p$ for each for a fixed prime p[Ch].

Lemma 3. *The sequence* $(x_n)_{n=0}^{\infty}$ *in* $\Delta_{\mathbf{a}}$ *is asymptotically distributed if and only if for each* χ_t *in* $\mathbf{Z}(\mathbf{a}^{\infty})$

$$c_t = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \chi_t(x_n)$$

exists. Moreover if

$$\mu_r(\alpha) = \lim_{N \to \infty} \frac{A(\alpha + \Lambda_r;\ N)}{N},$$

then

$$c_t = \frac{1}{A_r} \sum_{j=0}^{A_r - 1} \mu_r(j) e(tj)$$

for each $r > 1$ *and if* $Z_r = \{(l/A_s) \in \mathbf{Z}(\mathbf{a}^\alpha): 0 \leqslant s \leqslant r\}$,

$$\mu_r(\alpha) = \frac{1}{A_r} \sum_{t \in Z_r} c_t e(-t\alpha).$$

Proof. For each rational integer j an element y of Δ_a being in $j + \Lambda_k$ implies that $e(ty) = e(tj)$. So for any sequence $(y_n)_{n=0}^{\infty}$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} e(ty_n) = \sum_{j=0}^{A_r - 1} e(jt) N^{-1} |\{y_n \in j + \Lambda_r : 1 \leqslant n \leqslant N\}|.$$

Therefore if $(y_n)_{n=0}^{\infty}$ is asymptotically distributed

$$c_t = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_t(y_n)$$

exists. On the other hand letting

$$B_N = \sum_{t \in Z_r} e(-t\alpha) \sum_{n=0}^{N-1} e(ty_n) = \sum_{n=0}^{N-1} \sum_{t \in Z_r} e(t(y_n - \alpha)).$$

If y_n is in $j + \Lambda_k$ this sum is $A_r |\{y_n \in j + \Lambda_r : 1 \leqslant n \leqslant N\}|$. This means that,

$$\lim_{N \to \infty} \frac{1}{N} |\{y_n \in j + \Lambda_r : 1 \leqslant n \leqslant N\}| = \frac{1}{A_r} \sum_{t \in Z_r} e(-tj) c_t,$$

as required. ∎

Lemma 4. *Let* $j(\alpha) = \lim_{k \to \infty} \mu_k(\alpha)$, *which always exists as consequence of the monotonicity of* $\mu_k(\alpha)$ *as a function of* k. *Then if* $(y_n)_{n=0}^{\infty}$ *is asymptotically distributed on* Δ_a

$$\lim_{r \to \infty} \frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 = \sum_{\alpha \in D} j^2(\alpha),$$

where D *is the countable set where* $j(\alpha) \neq 0$.

Proof. By Lemma 3

$$|c_t|^2 = c_t \overline{c_t} = \sum_{0 \leqslant h \leqslant A_r - 1} e(th) \mu_r(h) \sum_{0 \leqslant m \leqslant A_r - 1} e(-tm) \mu_r(m).$$

Thus

$$\sum_{t \in Z_r} |c_t|^2 = \sum_{0 \leqslant h,\ m \leqslant A_r - 1} \mu_r(h) \mu_r(m) \sum_{t \in Z_r} e(t(h - m)).$$

Now $\Sigma_{t \in Z_r} e(t(h-m))$ is equal to A_r if $h = m$ and 0 otherwise. In addition $\mu_r(m)$ is the constant value of $\mu_r(\alpha)$ on $\alpha + \Lambda_k$. Therefore

$$\frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 = \sum_{0 \leqslant m \leqslant A_r - 1} \mu_r^2(m) = \int_{\Delta_a} \mu_r(\alpha) \mathrm{d}\bar{\mu}(\alpha),$$

where the measure $\bar{\mu}$ is that whose existence follows by Lemma 3 from the sequence $(c_t)_{t \in \mathbf{Z}(\mathbf{a}^\alpha)}$. Letting r tend to infinity, we have

$$\lim_{r \to \infty} \frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 \int_{\Delta_a} j(\alpha) \mathrm{d}\bar{\mu}(\alpha) = \sum_{\alpha \in D} j^2(\alpha)$$

as required.

Consider $G^{(1)}$: $\mathbf{Z}(\mathbf{a}^\infty)\to\mathbf{C}$ defined by

$$G^{(1)}(\chi_{l/A_r}) = G^{(1)}_{\alpha_0,\dots,\alpha_k}(\chi_{l/A_r}) = \frac{\chi_{l/A_r}(\alpha_0)}{D_r}\sum_{m=1}^{D_r} e^{2\pi i(\gamma(m)/D_r)},$$

for all l/A_r in $\mathbf{Z}(\mathbf{a}^\infty)$. Here the positive integer D_r and the polynomial γ of degree k with non-negative integer coefficients are described as follows. Let

$$l_j = l(\alpha_j(0) + \alpha_j(1)a_0 + \alpha_j(2)a_0a_1 + \cdots \alpha_j(r-1)a_0\cdots\alpha_{r-2}), \quad j=1,\dots,k).$$

Here $\alpha_j(r)$ denotes the rth terms of α_j viewed as a sequence. Let m_j/B_j with $(m_j, B_j)=1$ denote l_j/A_r in reduced form. We use D_r to denote the least common multiple of $B_1,\dots,B_k$ and define $\gamma(x)$ by

$$\frac{\gamma(x)}{D_r} = \frac{m_k}{B_k}x^k + \cdots + \frac{m_1}{B_1}x.$$

Also consider $G^{(2)}$:$\mathbf{Z}(\mathbf{a}^\infty)\to\mathbf{C}$ defined by

$$G^{(2)}(\chi_{l/A_r}) = G^{(2)}_{\alpha_0\cdots\alpha_l}(\chi_{l/A_r}) = \frac{\chi_{l/A_r}(\alpha_0)}{\phi(D_r)}\sum_{\substack{m=1\\(D_r,m)=1}}^{D_r} e^{2\pi i(\gamma(m)/D_r)},$$

where ϕ denotes Euler's totient function.

Lemma 5. *For all* χ_{l/A_r}, *we have*

$$G^{(1)}(\chi_{l/A_r}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(n)) \tag{1}$$

and

$$G^{(2)}(\chi_{l/A_r}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(p_n)). \tag{2}$$

Proof. We first prove (1). Note that

$$\begin{aligned}
&\chi_{l/A_r}(\rho(n)) = \chi_{l/A_r}(\alpha_0+\alpha_1 n+\cdots+\alpha_k n^k)\\
&= \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(\chi_{l/A_r}(\alpha_j))^{n^j}\\
&= \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(e^{2\pi i(l/A_r)n^j(\alpha_j(0)+\alpha_j(1)a_0+\cdots+\alpha_j(r-1)a_0\cdots a_{r-2})}),\\
&= \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(e^{2\pi i(l_j/A_r)n^j}) = \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(e^{2\pi i(m_j/B_j)n^j}) = \chi_{l/A_r}(\alpha_0)\,e^{2\pi i(\gamma(n)/D_r)}.
\end{aligned}$$

If we now partition the integers into their residue classes modulo D_r it follows immediately that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(n)) = \frac{\chi_{l/A_r}(\alpha_0)}{D_r}\sum_{m=1}^{D_r}e^{2\pi i(\gamma(m)/D_r)},$$

proving (1). We now show (2). Arguing as in (1)

$$\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(p_n)) = \frac{\chi_{l/A_r}(\alpha_0)}{\pi_N}\sum_{1\le p\le N}e^{2\pi i(\gamma(p)/D_r)}. \tag{3}$$

Let $\Lambda: \mathbf{N} \mapsto \mathbf{R}$ be the Von Mangoldt function, defined by $\Lambda(n) = \log_e p$ if $n = p^l$ for some prime p and positive integer l, and zero otherwise. Using partial summation we readily see that

$$\frac{1}{\pi_N} \sum_{1 \leqslant p \leqslant N} e^{2\pi i(\gamma(p)/D_r)} = \frac{1}{N} \sum_{1 \leqslant n \leqslant N} \Lambda(n) e^{2\pi i(\gamma(n)/D_r)} + O((\log N)^{-1}). \tag{4}$$

The Siegel–Walfisz prime number theorem for arithmetic progressions [D, p. 133] says that for fixed positive u, if $1 \leqslant D_r \leqslant (\log N)^u$, and $(m, D_r) = 1$, then for some $C > 0$,

$$\sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv m (\mathrm{mod}\, D_r)}} \Lambda(n) = \frac{N}{\phi(D_r)} + o(N e^{-C(\log N)^{1/2}}).$$

Now note that

$$\sum_{n=1}^{N} \Lambda(n) e^{2\pi i(\gamma(n)/D_r)} = \left(\sum_{\substack{m=1 \\ (m, D_r)=1}}^{D_r} e^{2\pi i(\gamma(m)/D_r)} \right) \left(\sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv m (\mathrm{mod}\, D_r)}} \Lambda(n) \right) + O\left(\sum_{p^l \leqslant N;\ p \mid D_r} \Lambda(p^l) e^{2\pi i(\gamma(p^l)/D_r)} \right).$$

Using the fact that the third sum on the right is $O((\log N)(\log\log N))$ and the Siegel–Walfisz theorem

$$\frac{1}{N} \sum_{n=1}^{N} \Lambda(n) e^{2\pi i(\gamma(n)/D_r)} = \left(\frac{1}{\phi(D_r)} \sum_{\substack{m=1 \\ (m, D_r)=1}}^{D_r} e^{2\pi i(\gamma(m)/D_r)} \right) + O\left(\frac{(\log N)(\log\log N)}{N} \right). \tag{5}$$

Combining (3), (4) and (5) and letting N tend to infinity (4) is proved. ∎

To complete the proof of Theorem 1 we need Weyl's inequality [Va].

Lemma 6. *Suppose that the integers a and q are coprime and that $|\alpha_k - (a/q)| \leqslant q^{-2}$ where*

$$\phi(x) = \alpha_k x^k + \cdots + \alpha_1 x + \alpha_0.$$

Then

$$\sum_{x=1}^{Q} e(\phi(x)) \ll Q^{1+\varepsilon}(q^{-1} + Q^{-1} + qQ^{-k})^{1/K},$$

where $K = 2^{k-1}$.

We now complete the proof of Theorem 1. As a consequence of Lemma 6 we see that there exists a $\delta > 0$ such that $|G^{(1)}| \ll A_r^{-\delta}$. Also

$$\sum_{\substack{1 \leqslant m \leqslant q \\ (m,q)=1}} e(a\gamma(m)q^{-1}) = \sum_{1 \leqslant m \leqslant q} e(a\gamma(m)q^{-1}) - \sum_{p \mid q} \sum_{\substack{1 \leqslant m \leqslant q \\ p \mid m}} e(a\gamma(m)q^{-1}).$$

Hence we also have $|G^{(2)}| \ll A_r^{-\delta}$ for a possibly different δ. This means that in either case

$$\frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 \leqslant \frac{1}{A_r} \sum_{s=0}^{r} A_s^{1-2\delta},$$

which tends to zero as r tends to infinity because $A_r \geqslant 2^r (r = 1, 2, \ldots)$, completing the proof of Theorem 1. ∎

3. Proof of Theorem 2 (i)

We now turn to the proof of Theorem 2. We begin with two classical inequalities.

Let $(r_n(x))_{n=0}^{\infty}$ denote the sequence of Rademacher functions. That is we define $r_0(x)$ to be 1 on $(0, \frac{1}{2}]$, $r_0(x)$ to be -1 on $(\frac{1}{2}, 1]$, extend this definition periodically to the whole of $\mathbf{R}$ and define $r_n(x)$ to be $r_0(2^n x)$ $(n = 1, 2, \ldots)$. We have the following well-known inequality of Khinchine [Z1, p. 213].

Lemma 7. *Suppose* $q \geqslant 1$. *Then for each set of complex scalars* $(a_n)_{n=0}^{\infty}$, *we have positive constants* A_q *and* B_q, *such that*

$$A_q \left(\sum_{n=0}^{N} |a_n|^2 \right)^{1/2} \leqslant \left(\int_0^1 \left| \sum_{n=0}^{N} a_n r_n(x) \right|^q \mathrm{d}x \right)^{1/q} \leqslant B_q \left(\sum_{n=0}^{N} |a_n|^2 \right)^{1/2}.$$

Further A_q *is absolute (that is independent of* q*) and* $B_q \leqslant cq^{1/2}$, *where* c *is absolute.*

The following inequality is due to Bernstein [Z2, p. 11].

Lemma 8. *There is an absolute constant* $A > 0$ *such that*

$$\| f' \|_{\infty} \leqslant AN \| f \|_{\infty}$$

for every positive integer N, *and every trigonometric polynomial* f *with spectrum in* $[-N, N]$.

To prove Theorem 2 we need a series of lemmas, which in their turn need some additional ideas to be stated. For a real valued function F defined on the probability space $P = (\Lambda, \beta, \pi)$ we let $m_{F,P}$ denote its distribution function. That is

$$m_{F,P}(x) = \pi\{\lambda \in \Lambda : F(\lambda) \leqslant x\}.$$

We say $m_{F,P}$ is symmetric if $m_{F,P}(x) + m_{F,P}(-x) = c$ for some constant independent of x and that the function F is symmetric with respect to probability space P if the corresponding distribution $m_{F,P}$ is symmetric. Note that the function $F^s(\lambda, \lambda') = F(\lambda) - F(\lambda')$, called the symmetrization of F, is symmetric with respect to the product probability space $P \otimes P = (\Lambda \otimes \Lambda, \beta \otimes \beta, \pi \otimes \pi)$. Let $f_n(\lambda) = I_{c_n}(\lambda) - q_n$, where I_{c_n} is the characteristic function of the cylinder

$$c_n = \{\lambda \in \Lambda : \lambda_n = 1\}.$$

Lemma 9. *If* $(r_k(x))_{n=1}^{\infty}$ *is the sequence of Rademacher functions and* $(f_k^s(\lambda, \lambda'))_{n=1}^{\infty}$ *is the sequence of symmetrizations of the sequence of functions* $(f_k(\lambda))_{n=1}^{\infty}$ *then for* $p \geqslant 1$,

$$\left(\int_0^1 \int_{\Lambda \otimes \Lambda} \left| \sum_{k=1}^{N} r_k(t) f_k^s(\lambda, \lambda') \right|^p \mathrm{d}\pi(\lambda) \mathrm{d}\pi(\lambda') \right)^{1/p}$$
$$= \left(\int_0^1 \int_{\Lambda \otimes \Lambda} \left| \sum_{k=1}^{N} f_k^s(\lambda, \lambda') \right|^p \mathrm{d}\pi(\lambda) \mathrm{d}\pi(\lambda') \right)^{1/p}.$$

Proof. First note that for any two distinct numbers t_1 and t_2 in $[0, 1)$ which are not diadic rationals, the functions $r_k(t_1) f_k^s(\lambda, \lambda')$ and $r_k(t_2) f_k^s(\lambda, \lambda')$ have the same

distribution function, that is

$$m_{r_k(t_1)f_k^s,\pi\otimes\pi} = m_{r_k(t_2)f_k^s,\pi\otimes\pi}, \quad (k=1,2,\ldots). \tag{6}$$

For an arbitrary measurable subset T of $\mathbf{R}$ let

$$A=\{(x_1,\ldots,x_N)\in\mathbf{R}^N : x_1+\cdots+x_N\in T\}.$$

In addition for non-diadic t_1 in $[0,1)$, the sequence of functions $(r_k(t_1)f_k^s(\lambda,\lambda'))_{n=1}^{\infty}$ is independent on $P\otimes P$, which means that

$$\pi\otimes\pi(\{(\lambda,\lambda'):(r_1(t_1)f_1^s(\lambda,\lambda'),\ldots,r_N(t_1)f_N^s(\lambda,\lambda')\in A\})$$

$$=\int\cdots\int_A \mathrm{d}m_{r_1(t_1)f_1^s,\pi\otimes\pi}\cdots\mathrm{d}m_{r_N(t_1)f_N^s,\pi\otimes\pi}.$$

But by (6) this is

$$\int\cdots\int_A \mathrm{d}m_{r_1(t_2)f_1^s,\pi\otimes\pi}\cdots\mathrm{d}m_{r_N(t_2)f_N^s,\pi\otimes\pi}$$

which in consequence tells us that

$$\pi\otimes\pi(\{(\lambda,\lambda'):(r_1(t_1)f_1^s(\lambda,\lambda'),\ldots,r_N(t_1)f_N^s(\lambda,\lambda')\in A\})$$
$$=\pi\otimes\pi(\{(\lambda,\lambda'):(r_1(t_2)f_1^s(\lambda,\lambda'),\ldots,r_N(t_2)f_N^s(\lambda,\lambda')\in A\}).$$

This tells us that

$$\int_{\Lambda\otimes\Lambda}\left|\sum_{k=1}^N r_k(t_1)f_k^s(\lambda,\lambda')\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda)\mathrm{d}t$$
$$=\int_{\Lambda\otimes\Lambda}\left|\sum_{k=1}^N r_k(t_2)f_k^s(\lambda,\lambda')\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')$$

and hence that

$$\int_0^1\int_{\Lambda\otimes\Lambda}\left|\sum_{k=1}^N r_k(t)f_k^s(\lambda,\lambda')\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')\mathrm{d}t$$
$$=\int_{\Lambda\otimes\Lambda}\left|\sum_{k=1}^N r_k(t_2)f_k^s(\lambda,\lambda')\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda').$$

This means that choosing t_2 such that $r_k(t_2)=1$ for all k in $[1,N]\cap\mathbf{Z}$ completes the proof of the lemma. ■

Lemma 10. *Suppose $p\geqslant 1$ and that for the function F is defined on the probability space $P=(\Lambda,\beta,\pi)$. Suppose also that $\int_\Lambda |F(\lambda)|^p\mathrm{d}\pi(\lambda)=1$. Suppose further that $\int_\Lambda |F(\lambda)|\mathrm{d}\pi(\lambda)\leqslant\frac{1}{4}$. Then $(\int_{\Lambda\otimes\Lambda}|F^s(\lambda,\lambda')|^p\mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda'))^{1/p}\geqslant c>0$ for some absolute constant c.*

Proof. By Minkowski's inequality

$$\left(\int_{[|F|\leqslant 1]}|F^s|^p\mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p}\geqslant\left(\int_{[|F|\leqslant 1]}|F(\lambda)|^p\mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p}$$
$$-\left(\int_{[|F|\leqslant 1]}|F(\lambda')|^p\mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p}.$$

Note that $F(\lambda)$ and $F(\lambda')$ are independent on $P \otimes P$, hence

$$\left(\int_{[|F|\leqslant 1]} |F(\lambda')|^p \mathrm{d}\pi(\lambda)\pi(\lambda')\right) = \pi([|F|\leqslant 1])\left(\int_\Lambda |F(\lambda')|^p \pi(\lambda')\right)^{1/p}$$

$$= \pi([|F|\leqslant 1]) \geqslant \frac{3}{4}.$$

This means that

$$\left(\int_{[|F|\leqslant 1]} |F^s|^p \mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p} \geqslant \left(\frac{3}{4}\right)^{1/p} - \left(\frac{1}{4}\right)^{1/p},$$

which completes the proof of the lemma. ∎

Lemma 11. *Suppose* $p \geqslant 1$ *and that for the function* $F(\lambda)$ *defined on the probability space* $P = (\Lambda, \beta, \pi)$ *we have* $\int_\Lambda |F(\lambda)|^p \mathrm{d}\pi(\lambda) = 1$ *and* $\int_\Lambda F(\lambda)\mathrm{d}\pi(\lambda) = 0$ *Suppose also that* $\int_\Lambda |F(\lambda)|\mathrm{d}\pi(\lambda) \geqslant \frac{1}{4}$. *Then* $(\int_{\Lambda\otimes\Lambda} |F^s(\lambda,\lambda')|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda'))^{1/p} \geqslant c > 0$ *for some constant* c *dependent only on* p.

Proof. Note that

$$\int_{\Lambda\otimes\Lambda} |F^s(\lambda,\lambda')|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda') \geqslant \int_{[F(\lambda)\leqslant 0]\cap[F(\lambda')>0]} |F^s(\lambda,\lambda')|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')$$

and that using the independence of $F(\lambda)$ and $F(\lambda')$ on $P \otimes P$ we have

$$\geqslant \int_{[F(\lambda)\leqslant 0]\cap[F(\lambda')>0]} F(\lambda')^p \mathrm{d}\pi(\lambda') = \pi[F(\lambda)\leqslant 0] \int_{[F(\lambda')>0]} F(\lambda')^p \mathrm{d}\pi(\lambda').$$

Now suppose $\pi[F(\lambda')>0] \geqslant \frac{1}{2}$ then noting that $\int_{[F(\lambda')>0]} F(\lambda')^p \mathrm{d}\pi(\lambda')$ is not less than $1/2$ $(1/8)^p$ we have

$$\int_{\Lambda\otimes\Lambda} |F^s(\lambda,\lambda')|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda') \geqslant \frac{1}{2}\left(\frac{1}{8}\right)^p.$$

If $\pi[F(\lambda')>0] < \frac{1}{2}$, the same estimate follows by reversing the roles of λ and λ'. This completes the proof of the lemma.

We now complete the proof of Theorem 2 (i). The following argument is adapted from [S, p. 523–4]. See also [Bo] for a similar argument.

For $p \geqslant 1$, let

$$B_p = \left(\int_\Lambda \left|\sum_{n=1}^N I_{c_n}\right|^p \mathrm{d}\pi(\lambda)\right)^{1/p}.$$

Then by Minkowski's inequality

$$B_p \leqslant \sum_{k=1}^N q_k + \left(\int_\Lambda \left|\sum_{k=1}^N (I_{c_k} - q_k)\right|^p \mathrm{d}\pi(\lambda)\right)^{1/p},$$

which by lemmas 10 and 11 is

$$\leqslant \sum_{k=1}^N q_k + 2\left(\int_0^1 \int_\Lambda \left|\sum_{k=1}^N r_k(t)(I_{c_k} - q_k)\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}t\right)^{1/p}.$$

Using Lemma 7 this is

$$\leqslant \sum_{k=1}^{N} q_k + 2\sqrt{p}\left(\int_\Lambda \left(\sum_{k=1}^{N} |I_{c_k} - q_k|^2\right)^{p/2} \mathrm{d}\pi(\lambda)\right)^{1/p},$$

which is

$$\leqslant \sum_{k=1}^{N} q_k + 4\sqrt{p}B_p^{1/2}.$$

It therefore follows that there exists a positive constant C such that

$$B_p \leqslant C \max\left(p, \sum_{k=1}^{N} q_k\right) \tag{7}$$

and that

$$\left(\int_\Lambda \left|\sum_{k=1}^{N} \chi_{l/A_r}(k\eta)(I_{c_k} - q_k)\right|^p \mathrm{d}\pi(\lambda)\right)^{1/p} \leqslant C\max\left(p, p^{1/2}\left(\sum_{k=1}^{N} q_k\right)^{1/2}\right). \tag{8}$$

Let

$$E(\lambda) = \frac{1}{S_N}\sum_{n\in S_N} \chi_{l/A_r}(n\eta) - \frac{1}{\sum_{l=1}^{N} q_l}\sum_{k=1}^{N} q_k \chi_{l/A_r}(k\eta).$$

By the triangle inequality this satisfies

$$|E(\lambda)| \leqslant \frac{1}{\sum_{k=1}^{N} p_k}\left(\left|\sum_{k=1}^{N}(I_{c_k} - q_k)\right| + \left|\sum_{k=1}^{N} \chi_{l/A_r}(k\eta)(I_{c_k} - q_k)\right|\right).$$

Now by the mean value theorem there exists θ_0 in the interval $[\theta_1, \theta_2]$ such that

$$E(\lambda,\theta_0) - |\theta_1 - \theta_2||E'(\lambda,\theta_0)| \leqslant |E(\lambda,\theta_1)| \leqslant E(\lambda,\theta_0) + |\theta_1 - \theta_2||E'(\lambda,\theta_0)|.$$

We therefore have by Lemma 8

$$E(\lambda,\theta_0) - |\theta_1 - \theta_2| N \sup_{\theta\in[0,2\pi)} |E(\lambda,\theta)| \leqslant |E(\lambda,\theta_1)|$$

$$\leqslant E(\lambda,\theta_0) + |\theta_1 - \theta_2| N \sup_{\theta\in[0,2\pi)} |E(\lambda,\theta)|.$$

Hence, for an appropriate choice of the positive constant c, if $|\theta_1 - \theta_2| \leqslant c/N$, there exist positive constants c_1 and c_2 such that

$$c_1|E(\lambda,\theta_1)| \leqslant |E(\lambda,\theta_2)| \leqslant c_2|E(\lambda,\theta_1)|.$$

This means that if for some θ in $[0,\pi)$ we let

$$D = \left\{\theta, \theta + \frac{c}{N}, \ldots, \theta + \frac{c(N-1)}{N}\right\}$$

and hence we have

$$\left(\int_\Lambda \sup_{\theta\in[0,2\pi)} |E(\lambda,\theta)|^p \mathrm{d}\pi(\lambda)\right)^{1/p} \leqslant C\left(\int_\Lambda \sup_{\theta\in D} |E(\lambda,\theta)|^p \mathrm{d}\pi(\lambda)\right)^{1/p}$$

$$\leqslant 2C \sup_{\theta\in D}\left(\int_\Lambda |E(\lambda,\theta)|^p \mathrm{d}\pi(\lambda)\right)^{1/p}$$

which by (7) and (8) is

$$\leqslant C\left\{\frac{\log N}{\Sigma_{k=1}^{N} q_n}\right\}^{1/2}.$$

Applying Markov's inequality we have

$$\pi\left(\left\{\lambda: \left|\frac{1}{|S_N|}\sum_{n\in S_N}\chi_{l/A_r}(n\eta) - \frac{1}{\Sigma_{l=1}^{N} q_l}\sum_{k=1}^{N} q_k \chi_{l/A_r}(k\eta)\right| \geqslant B\left\{\frac{\log N}{\Sigma_{l=1}^{N} q_l}\right\}^{1/2}\right\}\right)$$
$$\leqslant c^{-p}\left\{\frac{\Sigma_{l=1}^{N} q_l}{\log N}\right\}^{-p/2}\int_{\Lambda}\left(\sup_{\theta\in[0,2\pi)}|E(\lambda,\theta)|\right)^{p} d\pi(\lambda).$$

For appropriate choice of B this is

$$\leqslant N^{-\alpha}. \tag{9}$$

Because Δ_a is a countable group to prove Theorem 2(i) it is sufficient to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(k_n\eta) = 0$$

for almost all S in Λ. Remember that $\lim_{n\to\infty} q_n n = \infty$ and $\lim_{N\to\infty} \log N/\Sigma_{l=1}^{N} q_l = 0$. In addition q_n decreases monotonically to zero and $\Sigma_{k=1}^{\infty} q_k = \infty$ for η not equal to zero, we have

$$\lim_{N\to\infty}\frac{1}{\Sigma_{k=1}^{N} q_k}\sum_{k=1}^{N} q_k \chi_{l/A_r}(k\eta) = 0.$$

Hence as a consequence of (9) the proof of Theorem 2 (i) is complete.

4. Proof of Theorem 2 (ii)–(ix)

To demonstrate Theorem 2 (ii)–(ix) we need a number of lemmas. The first appears in [Na3].

Lemma 12. *Suppose for a sequence of real numbers* $(a_n)_{n=1}^{\infty}$ *and a real number* α, *for all* β *in* $\{\alpha + m : m\in\mathbf{Z}\}\setminus\{\mathbf{Z}\}$, *that we have*

$$E(\beta, N) = \sum_{n=1}^{N} e(\beta a_n) = o(N).$$

Then we have

$$\sum_{n=1}^{N} e(\alpha[a_n]) = o(N).$$

We now state the well known uniform distribution results of Weyl [W] and Vinogradov [Vi, p. 287] as in Lemmas 13 and 14 respectively.

Lemma 13. *Let*

$$P(x) = \alpha_k x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_0.$$

Suppose the real numbers $\alpha_1, \ldots, \alpha_k$ are not all rational multiples of a fixed real number. Then for each real number β other than zero we have

$$\sum_{n=1}^{N} e(\beta P(n)) = o(N).$$

Lemma 14. *Suppose that the polynomial $P(x)$ satisfies the hypothesis of Lemma 13 and that $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes. Then again*

$$\sum_{n=1}^{N} e(\beta P(p_n)) = o(N).$$

Lemmas 15 and 16 which follow appear in [Ba].

Lemma 15. *Suppose $f(z)$ is a non-polynomial entire function which is real valued on the real numbers and such that $|f(z)| \ll e^{(\log|z|)^{\alpha}}$ with $\alpha < \frac{4}{3}$, then if $\beta \neq 0$*

$$\sum_{n=1}^{N} e(\beta f(n)) = o(N).$$

Similarly we have the following lemma.

Lemma 16. *Suppose $f(z)$ is as in Lemma 15 and that $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes, then if $\beta \neq 0$*

$$\sum_{n=1}^{N} e(\beta f(p_n)) = o(N).$$

The following lemma is quoted from [Co].

Lemma 17. *Let a, b, k be integers. Suppose that $a < b$, $k \geqslant 2$, and set $s = 2^k$. Suppose the real valued function g is k times differentiable in the interval $[a, b]$ and that $|g^{(k)}(x)| \geqslant \lambda$ for some positive constant λ which is independent of x in $[a, b]$. Then letting*

$$R = \frac{1}{a-b}|g^{(k-1)}(a) - g^{(k-1)}(b)|$$

we have

$$\left|\sum_{u=a}^{b} e(g(u))\right| < C(b-a)\left(\left(\frac{R^2}{\lambda}\right)^{1/(k-2)} + \frac{1}{(\lambda(b-a)^k)^{2/k}} + \left(\frac{R}{\lambda(b-a)}\right)^{2/k}\right),$$

for some $C > 0$.

Setting $g(x) = \beta x^{\alpha}$, $a = 1, b = 1$ and $[\alpha] + 1$ we readily see that the following lemma is true.

Lemma 18. *For fixed ε, if $0 < \lambda < k - \alpha$ there exists $t \in (0, \frac{\varepsilon}{2})$ such that if $N^{\varepsilon - \alpha} \leqslant |\beta| \leqslant N^{\varepsilon}$ then*

$$\left|\sum_{u=a}^{b} e(\beta u^{\alpha})\right| \leqslant CN^{1-t},$$

where the constant depends only on α and ε.

For a sequence of real numbers $(x_n)_{n=1}^N$ we call

$$D(x_1,\ldots,x_N) = \sup_{I\subseteq[0,1)} \left| \frac{1}{N}\sum_{n=1}^{N} \chi_I(\langle x_n\rangle) - |I| \right|$$

its discrepancy. Here the supremum is taken over all intervals I contained in $[0,1)$ which are closed on the left and open on the right. The next few lemmas are metrical in nature. Lemmas 19 and 20 in the case where β is an integer other than zero appear in [Na1]. The general case is virtually identical and the proof is therefore forgone.

Lemma 19. *Suppose that* $(a_n)_{n=1}^{\infty}$ is a strictly increasing sequence of integers, β *is a real number other than zero and that we are given* $\varepsilon > 0$. *Then for almost all* x *with respect to Lebesgue measure we have*

$$D(\beta a_1 \cos(a_1 x),\ldots,\beta a_N \cos(a_N x)) = o(N^{-1/4}(\log N)^{(3/2)+\varepsilon}).$$

A refinement is the following lemma.

Lemma 20. *Suppose that* $(a_n)_{n=1}^{\infty}$ *are as in Lemma* 18 *and further that* $a_n = O(n^p)$ *for some* $p > 1$ *then*

$$D(\beta a_1 \cos(a_1 x),\ldots,\beta a_N \cos(a_N x)) = o(1)$$

except on a set of Hausdorff dimension at most $1-(1/(4p+(1/2)))$.

Lemmas 21 and 22 which follow appear in [Na2].

Lemma 21. *Suppose the sequence* $(g_n(x))_{n=1}^{\infty}$ *of continuously differentiable functions are all defined on the same interval* $[a,b]$ *and for each pair of distinct natural numbers* m *and* n *that we have*:

(a) $g_n'(x) - g_m'(x)$ *is monotone on* $[a,b]$ *and*
(b) *there is an absolute constant* λ *such that*

$$|g_n'(x) - g_m'(x)| \geqslant \lambda > 0.$$

Then given $\varepsilon > 0$, *for almost all* x *with respect to Lebesgue measure we have*

$$D(\beta g_1(x),\ldots,\beta g_N(x)) = o(N^{-1/2}(\log N)^{(5/2)+\varepsilon}).$$

Analogous to Lemma 21 is the following lemma.

Lemma 22. *Suppose that the sequence of functions* $(g_n(x))_{n=1}^{\infty}$ *satisfies the hypothesis of Lemma* 22 *and further that the following is true*:

(c) *for all* x *in* $[a,b]$ *we have*

$$|g_n'(x)| = O(n^p)$$

for some $p > 1$ *and with an implied constant independent of* x *and*

(d) *for each pair of distinct positive integers* m *and* n *the functions*

$$\frac{g_n'(x)g_m'(x)}{g_m'(x) - g_n'(x)}$$

are monotonic on $[a, b]$, *then*

$$D(\beta g_1(x), \ldots, \beta g_N(x)) = o(1)$$

except on a set of Hausdorff dimension at most $1 - (1/p)$.

We have the following lemma due to Koksma [Ko].

Lemma 23. *Suppose that the function* f *is of bounded variation* $V(f)$ *on* $[0, 1)$ *then for a finite set of reals* $(x_n)_{n=1}^N$ *we have*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\langle x_n \rangle) - \int_0^1 f(t)\mathrm{d}t \right| \leqslant V(f) D(x_1, \ldots, x_N).$$

Appplying Lemma 12 and quoting Lemmas 13 to 18 we have Theorem 2 in the cases (ii)–(vi). The remaining cases of Theorem 2 follow on applying Lemma 23 to $f(x) = e(x)$ and quoting Lemmas 12 and 19–23.

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A new approximate functional equation for Hurwitz zeta function for rational parameter

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Abstract. For Hurwitz zeta function $\zeta(s,(a/k))$ with $a=1,2,3,\ldots,k$, we obtain a new simple approximate functional equation (uniform in k and t) in critical strip. Our method should prove to be an alternative approach to Atkinson's method in dealing with $\Sigma_{\chi(\mathrm{mod}\,q)}\int_0^T|L(s,\chi)|^2\,\mathrm{d}t$, where $L(s,\chi)$ is Dirichlet L-series modulo q and $s=\sigma+it$.

Keywords. Hurwitz zeta function; Dirichlet L-series.

1. Introduction

For integer $q\geqslant 1$, let $\chi(\mathrm{mod}\,q)$ be a Dirichlet character modulo q and let $L(s,\chi)$ be the corresponding Dirichlet L-series, where $s=\sigma+it$ with σ,t real. In what follows, we shall have $t>0$. For $0<\alpha\leqslant 1$. Let $\zeta(s,\alpha)$ be Hurwitz zeta function defined by $\zeta(s,\alpha)=\Sigma_{n\geqslant 0}(n+\alpha)^{-s}$ for Re $s>1$; and its analytic continuation. Let $\zeta_1(s,\alpha)=\zeta(s,\alpha)-\alpha^{-s}$. In what follows $\ll$ and 0-constants are absolute. Also $[u]$ for real u, shall denote integral part of u. In the light of the fact

$$\Sigma_{\chi(\mathrm{mod}\,q)}|L(\sigma+it,\chi)|^2=\frac{\phi(q)}{q^{2\sigma}}\cdot\sum_{k/q}\mu(q/k)\left(\sum_{a=1}^{k}\left|\zeta\left(\sigma+it,\frac{a}{k}\right)\right|^2\right),$$

where ϕ and μ are Euler and Moebius functions respectively, one needs an estimate for $\Sigma_{a=1}^{k}|\zeta(s,(a/k))|^2$ for a divisor k of q. The object of this paper is to prove the following theorem, which gives a new form of the approximate functional equation for $\zeta(s,(a/k))$ in critical strip (for $a=1,2,\ldots,k$), with a mean value estimate for the error term.

Theorem. *Let $s=\sigma+it$ with $0<\sigma<1$ and $x\geqslant 1$ an integer. Let $2\pi xy=t$ with $y>0$. Let $K(s)=(2\pi/i)^{s-1}.|^{-}(1-s)$. Then, we have*

1) *for $y\geqslant k$,*

$$\zeta\left(s,\frac{a}{k}\right)=\sum_{0\leqslant m\leqslant x-1}\left(m+\frac{a}{k}\right)^{-s}+K(s)\sum_{n\leqslant y}\mathrm{e}^{-(2\pi ina)/k}.n^{s-1}$$
$$+\sum_{1\leqslant n\leqslant y+k}b_n(s)\mathrm{e}^{-(2\pi ina)/k}+R\left(s,x,\frac{a}{k}\right),$$

where

$$b_n(s)=-\int_0^x\frac{\mathrm{e}^{2\pi inu}}{u^s}\,\mathrm{d}u \qquad \text{for} \qquad n\leqslant y$$
$$=\int_x^\infty\frac{\mathrm{e}^{2\pi inu}}{u^s}\,\mathrm{d}u \qquad \text{for} \qquad y<n\leqslant y+k;$$

with

$$\sum_{a=1}^{k}\left|R\left(s,x,\frac{a}{k}\right)\right|^2 \ll x^{-2\sigma}\log^2 kt.$$

2) *for* $y<k$,

$$\zeta\left(s,\frac{a}{k}\right)=\sum_{0\leqslant m\leqslant x-1}\left(m+\frac{a}{k}\right)^{-s}+K(s)\sum_{n\leqslant y}e^{-(2\pi ina)/k}\cdot n^{s-1}$$
$$+\frac{x^{1-s}}{s-1}+\sum_{1\leqslant n\leqslant y+k}b_n(s)e^{-(2\pi ina)/k}+\sum_{1\leqslant n\leqslant k}c_n(s)\,e^{(2\pi ina)/k}+R'\left(s,x,\frac{a}{k}\right),$$

where

$$c_n(s)=-\int_0^x\frac{e^{-2\pi inu}}{u^s}du \qquad \textit{for} \qquad n\leqslant y$$
$$=\int_x^\infty\frac{e^{-2\pi inu}}{u^s} \qquad \textit{for} \qquad y<n\leqslant k$$

with

$$\sum_{a=1}^{k}\left|R'\left(s,x,\frac{a}{k}\right)\right|^2 \ll x^{-2\sigma}\log^2 kt$$

and

$$\sum_{a=1}^{k}\left|\sum_{n\leqslant k}c_n(s)\cdot e^{(2\pi ina)/k}\right|^2 \ll kx^{-2a}\cdot\min(1,1/y)$$

Our approximate functional equation can be used to obtain asymptotic formula for $\Sigma_{\chi(\mathrm{mod}\,q)}|L(s,\chi)|^2$ and $\Sigma_{\chi(\mathrm{mod}\,q)}\int_0^T|L(s,\chi)|^2\,dt$ (uniform in q and t (or T)) on critical line. In fact, our approach may serve as an alternative to Atkinson's method, while dealing with $\int_0^T dt\Sigma_{\chi(\mathrm{mod}\,q)}|L(s,\chi)|^2$ on critical line. Our approach also gives an interesting expression for $\int_0^1|\zeta_1(s,\alpha)|^2\,d\alpha$ in critical strip and an expression for $h/k-[h/k]-1/2$ for integers h and k with $k\geqslant 1$ (see Remarks below).

Our Theorem may be viewed as the continuation of [1]. In the context of our Theorem, it is pertinent to compare our Theorem to Riemann–Siegel type formula for $\zeta((1/2)+it,\alpha)$ of [2]. Also see [3] and [4].

In what follows, we shall be using the following facts. For integer $n\geqslant 1$ and sufficiently large integer N and with $y=t/2\pi x$, we have

a) for $n>y$, $\displaystyle\int_x^N\frac{e^{2\pi inu}}{u^{s+1}}du\ll\frac{x^{-\sigma-1}}{n-y}$; and $\displaystyle\int_x^N\frac{e^{2\pi inu}}{u^s}du\ll\frac{x^{-\sigma}}{n-y}$,

b) $\displaystyle\int_x^N\frac{e^{-2\pi inu}}{u^{s+1}}du\ll\frac{x^{-\sigma-1}}{n+y}$ and $\displaystyle\int_x^N\frac{e^{-2\pi inu}}{u^s}du\ll\frac{x^{-\sigma}}{n}$ for $n\geqslant 1$,

c) $\displaystyle\int_0^x\frac{e^{2\pi inu}}{u^s}du\ll\frac{x^{-\sigma}}{y-n}$ for $n<y$; and $\ll t^{(1/2)-\sigma}\cdot y^{\sigma-1}$ for $n\geqslant 1$,

d) $\displaystyle\int_0^x\frac{e^{-2\pi inu}}{u^s}du\ll\frac{x^{-\sigma}}{y}$ for $n\geqslant 1$,

e) For $n \geqslant 1$, we have $\int_0^\infty \frac{e^{2\pi i n u}}{u^s} du = K(s) \cdot n^{s-1}$ and

$$\int_0^\infty \frac{e^{-2\pi i n u}}{u^s} du = \overline{K(\bar{s})} \cdot n^{s-1} \ll n^{\sigma-1} \cdot e^{-\pi t} \cdot t^{(1/2)-\sigma},$$

f) $\sum_{n \leqslant y+k} |b_n(s)| \ll x^{-\sigma} \cdot \log k(y+2) + t^{(1/2)-\sigma} \cdot y^{\sigma-1}$ and $\sum_{n \leqslant k} |c_n(s)| \ll x^{-\sigma} \cdot \log kt$.

Remarks on Theorem. 1) From the proof of Theorem, we shall find that for $0 < \operatorname{Re} s < 1$ with fixed s, $\zeta_1(s, \alpha)$ as a function of α has a Fourier series $\zeta_1(s, \alpha) = \sum_{n=-\infty}^{\infty} a_n(s) e^{2\pi i n \alpha}$, where $a_0(s) = 1/(s-1)$; and for $n \neq 0$, $a_n(s) = \int_1^\infty (e^{-2\pi i n u}/u^s)\, du$. As a function of α for $0 < \operatorname{Re} s < 1$, $\zeta_1(s, \alpha)$ is a function of bounded variation and hence it satisfies Parseval's result

$$\int_0^1 |\zeta_1(s, \alpha)|^2 d\alpha = \sum_{n=-\infty}^{\infty} |a_n(s)|^2.$$

2) In the light of the proof of our Theorem it is interesting to find an expression for $h/k - [h/k] - 1/2$, where h, k are integers with $k \geqslant 1$.

$$\frac{h}{k} - \left[\frac{h}{k}\right] - \frac{1}{2} = -\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n h/k}}{2\pi i n} = \frac{1}{2\pi i} \sum_{r=1}^{k-1} a(r, k) e^{2\pi i h r/k},$$

where

$$a(r, k) = -\sum_{\substack{n=-\infty \\ n \neq 0 \\ n \equiv r (\mathrm{mod}\, k)}}^{\infty} \frac{1}{n} = \sum_{m=1}^{\infty} \frac{2r}{m^2 k^2 - r^2} - \frac{1}{r}.$$

This expression should be useful in problems related to Dedekind sums.

2. *Proof of Theorem.* We shall write

$$\zeta_x(s, \alpha) = \zeta(s, \alpha) - \sum_{0 \leqslant m \leqslant x-\alpha} (m+\alpha)^{-s}.$$

Thus for $\operatorname{Re} s = \sigma > 0$, as in [1] for large N and integer x, we have

$$\begin{aligned}\zeta_x(s, \alpha) &= -s \cdot \int_x^N \frac{(u - \alpha - [u-\alpha] - (1/2))}{u^{s+1}} du \\ &\quad + x^{-s} \cdot \left(x - \alpha - [x-\alpha] - \frac{1}{2}\right) + \frac{x^{1-s}}{s-1} + O(tN^{-\sigma}) \\ &= \sum_{n=-\infty}^{\infty}{}' \frac{e^{-2\pi i n \alpha}}{2\pi i n} \left(\int_x^N \frac{s e^{2\pi i n u}}{u^{s+1}} du - x^{-s} \right) + \frac{x^{1-s}}{s-1} + O(tN^{-\sigma}),\end{aligned}$$

where dash over the summation sign $\sum_{n=-\infty}^{\infty}{}'$ indicates exclusion of $n = 0$. On integration by parts, we get

$$\begin{aligned}\zeta_x(s, \alpha) &= \sum_{n=-\infty}^{\infty}{}' \frac{e^{-2\pi i n \alpha}}{2\pi i n} \left(-\frac{e^{2\pi i n N}}{N^s} + 2\pi i n \int_x^N \frac{e^{2\pi i n u}}{u^s} du \right) + \frac{x^{1-s}}{s-1} + O(tN^{-\sigma}) \\ &= -N^{-s} \cdot \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n (N-\alpha)}}{2\pi i n} + \sum_{n=-\infty}^{\infty}{}' e^{-2\pi i n \alpha} \cdot \int_x^N \frac{e^{2\pi i n u}}{u^s} du\end{aligned}$$

$$+\frac{x^{1-s}}{s-1}+O(tN^{-\sigma})=N^{-s}\cdot\left(N-\alpha-[N-\alpha]-\frac{1}{2}\right)$$

$$+\sum_{n=-\infty}^{\infty}{}' e^{-2\pi in\alpha}\cdot\int_x^N\frac{e^{2\pi inu}}{u^s}du+\frac{x^{1-s}}{s-1}+O(t\cdot N^{-\sigma})$$

$$=\sum_{n=-\infty}^{\infty}{}' e^{-2\pi in\alpha}\cdot\int_x^N\frac{e^{2\pi inu}}{u^s}du+\frac{x^{1-s}}{s-1}+O(t\cdot N^{-\sigma}).$$

Taking $x=1$ and letting $N\to\infty$, we get

$$\zeta_1(s,\alpha)=\sum_{n=-\infty}^{\infty}a_n(s)\cdot e^{2\pi in\alpha},$$

with $a_0(s)=1/(s-1)$ and $a_n(s)=\int_1^\infty e^{-2\pi inu}/u^s du$ for $n\neq 0$.
Now, we continue with the proof of the Theorem. For $\sigma>0$, for large N, we have

$$\zeta_x\left(s,\frac{a}{k}\right)=-s\int_x^N\frac{(u-(a/k)-[u-(a/k)]-(1/2))}{u^{s+1}}du$$

$$+\left(x-\frac{a}{k}-\left[x-\frac{a}{k}\right]-\frac{1}{2}\right)\cdot x^{-s}+\frac{x^{1-s}}{s-1}+O(tN^{-\sigma}).$$

$$=\frac{x^{1-s}}{s-1}+\left(x-\frac{a}{k}-\left[x-\frac{a}{k}\right]-\frac{1}{2}\right)x^{-s}$$

$$+\sum_{n=-\infty}^{\infty}{}'\frac{e^{-2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du+O(tN^{-\sigma}).$$

Next,

$$\sum_{n\geqslant 1}\frac{e^{-2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du=\sum_{n\leqslant y+k}+\sum_{n>y+k},\ \text{say.}$$

Now

$$\sum_{a=1}^{k}\left|\sum_{n>y+k}\frac{e^{-2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du\right|^2$$

$$=k\sum_{a=1}^{k}\left|\sum_{\substack{n>y+k\\ n\equiv a(\mathrm{mod}\,k)}}\frac{1}{2\pi in}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du\right|^2$$

$$\ll k\sum_{a=1}^{k}\left|x^{-\sigma}\cdot\sum_{\substack{n>y+k\\ n\equiv a(\mathrm{mod}\,k)}}\frac{y}{n(n-y)}\right|^2$$

$$=kx^{-2\sigma}\cdot\sum_{a=1}^{k}\left|\sum_{m>1+((y-a)/k)}\frac{y}{(mk+a)(mk+a-y)}\right|^2$$

$$\ll\frac{x^{-2\sigma}}{k}\cdot\sum_{a=1}^{k}\left|\sum_{m>1+((y-a)/k)}\frac{y/k}{(m+(a/k))(m-(y-a)/k)}\right|^2\ll x^{-2\sigma}\cdot\log^2 kt.$$

Next

$$\sum_{a=1}^{k}\left|\sum_{n>k}\frac{e^{2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{-2\pi inu}}{u^{s+1}}du\right|^2$$

$$=k\cdot\sum_{a=1}^{k}\left|\sum_{\substack{n>k\\ n\equiv a(\mathrm{mod}\,k)}}\frac{1}{2\pi in}\cdot\int_x^N\frac{se^{-2\pi inu}}{u^{s+1}}du\right|^2$$

$$\ll kx^{-2\sigma}\cdot\sum_{a=1}^{k}\left|\sum_{\substack{n>k\\ n\equiv a(\mathrm{mod}\,k)}}\frac{y}{n(n+y)}\right|^2$$

$$\ll kx^{-2\sigma}\cdot\sum_{a=1}^{k}\left|\sum_{m\geqslant 1}\frac{y}{(mk+a)(mk+a+y)}\right|^2$$

$$\ll\frac{x^{-2\sigma}}{k}\cdot\sum_{a=1}^{k}\left|\sum_{m\geqslant 1}\frac{y/k}{(m+(a/k))(m+(a/k)+(y/k))}\right|^2\ll x^{-2\sigma}\log^2 kt.$$

Thus, we have

$$\zeta_x\left(s,\frac{a}{k}\right)-\frac{x^{1-s}}{s-1}=\left(x-\frac{a}{k}-\left[x-\frac{a}{k}\right]-\frac{1}{2}\right)x^{-s}$$
$$+\frac{1}{2\pi i}\left(\sum_{n\leqslant y+k}\frac{e^{-2\pi ina/k}}{n}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du\right.$$
$$\left.-\sum_{n\leqslant k}\frac{e^{2\pi ina/k}}{n}\cdot\int_x^N\frac{se^{-2\pi inu}}{u^{s+1}}du\right)+O(tN^{-\sigma})+R_1\left(s,x,\frac{a}{k}\right),$$

where

$$\sum_{a=1}^{k}\left|R_1\left(s,x,\frac{a}{k}\right)\right|^2\ll x^{-2\sigma}\cdot\log^2 kt.$$

Next,

$$\sum_{n\leqslant y+k}\frac{e^{-2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du$$

$$=x^{-s}\sum_{n\leqslant y+k}\frac{e^{2\pi in(x-(a/k))}}{2\pi in}+\sum_{n\leqslant y+k}e^{-2\pi ina/k}\cdot\int_x^N\frac{e^{2\pi inu}}{u^s}du+O(N^{-\sigma}\cdot\log kt).$$

Similarly,

$$\sum_{n\leqslant k}\frac{e^{2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{-s\cdot e^{-2\pi inu}}{u^{s+1}}du$$

$$=-x^{-s}\cdot\sum_{n\leqslant k}\frac{e^{-2\pi in(x-(a/k))}}{2\pi in}+\sum_{n\leqslant k}e^{2\pi ina/k}\cdot\int_x^N\frac{e^{-2\pi inu}}{u^s}du+O(N^{-\sigma}\cdot\log kt).$$

Next,

$$\sum_{a=1}^{k}\left|\sum_{k<n\leqslant y+k}\frac{e^{2\pi in(x-(a/k))}}{2\pi in}\right|^2$$

$$= k \sum_{a=1}^{k} \left| \sum_{\substack{k < n \leqslant y+k \\ n \equiv a (\mathrm{mod}\, k)}} \frac{e^{2\pi i n x}}{2\pi i n} \right|^2$$

$$= k \sum_{a=1}^{k} \left| \sum_{\substack{k < n \leqslant y+k \\ n \equiv a (\mathrm{mod}\, k)}} \frac{1}{2\pi i n} \right|^2 \text{ as } x \text{ is an integer and thus } \ll \log^2 kt.$$

As x is an integer,

$$x^{-s} \cdot \left(x - \frac{a}{k} - \left[x - \frac{a}{k} \right] - \frac{1}{2} \right) = x^{-s} \cdot \left(\frac{1}{2} - \frac{a}{k} \right).$$

Thus,

$$\left(\frac{1}{2} - \frac{a}{k} \right) + \sum_{n \leqslant k} \frac{e^{-2\pi i n a/k}}{2\pi i n} - \sum_{n \leqslant k} \frac{e^{2\pi i n a/k}}{2\pi i n}$$

$$= - \sum_{|n| > k} \frac{e^{-2\pi i n a/k}}{2\pi i n}.$$

Hence,

$$\zeta_x \left(s, \frac{a}{k} \right) = \frac{x^{1-s}}{s-1} - x^{-s} \cdot \sum_{|n| > k} \frac{e^{-2\pi i n a/k}}{2\pi i n}$$

$$+ \sum_{n \leqslant y+k} e^{-2\pi i n a/k} \cdot \int_x^N \frac{e^{2\pi i n u}}{u^s} du$$

$$+ \sum_{n \leqslant k} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du + O(tN^{-\sigma})$$

$$+ R_2 \left(s, x, \frac{a}{k} \right),$$

where $\Sigma_{a=1}^{k} |R_2(s, x, (a/k))|^2 \ll x^{-2\sigma} \log^2 kt$.

Next, we show

$$\sum_{a=1}^{k} \left| \sum_{|n| > k} \frac{e^{-2\pi i n a/k}}{2\pi i n} \right|^2 = \sum_{a=1}^{k-1} \left| \sum_{|n| > k} \frac{e^{-2\pi i n a/k}}{2\pi i n} \right|^2 \ll 1.$$

We know for real x, a and b,

$$\sum_{a < n \leqslant b} e^{\pm 2\pi i n x} \ll \frac{1}{\| x \|}.$$

Here $\| x \| = \min_n |x - n|$, where n varies over integers. This gives, by partial summation for large M,

$$\sum_{n=k+1}^{M} \frac{e^{\pm 2\pi i n x}}{2\pi i n} \ll \frac{1}{k \| x \|}.$$

Thus

$$\sum_{a=1}^{k-1}\left|\sum_{|n|>k}\frac{e^{-2\pi ina/k}}{2\pi in}\right|^2 \ll \sum_{a=1}^{k-1}\left\|\frac{a}{k}\right\|^{-2}\cdot k^{-2} \ll \sum_{\substack{a\neq 0\\ -k/2<a\leqslant k/2}} k^{-2}\cdot\frac{k^2}{a^2}\ll 1.$$

Thus

$$\zeta_x\left(s,\frac{a}{k}\right)-\frac{x^{1-s}}{s-1}=\sum_{n\leqslant y+k} e^{-2\pi ina/k}\cdot\int_x^N\frac{e^{2\pi inu}}{u^s}du + \sum_{n\leqslant k} e^{2\pi ina/k}\cdot\int_x^N\frac{e^{-2\pi inu}}{u^s}du + O(t\cdot N^{-\sigma})+R_3\left(s,x,\frac{a}{k}\right),$$

where $\Sigma_{a=1}^{k}|R_3(s,x,(a/k))|^2\ll x^{-2\sigma}\log^2 kt$.
Thus,

$$\zeta_x\left(s,\frac{a}{k}\right)-\frac{x^{1-s}}{s-1}=\sum_{n\leqslant y} e^{-2\pi ina/k}\cdot\left(\int_0^\infty-\int_0^x-\int_N^\infty\right)\frac{e^{2\pi inu}}{u^s}du + \sum_{y<n\leqslant y+k} e^{-2\pi ina/k}\cdot\int_x^N\frac{e^{2\pi inu}}{u^s}du + \sum_{n\leqslant k} e^{2\pi ina/k}\cdot\int_x^N\frac{e^{-2\pi inu}}{u^s}du + O(tN^{-\sigma})+R_3\left(s,x,\frac{a}{k}\right).$$

If $k\leqslant y$, we write

$$\sum_{n\leqslant k} e^{2\pi ina/k}\cdot\int_x^N\frac{e^{-2\pi inu}}{u^s}du = \sum_{n\leqslant k} e^{2\pi ina/k}\cdot\left(\int_0^\infty-\int_0^x-\int_N^\infty\right)\frac{e^{-2\pi inu}}{u^s}du.$$

Noting

$$\int_0^\infty\frac{e^{2\pi inu}}{u^s}du=K(s)\cdot n^{s-1};\quad \int_N^\infty\frac{e^{2\pi inu}}{u^s}du\ll N^{-\sigma},$$

$$\int_0^\infty\frac{e^{-2\pi inu}}{u^s}du=\overline{K(\bar s)}\cdot n^{s-1};\quad \int_N^\infty\frac{e^{-2\pi inu}}{u^s}du\ll N^{-\sigma},$$

so that $|K(\bar s)\cdot\Sigma_{n\leqslant k} e^{2\pi ina/k}\cdot n^{s-1}|$ is small; and noting for $y\geqslant k$,

$$\sum_{a=1}^{k}\left|\sum_{n\leqslant k} e^{2\pi ina/k}\int_0^x\frac{e^{-2\pi inu}}{u^s}du\right|^2 = k\sum_{n\leqslant k}\left|\int_0^x\frac{e^{-2\pi inu}}{u^s}du\right|^2\ll k\sum_{n\leqslant k}\frac{x^{-2\sigma}}{y^2}\ll x^{-2\sigma}$$

and $\Sigma_{a=1}^{k} |x^{1-s}/(s-1)|^2 \ll x^{-2\sigma}$ and on letting $N \to \infty$, we find that for $y \geqslant k$,

$$\zeta\left(s, \frac{a}{k}\right) = \sum_{m \leqslant x-1} \left(m + \frac{a}{k}\right)^{-s} + K(s) \cdot \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot n^{s-1}$$
$$- \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot \int_0^x \frac{e^{2\pi i n u}}{u^s} du + \sum_{y < n \leqslant y+k} e^{-2\pi i n a/k} \cdot \int_x^\infty \frac{e^{2\pi i n u}}{u^s} du$$
$$+ R_4\left(s, x, \frac{a}{k}\right),$$

where $\Sigma_{a=1}^{k} |R_4(s, x, (a/k))|^2 \ll x^{-2\sigma} \log^2 kt$. Next let $y < k$. In that case

$$\sum_{n \leqslant k} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du = \left(\sum_{n \leqslant y} + \sum_{y < n \leqslant k}\right) e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du.$$

Next

$$\sum_{a=1}^{k} \left| \sum_{y < n \leqslant k} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du \right|^2$$
$$= k \sum_{y < n \leqslant k} \left| \int_x^N \frac{e^{-2\pi i n u}}{u^s} du \right|^2 \ll k \sum_{y < n \leqslant k} \frac{x^{-2\sigma}}{n^2} \ll k x^{-2\sigma} \cdot \min(1, 1/y).$$

As before, we find

$$\sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du$$
$$= \overline{K(\bar{s})} \cdot \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot n^{s-1} - \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_0^x \frac{e^{-2\pi i n u}}{u^s} du + O(N^{-\sigma} \log kt).$$

Noting $|\overline{K(\bar{s})} \cdot \Sigma_{n \leqslant y} e^{2\pi i n a/k} \cdot n^{s-1}|$ is very small and

$$\sum_{a=1}^{k} \left| \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_0^x \frac{e^{-2\pi i n u}}{u^s} du \right|^2$$
$$= k \sum_{n \leqslant y} \left| \int_0^x \frac{e^{-2\pi i n u}}{u^s} du \right|^2 \ll k \sum_{n \leqslant y} \frac{x^{-2\sigma}}{y^2} \ll \frac{k}{y} \cdot x^{-2\sigma};$$

and

$$\sum_{a=1}^{k} \left| \frac{x^{1-s}}{s-1} \right|^2 \ll \frac{k}{y^2} \cdot x^{-2\sigma},$$

we find that for $y < k$,

$$\zeta\left(s, \frac{a}{k}\right) = \sum_{m \leqslant x-1} \left(m + \frac{a}{k}\right)^{-s} + K(s) \cdot \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot n^{s-1}$$
$$- \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot \int_0^x \frac{e^{2\pi i n u}}{u^s} du + \sum_{y < n \leqslant y+k} e^{-2\pi i n a/k} \cdot \int_x^\infty \frac{e^{2\pi i n u}}{u^s} du$$
$$+ \left(\frac{x^{1-s}}{s-1} - \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_0^x \frac{e^{-2\pi i n u}}{u^s} du + \sum_{y < n \leqslant k} e^{-2\pi i n u/k} \cdot \int_x^\infty \frac{e^{-2\pi i n u}}{u^s} du\right)$$
$$+ R_4\left(s, x, \frac{a}{k}\right),$$

where

$$\sum_{a=1}^{k}\left|R_4\left(s,x,\frac{a}{k}\right)\right|^2 \ll x^{-2\sigma}\log^2 kt,$$

and

$$\sum_{a=1}^{k}\left(\left|\sum_{n\leqslant y}\mathrm{e}^{2\pi ina/k}\cdot\int_0^x\frac{\mathrm{e}^{-2\pi inu}}{u^s}\mathrm{d}u\right|^2\right.$$
$$\left.+\left|\sum_{y<n\leqslant k}\mathrm{e}^{2\pi ina/k}\cdot\int_x^\infty\frac{\mathrm{e}^{-2\pi inu}}{u^s}\mathrm{d}u\right|^2\right)\ll kx^{-2\sigma}\cdot\min(1,1/y).$$

This completes the proof of our theorem.

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A weak Brun–Titchmarsh theorem for multiplicative functions

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Abstract. An important theorem of Shiu gives a (precise) bound for the average of values of multiplicative functions, of a certain class, over 'short' intervals. Here we obtain, by simple means, the above result of same qualitative order.

Keywords. Multiplicative functions; short intervals.

1. Introduction

In the sequel, $f(n)$ denotes a multiplicative function satisfying (i) $0 \leqslant f(n) = O_\varepsilon(n^\varepsilon)$, for every $\varepsilon > 0$; (ii) $f(p^l) \leqslant A_1^l$ with a positive constant A_1. Here and throughout, the letter p denotes primes. Under these conditions the following result was proved in [1]:

Theorem 1. *Let $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, and let a, k be integers satisfying $0 < a < k$, $(a, k) = 1$. Then, as $x \to \infty$, under* (i) *and* (ii)

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a(\mathrm{mod}\, k)}} f(n) \ll \frac{y}{\varphi(k)} \frac{1}{\log x} \exp\left(\sum_{\substack{p\leqslant x \\ p\neq k}} \frac{f(p)}{p} \right), \tag{1}$$

uniformly in a, k and y provided that

$$1 \leqslant k < y^{1-\alpha}, \quad x^\beta \leqslant y \leqslant x. \tag{2}$$

Here we present a weaker version of the above result noted while attempting for a simpler proof of it. Actually, we obtain mainly the following

Theorem 1′. *Let $0 < \alpha (< 1)$, $0 < \beta (< 1)$. Then, as $x \to \infty$, under* (i) *and* (ii),

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a(\mathrm{mod}\, k)}} f(n) \ll (a, k) \frac{y}{k} \exp\left(\sum_{\substack{p\leqslant y/k \\ (p,k)\mid(a,k)}} \frac{(f(p))^R}{p} \right) \tag{1′}$$

with a certain constant R depending only on α, β, f, uniformly in y and integers a, k provided that

$$1 \leqslant k < y^{1-\alpha}, \quad x^\beta \leqslant y \leqslant x. \tag{2′}$$

Also we note in course that Theorem 1′ and its proof, contain (qualitative) consequences similar to Theorems 2 and 3 of [1], apart from implying Theorem 4 there.

2. Proof of Theorem 1′

Let $n \in P_Y$ mean that there exist p and b such that $p \leqslant Y < p^b$, $p^b \| n$. Set $Y = (y/k)^\delta$, where δ denotes a small positive constant, to be fixed shortly. We start by observing

that

$$|\{n: M-y<n\leqslant M, n\equiv a(\mathrm{mod}\, k);\ n\in P_Y\}| = O\left(Y+(a,k)\left(\frac{y}{k}\right)^{1-\frac{1}{2}\delta}\right), \quad (3)$$

holds, with an absolute constant, uniformly in M. For this, obviously we have that the quantity on the left side is

$$\leqslant \sum_{p\leqslant Y}\left\{1+\sum_{\substack{b\\ Y<p^b\leqslant (y/k)(p^b,k)}}\left(1+\frac{y}{[p^b,k]}\right)\right\} = O\left(Y+\frac{(a,k)y}{k}\sum_{p\leqslant Y<p^b}\frac{1}{p^b}\right),$$

on noting that if p^b gives a non-zero contribution then $(p^b, k)|(a, k)$. This implies (3). Therefore, for a small enough $\delta>0$,

$$\sum_{n,n\in P_Y}{}' f(n) = O\left((a,k)\frac{y}{k}c_{\alpha,\beta,f,\delta}\right) = O_{\alpha,\beta,f,\delta}\left((a,k)\frac{y}{k}\right), \quad (4)$$

where $'$ denotes (here and hereafter) the condition $x-y<n\leqslant x, n\equiv a(\mathrm{mod}\, k)$. Now we shall assume that δ is fixed to yield (4).

Next, notice that $n\notin P_Y$ implies that n has factorization of the form $n=m_1\cdots m_S p_1^{b_1}\cdots p_r^{b_r}$, such that (a) $m_1^\delta<\cdots<m_S^\delta\leqslant Y<p_1<\cdots<p_r$, (b) $m_j>(y/k)^{1-\delta}$, $2\leqslant j\leqslant S$, and (c) m_j's are composed of distinct prime powers not exceeding Y, and are mutually relatively prime. Therefore, by using (ii),

$$f(n)\ll_{\alpha,\beta,f}\left(\max_{1\leqslant j\leqslant S} f(m_j)\right)^S;\quad S\leqslant R(\alpha,\beta,f)=:R.$$

Now, any $m\leqslant y/k$ can occur as an $m_j|n$ for at most $(1+(y/[k,m]))$ values of $n\notin P_Y$, coming under the condition $'$. Thus, using $(m_j, k)|(a, k)$, we obtain

$$\sum_{n,n\notin P_Y}{}' f(n)\ll_{\alpha,\beta,f}\sum_{\substack{m\leqslant y/k\\ (m,k)|(a,k)}}(f(m))^R\frac{y}{[k,m]}\ll\frac{(a,k)y}{k}\sum_{\substack{m\leqslant y/k\\ (m,k)|(a,k)}}\frac{(f(m))^R}{m}. \quad (5)$$

Here, on using (i), we observe that

$$\sum_{\substack{m\leqslant Z\\ (m,k)|(a,k)}}\frac{(f(m))^R}{m}\leqslant\exp\left(\sum_{\substack{p\leqslant Z\\ (p,k)|(a,k)}}\frac{(f(p))^R}{p}+c_f\right). \quad (6)$$

By collecting (4), (5) and (6), we obtain (1′) thereby completing the proof of Theorem 1′.

3. Concluding Remarks

From Theorem 1′, by using $f(p)\leqslant A_1$, we immediately deduce the following Corollary.

COROLLARY 1

Under conditions of Theorem 1′ we have

$$\sum_{\substack{x-y<n\leqslant x\\ n\equiv a(\mathrm{mod}\, k)}} f(n)\leqslant D_1\frac{(a,k)y}{k}(\log x)^{D_2}, \quad (1'')$$

where the constants D_1, D_2 depend at most on α, β, f and ε.

This corollary contains qualitative versions of Theorems 2 and 3 of [1]. Here, we note that (1″) is false without the factor (a, k) in the upper bound. To see this we may take $f(n) = d(n)$, the number of positive divisors of n, $a = 0$ and assume that k is such that $d(k) > \exp((\log\log k)^2)$, say. Then we would obtain the lower bound $d(k)(y/k)$ for the sum in (1″), by using $d(k) \leqslant d(n)$ whenever k divides n, thereby leading to a contradiction if the factor (a, k) were absent on the right side in (1″). Further, observe that from (3) and (5) we obtain the following Corollary 2, which incidentally implies Theorem 4 of [1].

COROLLARY 2

Suppose, further that, f has support over a sequence A with $|\{n: n \leqslant x, n \in A\}| = O(x^{1-\eta})$, for some $\eta > 0$, then

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a(\mathrm{mod}\, k)}} f(n) \leqslant D\frac{(a,k)\,y}{k}, \qquad D = D(\alpha, \beta, f). \tag{7}$$

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The author thanks Professor Aleksandar Ivić, for suggesting the problem of a shorter, alternative approach to the important Theorem 1 considered here.

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Note added in proof.

The reader would find in a forthcoming paper, [2], significant advances on the result of [1].

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Abelian and Tauberian theorems for a new trigonometric method of summation

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Abstract. We first introduce a new trigonometric method of summation and then prove some Abelian and Tauberian theorems for this method.

Keywords. (R_2) method; $(R, 2)$ method; $(R^*_{2+\varepsilon})$ method; (A_x) method; Fourier effective.

1. Introducing the method

We know ([5], vol. I, p. 186) that for $0 < \varepsilon < 1$,

$$\sum_{k=0}^{\infty} A_k^{-\varepsilon} e^{ikx} = \left(2\sin\frac{1}{2}x\right)^{\varepsilon-1} \exp\left\{\frac{1}{2}i(\pi - x)(1-\varepsilon)\right\}, \quad 0 < x < 2\pi \tag{1.1}$$

where A_n^α is defined by

$$\frac{1}{(1-r)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha r^n, \quad \alpha > -1. \tag{1.2}$$

Equating real part on either side of (1.1), we get

$$\sum_{k=0}^{\infty} A_k^{-\varepsilon} \cos kx = \left(2\sin\frac{1}{2}x\right)^{\varepsilon-1} \cos\frac{1}{2}(\pi - x)(1-\varepsilon)$$

$$= x^{\varepsilon-1}\sin\frac{1}{2}\pi\varepsilon + O(1) \text{ as } x \to 0^+. \tag{1.3}$$

Integrating (1.3) twice, we obtain

$$A_0^\varepsilon \frac{x^2}{2} + \sum_{k=1}^{\infty} A_k^{-\varepsilon} \frac{1-\cos kx}{k^2} = \frac{x^{\varepsilon+1}}{\varepsilon(\varepsilon+1)} \sin\frac{1}{2}\pi\varepsilon + O(x^2). \tag{1.4}$$

We can rewrite (1.4) as

$$\frac{x^{1-\varepsilon}}{D(\varepsilon)} \sum_{k=1}^{\infty} A_k^{-\varepsilon} \left(\frac{\sin(1/2)kx}{(1/2)kx}\right)^2 = 1 + O(x^{1-\varepsilon}), \tag{1.5}$$

where

$$D(\varepsilon) = \frac{\pi}{(1+\varepsilon)} \left(\frac{\sin(1/2)\pi\varepsilon}{(1/2)\pi\varepsilon}\right). \tag{1.6}$$

The expression (1.5) motivates to a new method of summation as follows:

Given an infinite series $\Sigma_{n=0}^{\infty} a_n$ with the sequence of partial sums (s_n), we define a trigonometric mean of the sequence (s_n) by

$$F(h) = \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{k=1}^{\infty} A_k^{-\varepsilon} \left(\frac{\sin(1/2)kh}{(1/2)kh} \right)^2 s_k \tag{1.7}$$

provided the right side series is convergent for all small h. if

$$\lim_{h \to 0} F(h) = s$$

then we say that (s_n) is $(R_{2+\varepsilon})$ summable to s, where $0 < \varepsilon < 1$. We may take $h = 2/n$ and write

$$F\left(\frac{2}{n}\right) = \frac{2^{1-\varepsilon}}{D(\varepsilon) n^{1-\varepsilon}} \sum_{k=1}^{\infty} A_k^{-\varepsilon} \left(\frac{\sin(k/n)}{k/n} \right)^2 s_k \tag{1.8}$$

which is the discrete version of $(R_{2+\varepsilon})$ mean and the same will be denoted by $(R^*_{2+\varepsilon})$ mean. Though continuous method always implies its discrete version, the converse is not necessarily true.

The choice of the symbol $(R_{2+\varepsilon})$ is due to the fact that when $\varepsilon \to 0$ then

$$D(\varepsilon) \to \pi$$

and the method (1.7) in the limiting case reduces to the method defined by

$$t(h) = \frac{4}{\pi h} \sum_{k=0}^{\infty} \frac{\sin^2(1/2)kh}{k^2} s_k \tag{1.9}$$

and this is known as (R_2) method ([3], p. 89). Thus (R_2) method can be considered as a limiting case of $(R_{2+\varepsilon})$ method when $\varepsilon = 0$, though for the definition of $(R_{2+\varepsilon})$ method it is assumed that $0 < \varepsilon < 1$.

It may be remarked that (R_2) method is closely connected with Riemann method of summation $(R, 2)$ ([3], p. 89). It has been proved by Marcinkiewicz, and also Kuttner that (R_2) and $(R, 2)$ are not comparable. In fact it has been shown by Hardy and Rogosinski that the methods are incomparable even for Fourier series, that is, there are two examples of Fourier series of which one is summable $(R, 2)$, but not (R_2), and the other (R_2), but not $(R, 2)$. See Hardy ([3], p. 93) where relevant works of Marcinkiewicz, Kuttner, Hardy and Rogosinski are mentioned.

In what follows, we state two known methods of summation which we need in the sequel.

Let (p_n) be a sequence of constants. The Nörlund (N, p) mean of (s_n) is given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \tag{1.10}$$

where $P_n = p_0 + p_1 + \cdots + p_n$, $p_{-1} = P_{-1} = 0$ and $P_n \neq 0$, for $n \geqslant 0$.

The sequence (s_n) is (N, p)-summable to s when $t_n \to s$. The Cesàro method (C, α) $(\alpha > 0)$ is the special case $p_n = A_n^{\alpha - 1}$ (as defined by (1.2)).

Associated with (p_n) we formally write the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$

and define the sequence of constants (C_n) by

$$\sum_{n=0}^{\infty} C_n x^n = \left(\sum_{n=0}^{\infty} p_n x^n \right)^{-1}. \tag{1.11}$$

We write $(p_n) \in \mathscr{M}$, if

$$p_n > 0, \quad \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1 \ (n = 0, 1, 2, \ldots). \tag{1.12}$$

The power series method (A_α) of (s_n) is defined by the mean

$$A(x) = (1-x)^{\alpha+1} \sum_{n=0}^{\infty} A_n^\alpha s_n x^n, \quad 0 < x < 1 \tag{1.13}$$

where A_n^α is defined as in (1.2).

If $A(x) \to s$ as $x \to 1-$ then we say that (s_n) is summable by (A_α) method to s. It is obvious that (A_0) is the same as the Abel method (A). It is further known that

$$(A_\delta) \subset (A) \subset (A_\alpha)$$

for $\delta > 0$, $-1 < \alpha < 0$ ([1]).

In § 2, we consider Abelian theorems and examine the regularity and Fourier effectiveness of $(R_{2+\varepsilon})$ method. In § 3, we establish a Tauberian theorem.

2. Abelian theorems

In this section we prove the following two abelian theorems:

Theorem 1. *Let* $(p_n) \in \mathscr{M}$ *and* $0 \leqslant \varepsilon < 1$. *Then*

$$(N, p) \subset (R_{2+\varepsilon}).$$

In particular, for $0 < \alpha \leqslant 1$,

$$(C, \alpha) \subset (R_{2+\varepsilon}).$$

Theorem 2. *Let* $0 \leqslant \varepsilon < 1$. *Then*

$$(R_{2+\varepsilon}) \subset (A_{-\varepsilon}).$$

In connection with Theorem 1 we may remark that since (C, α) method is regular and Fourier effective for $\alpha > 0$, it follows that

(i) $(R_{2+\varepsilon})$ is regular

(ii) $(R_{2+\varepsilon})$ is Fourier effective.

Regularity of (R_2^*) method has been established by Carmichael and Herwitz (see [4]). Wiener [4] has shown that (R_2^*) method is Fourier effective.

Theorem 2 generalizes a result of Kuttner who proved $(R_2) \subset (A)$ (see Hardy [3], p. 369 for a proof).

For a proof of Theorem 1, we need the following lemmas:

Lemma 1 ([3], Theorem 22). *Let* $(p_n) \in \mathscr{M}$. *Then with the notation of* (1.11)

(i) $C_0 > 0,\ C_n \leqslant 0 \quad (n = 1, 2, 3, \ldots)$

(ii) $\sum_{n=0}^{\infty} C_n x^n$ *is absolutely convergent for* $|x| \leqslant 1$

(iii) $\sum_{n=0}^{\infty} C_n > 0$ *except when* $\Sigma\, p_n = \infty$ *in which case* $\sum_{n=0}^{\infty} C_n = 0$.

Lemma 2 ([2], Lemmas 3 and 4). *Let* $(p_n) \in \mathscr{M}$. *Then*

(i) $C_n^{(1)} = C_0 + C_1 + \cdots + C_n \geqslant 0$ *and monotonic non-increasing,*

(ii) $\sum_{n=M+1}^{\infty} |C_{n-\nu}| \leqslant C_{M-\nu}^{(1)}$.

(iii) $\sum_{\nu=0}^{\infty} P_\nu \sum_{n=M+1}^{\infty} |C_{n-\nu}| \leqslant M + 1$.

Lemma 3. *Let* $(p_n) \in \mathscr{M}$ *and let* $M = [2\delta/h]$ *for some* $\delta: 0 < \delta < \pi/4$. *Then for* $h > 0$

$$\sum_{n=\nu}^{M} C_{n-\nu} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \geqslant 0.$$

Proof. We note that for $0 < \theta < \delta$, $(\sin\theta/\theta)$ is positive and $(\sin\theta/\theta)^2$ is decreasing. By Abel's transfomation

$$\sum_{n=\nu}^{M} A_n^{-\varepsilon} C_{n-\nu} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2$$

$$= \sum_{n=\nu}^{M-1} \Delta_n \left[A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right] C_{n-\nu}^{(1)} + A_M^{-\varepsilon} \left(\frac{\sin(1/2)Mh}{(1/2)Mh} \right)^2 C_{M-\nu}^{(1)}. \qquad (2.1)$$

By lemma 2, the second term of (2.1) is non-negative. Also

$$(A_n^{-\varepsilon} - A_{n+1}^{-\varepsilon}) \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \geqslant 0$$

and

$$A_{n+1}^{-\varepsilon} \left[\left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 - \left(\frac{\sin(1/2)(n+1)h}{(1/2)(n+1)h} \right)^2 \right] \geqslant 0.$$

It follows that the first term of (2.1) is also non-negative. This proves Lemma 3.

Proof of Theorem 1. Using the well-known inverse transformation of (1.11)

$$s_n = \sum_{\nu=0}^{n} C_{n-\nu} P_\nu t_\nu,$$

We obtain from (1.7) that

$$F(h) = \sum_{\nu=0}^{\infty} \mathrm{d}_\nu(h) t_\nu,$$

where

$$d_v(h) = \frac{h^{1-\varepsilon}}{D(\varepsilon)} P_v \sum_{n=v}^{\infty} A_n^{-\varepsilon} C_{n-v} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \tag{2.2}$$

To prove Theorem 1, we establish the regularity of the transformation (2.2). For this we have to prove that (see Hardy [3], p. 49)

(i) $\sum_{v=0}^{\infty} d_v(h) \to 1$ as $h \to 0^+$,

(ii) $d_v(h) \to 0$ as $h \to 0^+$ (fixed v),

(iii) $\sum_{v=0}^{\infty} |d_v(h)| = O(1)$ as $h \to 0^+$.

Since by lemma 1, $\Sigma |C_n| < \infty$, it follows that

$$d_v(h) = O(\Sigma |C_n|) h^{1-\varepsilon} \to 0 \text{ as } h \to 0^+.$$

Since

$$\sum_{v=0}^{n} P_v C_{n-v} = 1 \text{ (all } n) \tag{2.3}$$

it follows from (1.5) that

$$\sum_{v=0}^{\infty} d_v(h) = 1 + O(h^{1-\varepsilon}) \to 1 (h \to 0^+).$$

We have now only to prove (iii). Now

$$\begin{aligned}
\Sigma &= \sum_{v=0}^{\infty} |d_v(h)| \\
&= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{v=0}^{\infty} P_v \left| \sum_{n=v}^{\infty} A_n^{-\varepsilon} C_{n-v} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{v=0}^{\infty} \left| P_v \left(\sum_{n=v}^{M} + \sum_{n=M+1}^{\infty} \right) A_n^{-\varepsilon} C_{n-v} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&\leqslant \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{v=0}^{\infty} P_v \left| \sum_{n=v}^{M} A_n^{-\varepsilon} C_{n-v} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&\quad + \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{v=0}^{\infty} P_v \left| \sum_{n=M+1}^{\infty} A_n^{-\varepsilon} C_{n-v} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&= \Sigma_1 + \Sigma_2, \text{ say.}
\end{aligned} \tag{2.4}$$

Now in Σ_1, $v \leqslant M$ and so considering $v \geqslant M+1$ terms as empty, we have, by Lemma 3

$$\begin{aligned}
\Sigma_1 &= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{v=0}^{M} P_v \left| \sum_{n=v}^{M} C_{n-v} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{v=0}^{M} P_v \sum_{n=v}^{M} C_{n-v} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2
\end{aligned}$$

$$= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{n=0}^{M} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \sum_{\nu=0}^{n} P_\nu C_{n-\nu}$$

$$= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{n=0}^{M} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2$$

$$= 1 + O(h^{1-\varepsilon}), \tag{2.5}$$

by use of the identity (2.3) and the estimate (1.5). Using Lemma 2 (iii), we have

$$\Sigma_2 \leqslant \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{\infty} P_\nu \sum_{n=M+1}^{\infty} |C_{n-\nu}| A_n^{-\varepsilon}$$

$$\leqslant \frac{h^{1-\varepsilon}}{D(\varepsilon)} A_{M+1}^{-\varepsilon} \sum_{\nu=0}^{\infty} P_\nu \sum_{n=M+1}^{\infty} |C_{n-\nu}|$$

$$\leqslant \frac{h^{1-\varepsilon}(M+1)}{D(\varepsilon)(M+1)^{\varepsilon}\Gamma(1-\varepsilon)}$$

$$= \frac{[h(M+1)]^{1-\varepsilon}}{D(\varepsilon)\Gamma(1-\varepsilon)} = O(1).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We need the following lemma.

Lemma 4 ([3], p. 366, Theorem 258). *If* $\Sigma c_n(1-\cos nx)$ *is convergent for all* x *in an interval* (α, β), *then* Σc_n *is convergent.*

We are given that $s_n \to s(R_{2+\varepsilon})$, and we may suppose that $s_0 = 0$, $s = 0$. Then with the notation of (1.7)

$$T(h) = \frac{h^{1+\varepsilon}D(\varepsilon)}{4} F(h) = \sum_{n=1}^{\infty} A_n^{-\varepsilon} \frac{\sin^2(1/2)nh}{n^2} s_n \tag{2.6}$$

converges for small h, and

$$T(h) = o(h^{1+\varepsilon}); \tag{2.7}$$

and we have to prove that

$$(1-r)^{1-\varepsilon} \sum_{n=1}^{\infty} A_n^{-\varepsilon} s_n r^n = o(1) \text{ as } r \to 1- \tag{2.8}$$

By Lemma 4 convergence of (2.6) for small h ensures the convergence of the series $\Sigma_{n=1}^{\infty} (A_n^{-\varepsilon} s_n)/n^2$ and hence we can write series for $T(h)$ as

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nh, \tag{2.9}$$

where

$$\alpha_n = -\frac{1}{2} A_n^{-\varepsilon} n^{-2} s_n (n > 0),$$

$$\alpha_0 = \sum_{n=1}^{\infty} A_n^{-\varepsilon} n^{-2} s_n = -2 \sum_{n=1}^{\infty} \alpha_n. \tag{2.10}$$

The series (2.9) converges to $T(h)$ for small h whenever $s_n \to s(R_{2+\varepsilon})$. We shall only consider two particular cases where

(I) the series (2.9) is a Fourier series, or
(II) the series (2.9) converges uniformly to zero in a neighbourhood of h that includes the origin.

By an argument similar to those used in ([3], p. 368) it can be proved that the theorem is true generally if it holds in cases (I) and (II). Thus, it is enough to prove the theorem in two special cases.

(I) Since in the present case (2.9) is a Fourier series there exists an even periodic function $\phi(h)$ with period 2π so that

$$\phi(h) \sim \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nh. \tag{2.11}$$

As (2.11) converges to $T(h)$ for small h

$$\phi(h) = T(h)$$

for almost all such h, and we may suppose that

$$\phi(h) = T(h)$$

for all such h. By the hypothesis $T(h) = o(h^{1+\varepsilon})$ and hence

$$\phi(h) = o(h^{1+\varepsilon}) \text{ as } h \to 0. \tag{2.12}$$

We write

$$\Delta(r, \theta) = 1 - 2r\cos\theta + r^2,$$

$$P(r, \theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1}{2}\frac{1-r^2}{\Delta(r, \theta)}, \quad 0 < r < 1.$$

It is easily seen that

$$P_{22}(r, \theta) = \frac{\partial^2}{\partial\theta^2} P(r, \theta) = -\frac{r(1-r^2)[(1+r^2)\cos\theta - 2r(1+\sin^2\theta)]}{\Delta^3(r, \theta)}. \tag{2.13}$$

In the present context $r \to 1-$ and so we may assume without loss of generality that $1/4 \leqslant r < 1$. As $\sin\theta \geqslant (2/\pi)\theta$ for $0 \leqslant \theta \leqslant \pi/2$, we have

$$\Delta(r, \theta) = (1-r)^2 + 4r\sin^2\frac{1}{2}\theta \geqslant \pi^{-2}[(1-r)^2 + \theta^2]$$

which further ensures that

$$\Delta^{-1}(r, \theta) = O(1)\begin{cases}(1-r)^{-2} \\ \theta^{-2}\end{cases}. \tag{2.14}$$

Also we have

$$(1+r^2)\cos\theta - 2r(1+\sin^2\theta) = O(\Delta(r, \theta)),\ 0 < r < 1,\ 0 < \theta < \pi. \tag{2.15}$$

Using (2.14) and (2.15), we obtain

$$P_{22}(r,\theta) = O(1)\begin{cases}(1-r)^{-3} & (2.16)\\ (1-r)\theta^{-4}. & (2.17)\end{cases}$$

For $0<r<1$, we have

$$\begin{aligned}\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}\alpha_n r^n \cos nh &= \frac{2}{\pi}\int_0^{\pi}\phi(\theta)\left\{\frac{1}{2}\sum_{n=1}^{\infty} r^n \cos n\theta \cos nh\right\}\mathrm{d}\theta\\ &= \frac{1}{\pi}\int_0^{\pi}\phi(\theta)\left\{1+\sum_{n=1}^{\infty} r^n(\cos n(\theta-h)+\cos n(\theta+h))\right\}\mathrm{d}\theta\\ &= \frac{1}{\pi}\int_0^{\pi}\phi(\theta)\{P(r,\theta-h)+P(r,\theta+h)\}\mathrm{d}\theta\\ &= \frac{2}{\pi}\int_0^{\pi}\phi(\theta)P(r,\theta-h)\mathrm{d}\theta.\end{aligned}$$

Differentiating twice with respect to h, we get

$$\begin{aligned}-\sum_{n=1}^{\infty} n^2\alpha_n r^n \cos nh &= \frac{2}{\pi}\int_0^{\pi}\phi(\theta)\frac{\partial^2}{\partial h^2}P(r,\theta-h)\mathrm{d}\theta\\ &= \frac{2}{\pi}\int_0^{\pi}\phi(\theta)P_{22}(r,\theta-h)\mathrm{d}\theta.\end{aligned}$$

At $h=0$, we obtain

$$-\sum_{n=1}^{\infty} n^2\alpha_n r^n = \frac{2}{\pi}\int_0^{\pi}\phi(\theta)P_{22}(r,\theta)\mathrm{d}\theta.$$

Therefore from (2.10)

$$\begin{aligned}\sum_{n=1}^{\infty} A_n^{-\varepsilon}s_n r^n &= -2\sum_{n=1}^{\infty} n^2\alpha_n r^n\\ &= \frac{4}{\pi}\int_0^{\pi}\phi(\theta)P_{22}(r,\theta)\mathrm{d}\theta. \qquad (2.18)\end{aligned}$$

We choose a δ: $0<\delta<\pi$, such that by (2.12)

$$\phi(\theta) = o(\theta^{1+\varepsilon}) \text{ for } |\theta|\leqslant\delta. \qquad (2.19)$$

Using (2.18), we write

$$\sum_{n=1}^{\infty} A_n^{-\varepsilon}s_n r^n = \frac{4}{\pi}\left[\int_0^{1-r} + \int_{1-r}^{\delta} + \int_{\delta}^{\pi}\right]\phi(\theta)P_{22}(r,\theta)\mathrm{d}\theta.$$

Now using (2.19), (2.16) and (2.17), we get

$$\sum_{n=1}^{\infty} A_n^{-\varepsilon}s_n r^n = o((1-r)^{-3})\int_0^{1-r}\theta^{1+\varepsilon}\mathrm{d}\theta + o(1-r)\int_{1-r}^{\delta}\theta^{\varepsilon-3}\mathrm{d}\theta$$

$$+O(1-r)\int_{\delta}^{\pi}\theta^{\varepsilon-3}\mathrm{d}\theta$$
$$=o((1-r)^{\varepsilon-1})+O(1-r)$$
$$=o((1-r)^{\varepsilon-1}) \text{ as } r\to 1-, \tag{2.20}$$

which ensures (2.8) and hence theorem holds in case (I).

(II) In this case the series (2.9) converges uniformly to zero in a neighbourhood of h that includes the origin. We have

$$\frac{2}{\pi}\int_0^{\pi}P(r,\theta)\cos n\theta\,\mathrm{d}\theta$$
$$=\frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2}+\sum_{m=1}^{\infty}r^m\cos m\theta\right)\cos n\theta\,\mathrm{d}\theta$$
$$=\frac{1}{\pi}\int_0^{\pi}(1+\cos 2n\theta)r^n\mathrm{d}\theta=r^n.$$

Thus, integrating by parts twice, we obtain

$$r^n=\frac{2}{\pi}\int_0^{\pi}P(r,\theta)\cos n\theta\,\mathrm{d}\theta,\quad n\geqslant 1$$
$$=-\frac{2}{\pi}\int_0^{\pi}P_{22}(r,\theta)\frac{1-\cos n\theta}{n^2}\mathrm{d}\theta,$$

from which it follows that

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}s_nr^n=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{A_n^{-\varepsilon}s_n}{n^2}\int_0^{\pi}P_{22}(r,\theta)(1-\cos n\theta)\mathrm{d}\theta. \tag{2.21}$$

As $s_n\to s(R_{2+\varepsilon})$ the series $\Sigma_{n=1}^{\infty}A_n^{-\varepsilon}s_n/n^2$ is convergent and hence

$$\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}\alpha_n\cos n\theta=\frac{1}{2}\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}(1-\cos n\theta). \tag{2.22}$$

The last series in (2.22) converges uniformly to zero for small θ, say for $|\theta|\leqslant\theta_0$ and therefore

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}\int_0^{\theta_0}(1-\cos n\theta)P_{22}(r,\theta)\mathrm{d}\theta=0. \tag{2.23}$$

For the proof of the theorem, it remains to show that

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}\int_{\theta_0}^{\pi}(1-\cos n\theta)P_{22}(r,\theta)\mathrm{d}\theta=o\left(\frac{1}{(1-r)^{1-\varepsilon}}\right) \tag{2.24}$$

for fixed positive θ_0. As

$$\int_{\theta_0}^{\pi}P_{22}(r,\theta)\mathrm{d}\theta=O(1),$$

it is clear that

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}\int_{\theta_0}^{\pi}P_{22}(r,\theta)\mathrm{d}\theta=O(1). \tag{2.25}$$

By simple calculation, it can be seen that $P_{22}(r,\theta)$, $P_{222}(r,\theta)$ and $P_{2222}(r,\theta)$ are uniformly bounded in the interval (θ_0,π) and hence integrating by parts twice, we obtain for $n \geqslant 1$

$$\begin{aligned} j_n &= \int_{\theta_0}^{\pi} \cos n\theta P_{22}(r,\theta)\mathrm{d}\theta \\ &= -\frac{P_{22}(r,\theta_0)\sin n\theta_0}{n} - \frac{P_{222}(r,\theta_0)\cos n\theta_0}{n^2} \\ &\quad + \frac{1}{n^2}\int_{\theta_0}^{\pi} P_{2222}(r,\theta)\cos n\theta\,\mathrm{d}\theta \\ &= -P_{22}(r,\theta_0)\frac{\sin n\theta_0}{n} + O(1/n^2). \end{aligned} \tag{2.26}$$

As $s_n = o(n^2/A_n^{-\varepsilon})$, we have

$$\sum_{n=1}^{\infty} s_n A_n^{-\varepsilon}\frac{j_n}{n^2} = -P_{22}(r,\theta_0)\sum_{n=1}^{\infty}\frac{A_n^{-\varepsilon}s_n\sin n\theta_0}{n^3} + \sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right).$$

Since $\Sigma_{n=1}^{\infty} A_n^{-\varepsilon}/n^2 s_n\cos n\theta$ converges uniformly for $|\theta| \leqslant \theta_0$, the series $\Sigma_{n=1}^{\infty}(A_n^{-\varepsilon}s_n\sin n\theta_0)/n^3$ is convergent and hence

$$\sum_{n=1}^{\infty}\frac{A_n^{-\varepsilon}}{n^2}s_n j_n = O(1)$$

which together with (2.25) ensures that

$$\sum_{n=1}^{\infty} A_n^{-\varepsilon}\frac{s_n}{n^2}\int_{\theta_0}^{\pi}(1-\cos n\theta)P_{22}(r,\theta)\mathrm{d}\theta = O(1)$$

from which (2.24) follows and this completes the proof of the theorem.

3. Tauberian theorem

In this section we prove the following Tauberian theorem for $(R_{2+\varepsilon})$ summability which corresponds to a theorem on (R_2) summability due to Vijayaraghavan (see [3], p. 306). For this we first recall the definition.

DEFINITION ([3], p. 124)

A functon f, defined for $x > 0$, is said to be slowly decreasing if it is real and

$$\liminf\{f(y) - f(x)\} \geqslant 0$$

whenever $x \to \infty$, $y > x$, $y/x \to 1$.

Theorem 3. *Let*

(i) $s_n = O(1)(R_{2+\varepsilon})$, $0 < \varepsilon < 1$
(ii) s_n *be real and slowly decreasing.*

Then $s_n = O(1)$.

Let a method of summation be defined by

$$J(x) = \sum_{n=0}^{\infty} c_n(x) s_n. \tag{3.1}$$

We suppose that

$$c_n(x) \geqslant 0,\ c_n(x) \to 0\ (x \to \infty),\ \sum_{n=0}^{\infty} c_n(x) = 1 \tag{3.2}$$

so that the method is totally regular. We need the following lemma.

Lemma 5 ([3], p. 306, Theorem 238). *Suppose that the following conditions are satisfied.*

(i) $\phi(u)$ *is positive and differentiable for* $u \geqslant 1$

$$\phi \to \infty, \quad 0 < \phi' < K, \tag{3.3}$$

where K *is independent of* u,

$$\Phi(u) = \int_1^u \frac{\mathrm{d}t}{\phi(t)} \tag{3.4}$$

(*so that* $\Phi \to \infty$ *with* u).

(ii) *The co-efficient* $c_n(x)$ *have, in addition to those already stated, the properties;*

$$\sum_{n=0}^{M} c_n(x) \to 0, \tag{3.5}$$

if $M \to \infty$, $x \to \infty$, $\Phi(x) - \Phi(M) \to \infty$;

$$\sum_{n=N}^{\infty} c_n(x) \to 0, \tag{3.6}$$

$$\sum_{n=N}^{\infty} c_n(x)(\Phi(n) - \Phi(N)) \to 0, \tag{3.7}$$

if $N \to \infty$, $x \to \infty$, $\Phi(N) - \Phi(x) \to \infty$.

(iii) *If* $s(t) = s_n$ *for* $n \leqslant t < n+1$, *then*

$$\liminf(s(t) - s(u)) \geqslant 0 \tag{3.8}$$

when $u \to \infty$, $t > u$, $(t-u)/\phi(u) \to 0$.

(iv) $\tau(x) = \Sigma c_n(x)\, s_n$ *is bounded.*

Then s_n *is bounded.*

Proof of Theorem 3. We shall deduce Theorem 3 as a particular case of Lemma 5 by a special choice of $c_n(x)$ and $\phi(u)$.

We take

$$c_n(x) = \frac{x^{\varepsilon - 1}}{D(\varepsilon)} A_n^{-\varepsilon} \left(\frac{\sin(n/2x)}{n/2x} \right)^2$$

$$\phi(u) = u,\ \Phi(u) = \log u.$$

Clearly $c_n(x)$ corresponds to $(R_{2+\varepsilon})$ method and it satisfies condition (3.2) for total regularity. Condition (iv) of Lemma 5 is satisfied as $s_n = O(1)\,(R_{2+\varepsilon})$ by the hypothesis while condition (iii) follows from the fact that s_n is real and slowly decreasing. Now

$$\sum_{n=1}^{M} c_n(x) = \frac{x^{\varepsilon-1}}{D(\varepsilon)} \sum_{n=1}^{M} A_n^{-\varepsilon} \left(\frac{\sin(n/2x)}{n/2x}\right)^2$$

$$= O(1)x^{\varepsilon-1} \sum_{n=1}^{M} \frac{1}{n^{\varepsilon}}$$

$$= O(1)(M/x)^{1-\varepsilon} \to 0$$

as $M \to \infty$, $x \to \infty$, $M/x \to 0$ (i.e., $\log x - \log M \to \infty$); and

$$\sum_{n=N}^{\infty} c_n(x) = \frac{x^{\varepsilon-1}}{D(\varepsilon)} \sum_{n=N}^{\infty} A_n^{-\varepsilon} \left(\frac{\sin(n/2x)}{n/2x}\right)^2$$

$$= O(1)x^{\varepsilon-1} \sum_{n=N}^{\infty} \frac{1}{n^{\varepsilon}} \frac{4x^2}{n^2}$$

$$= O(1)x^{\varepsilon+1} \sum_{n=N}^{\infty} \frac{1}{n^{2+\varepsilon}}$$

$$= O(1)(x/N)^{1+\varepsilon} \to 0;$$

if $N \to \infty$, $x \to \infty$, $N/x \to \infty$ (i.e. $\log N - \log x \to \infty$). Finally

$$\sum_{n=N}^{\infty} c_n(x)(\Phi(n) - \Phi(N))$$

$$= \frac{x^{\varepsilon-1}}{D(\varepsilon)} \sum_{n=N}^{\infty} A_n^{-\varepsilon} \left(\frac{\sin(n/2x)}{n/2x}\right)^2 \log \frac{n}{N}$$

$$= O(1)x^{\varepsilon+1} \sum_{n=N}^{\infty} \frac{1}{n^{2+\varepsilon}} \log \frac{n}{N}$$

$$= O(1)x^{\varepsilon+1} \int_{N}^{\infty} \frac{\log t/N}{t^{2+\varepsilon}} \mathrm{d}t$$

$$= O(1)(x/N)^{\varepsilon+1} \int_{1}^{\infty} \frac{\log u}{u^{2+\varepsilon}} \mathrm{d}u$$

$$= O(1)(x/N)^{\varepsilon+1} \to 0;$$

if $N \to \infty$, $x \to \infty$, $N/x \to \infty$.

Thus the remaining conditions (i) and (ii) of Lemma 5 are satisfied. This completes the proof of Theorem 3.

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Dirichlet problem for some hypoelliptic operators

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Abstract. In this paper, the Dirichlet problem for hypoelliptic operators verifying Hörmander condition and the maximum principle is considered.

Keywords. Hypoelliptic operators; maximum principle; Dirichlet problem.

1. Introduction

Let

$$L = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) D_{ij} + \sum_{i=1}^{n} b_i(x) D_i + c(x)$$

be a degenerate elliptic operator of 2nd order with C^∞ coefficients with $c \leqslant 0$ defined on an open set Ω of $\mathbb{R}^n$. Let us suppose that there exist vector fields $X_1, X_2, \ldots, X_r, Y$ such that L can be written in the form

$$L = \sum_{k=1}^{k=r} X_k^2 + Y + c.$$

Further let us assume that the rank of the Lie algebra generated by $X_1, X_2, \ldots, X_r, Y$ at any point is of rank n.

Under the above assumption, Hörmander in [2] has proved that the operator is hypoelliptic, in the sense of the following definition.

DEFINITION

L is said to be *hypoelliptic* if every distribution u is C^∞ in an open set ω if Lu is C^∞ in ω.

For these operators, with the additional assumption that $\exists\, \lambda < 0$ such that $c(x) \leqslant \lambda$, $\forall x \in \Omega$, Bony [1] has proved the existence and uniqueness of the Dirichlet problem.

$$Lu = f \text{ in } \omega,$$

$$u = \phi \text{ on } \partial\omega,$$

where f and ϕ are continuous and ω is a relatively compact open subset of Ω, $\bar{\omega} \subset \Omega$, such that $\partial\omega$ admits a special kind of barrier to be specified later on.

In this article we remove the above restriction on c and assume just that $c \leqslant 0$, but assume that L satisfies the maximum principle as defined below and prove the existence and uniqueness of the Dirichlet problem when $f = 0$.

DEFINITION 1

Let ω be a relatively compact open subset of Ω such that $\bar{\omega} \subset \Omega$. The maximum principle for subsolutions of $L = 0$ is said to hold if $\forall u \in C^2(\omega)$, upper semi continuous in $\bar{\omega}$,

$$Lu \geqslant 0 \text{ in } \omega, \limsup u(y)_{y \to X,\ y \in \omega,\ X \in \partial\omega} \leqslant 0, \quad \forall X \in \partial\omega,$$

$$\Rightarrow u \leqslant 0 \text{ in } \omega.$$

DEFINITION 2

A vector ν is called an *exterior normal* to a closed set F at a point $x_0 \in F$ if there exists an open ball contained in $\Omega \setminus F$ centered at a point x_1 such that x_0 belongs to the closure of this ball and such that $\nu = \lambda (x_1 - x_0)$ with $\lambda > 0$.

It is easy to see by reducing the radius if necessary that x_0 is the only point of F that belongs to the closure of the ball.

Theorem. *Let L be an operator of the type described above. Let further L satisfy the maximum principle for subsolutions of $L = 0$. Let ω be a relatively compact open subset of Ω, $\bar{\omega} \subset \Omega$, such that $\forall X \in \partial\omega$, $\exists$ an exterior normal ν such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(X) \nu_i \nu_j > 0.$$

Then $\forall \phi$ continuous on $\partial\omega$, $\exists$ a unique continuous function u on $\bar{\omega}$ such that

$$Lu = 0 \text{ in the sense of distributions in } \omega$$

$$u = \phi \text{ on } \partial\omega.$$

Proof. The uniqueness comes from the fact that L is hypoelliptic and that the maximum principle is valid for subsolutions of $L = 0$.

Therefore it is sufficient to show the existence.

Let $\phi \in C(\partial\omega)$ be given. Let L_n be the operator $L - (1/n)$.

Therefore by the theorem 5.2 in Bony [1], $\forall n \in \mathbb{N}$, $\exists$ a function $u_n \in C^\infty(\omega)$ such that

$$L_n u_n = \left(L - \frac{1}{n}\right) u_n = 0 \text{ in } \omega$$

$$u_n = \phi \text{ on } \partial\omega.$$

Since the maximum principle is valid for $L_n \forall n$ we have $\forall x \in \bar{\omega}$,

$$|u_n(x)| \leqslant \sup_{x \in \partial\omega} |\phi(x)|.$$

Hence the u_n's are all uniformly bounded in $\bar{\omega}$. Therefore (u_n) has a subsequence which converges weakly in $L^2(\bar{\omega})$ to a function $u \in L^2(\bar{\omega})$. Without loss of generality, we shall suppose that (u_n) itself converges weakly to u in $L^2(\omega)$.

Let $\psi \in \mathscr{D}(\omega)$. Then,

$$\begin{aligned}\int_\Omega (Lu)\psi \, dx &= \int_\Omega u L^*(\psi) dx \\ &= \lim_{n\to\infty} \int u_n L^*(\psi) dx \\ &= \lim_{n\to\infty} \int (Lu_n)\psi \, dx \\ &= \lim_{n\to\infty} \frac{1}{n} \int u_n \psi \, dx \\ &= 0.\end{aligned}$$

Therefore $Lu = 0$ in the sense of distributions. Since L is hypoelliptic, $u \in C^\infty(\omega)$.

We shall now show that $u(x) \to \phi(X) \forall X \in \partial\omega$, when $x \in \omega \to X$.

For this we suppose that ϕ admits an extension as a function of class C^2 in a neighbourhood of $\bar{\omega}$. We denote by ϕ the extension also.

For $X \in \partial\omega$, $\exists$ by assumption a point $x^0 \in \Omega \setminus \bar{\omega}$, and a $r > 0$, such that

$$\overline{B(x_0;\, r)} \cap \partial\omega = X$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(X)(X_i - x_i^0)(X_j - x_j^0) > 0.$$

From this property it is easy to see that if w is the function

$$e^{-k(|x-x^0|)^2} - e^{-k(|X-x^0|)^2}$$

then, for k large, $L(w(X)) > 0$. Let us fix one such k. $L(w) > 0$ in a neighbourhood V of X.

$$L(u_n - \phi) = \frac{1}{n} u_n - L\phi.$$

As the u_n's are uniformly bounded and as $L\phi$ is bounded, $L(u_n - \phi)$ is also uniformly bounded.

Therefore, we can choose M_1 large such that $\forall n$,

$$L(M_1 w \pm (u_n - \phi)) > 0, \quad \text{in } V \cap \omega.$$

As $w(X) = 0$, and as $w < 0$ in $\overline{(V \cap \omega)} \setminus X$, and as the u_n's are all uniformly bounded and $u_n = \phi$ on $\partial\omega$, we can choose $M_2 \geqslant M_1$ such that $\forall n \in \mathbb{N}$,

$$M_2 w \pm (u_n - \phi) \leqslant 0 \quad \text{on} \quad \partial(V \cap \omega).$$

Therefore we have,

$$L(M_2 w \pm (u_n - \phi)) > 0 \quad \text{on} \quad V \cap \omega$$

and

$$M_2 w \pm (u_n - \phi) \leqslant \text{ on } \partial(V \cap \omega).$$

Therefore by the maximum principle,

$$M_2 w \pm (u_n - \phi) \leqslant 0 \text{ on } V \cap \omega.$$

Therefore $|u_n - \phi| \leqslant M_2 |w|$ on $V \cap \omega$.
As u_n converges weakly to u in L^2, we have for any measurable set $E \subseteq V \cap \omega$,

$$\begin{aligned}\int_E |u-\phi| dx &= \int_{E \cap (u-\phi) \geqslant 0} (u-\phi) + \int_{E \cap (u-\phi) < 0} -(u-\phi) dx \\ &= \lim_{n \to \infty} \int_{E \cap (u-\phi) \geqslant 0} (u_n - \phi) dx - \int_{E \cap (u-\phi) < 0} (u_n - \phi) dx \\ &\leqslant \int_E M_2 |w| dx.\end{aligned}$$

This implies that

$$|(u-\phi)| \leqslant M_2 |w| \text{ a.e. in } V \cap \omega.$$

As u, ϕ and w are continuous, we have the inequality

$$|(u-\phi)| \leqslant M_2 |w| \text{ everywhere in } V \cap \omega.$$

From this it is easy to see that $u(x) \to \phi(X)$ as $x \in \omega \to X$ as $w(X) = 0$, w and ϕ are continuous at X.

For a general ϕ, we approximate uniformly by a sequence of functions ϕ_n which admit an extension as a function of class C^∞ in $\bar{\omega}$. If u_n is the corresponding sequence such that

$$Lu_n = 0 \text{ on } \omega,$$

$$u_n = \phi_n \text{ on } \partial\omega,$$

then by the maximum principle, we see that the sequence u_n converges uniformly on $\bar{\omega}$ to a continuous function u such that

$$Lu = 0 \text{ on } \omega,$$

$$u = \phi \text{ on } \partial\omega.$$

Thus the proof of the theorem is complete.

In the theorem we assumed that the subsolutions of $L = 0$ satisfy the maximum principle. It is not true in general that the subsolutions of $L = 0$ where L is an hypoelliptic operator satisfy the maximum principle as the following example shows.

Example. Let Ω be the ball of center 0 and radius 3 in $\mathbb{R}^2$. Let L be the operator

$$\frac{\partial^2}{\partial \theta^2} + \alpha(r) \frac{\partial^2}{\partial r^2} + \beta(r) \frac{\partial}{\partial r},$$

where α and β do not vanish simultaneously, $\beta > 0$ in the interval $[0, 1]$, $\beta < 0$ in the interval $[2, 3]$, support of $\alpha \subseteq [1, 2]$, $0 \leqslant \alpha \leqslant 1$. It is clear that L is hypoelliptic as the condition of Hörmander is verified at each point.

Let u be a function of r such that $u(3)$ is 0, u is increasing in $[0,1]$, u is a positive constant in $[1,2]$, u is decreasing in $[2,3]$. Then it is easy to see that $Lu = \beta(r)(\partial u/\partial r)$ which is >0. Hence u is a subsolution of $L=0$. But the maximum principle is not verified.

We can take for example for u the following function:

$$u(r) = r^5 \text{ in } [0,1]$$
$$= 1 \text{ in } [1,2]$$
$$= (3-r)^5 \text{ in } [2,3].$$

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Transformation of chaotic nonlinear polynomial difference systems through Newton iterations

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Abstract. Chaotic sequences generated by nonlinear difference systems or 'maps' where the defining nonlinearities are polynomials, have been examined from the point of view of the sequential points seeking zeroes of an unknown function f following the rule of Newton iterations. Following such nonlinear transformation rule, alternative sequences have been constructed showing monotonic convergence. Evidently, these are maps of the original sequences. For second degree systems, another kind of possibly less chaotic sequences have been constructed by essentially the same method. Finally, it is shown that the original chaotic system can be decomposed into a fast monotonically convergent part and a principal oscillatory part showing sharp oscillations. The methods are exemplified by the well-known logistic map, delayed-logistic map and the Hénon map.

Keywords. Chaos; nonlinear polynomial difference systems; Newton iterations; convergent sequences; decomposition; principal oscillatory part.

1. Introduction

Nonlinear discrete dynamical systems described by a variety of difference equations have been studied in depth as iterative maps and the different associated (qualitative) aspects are described in several books, such as, Baker and Gollub [2], Lauwerier [4], Lichtenberg and Lieberman [5], the review article by Whitney [11] specially for bifurcations and Smale [10] for diffeomorphism of manifolds where many other references can be found. Chaos, the main objective of these studies has also been given stochastic interpretation (cf. [4, 5]).

Herein, we consider such systems as sequences defined by nonlinear equations, which must converge to some point, diverge to infinity or oscillate finitely or infinitely. Chaos belongs to the last category when the oscillations are finite aperiodic oscillations. We look into ways of constructing nonlinear transformations, which are maps of the original but masking chaos by way of reduced chaotic oscillations or even monotonic convergence. Due to such behaviour of the new sequences, the error in long iterative computer simulations is well contained, while it is not so in an original given chaotic sequence. Handling and storage of the generated data is also accomplished more efficiently. Thus in a stored form, the new sequences will be better working tools and when the original sequence is needed, the inverse of the original trnsformation can be invoked. The plotted form of the new sequences can also possess their own geometrical appeal. These aspects in a general sense, can also be of significance in the treatment of difficult nonlinear differential equations by finite differences.

For construction of such transformations, we assume that the nonlinear difference equation is given in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n), \tag{1}$$

where $\mathbf{x}=(x^1,x^2,\ldots,x^m)$, m usually $\leqslant 3$ and $\mathbf{F}=(F^1,F^2,\ldots,F^m)$ has polynomial forms as its components, and the oscillating sequence seeks certain set in a domain containing a fixed point of the map. Transferring origin to the fixed point, we can assume the form

$$F^i = x_n^i - \sum_{j=1}^{m} a_j^i x_n^j - \sum g^i_{i_1 i_2 \ldots i_m} (x_n^1)^{i_1}(x_n^2)^{i_2}\cdots(x_n^m)^{i_m}, \tag{2}$$

$2 \leqslant i_1 + i_2 + \cdots + i_m \leqslant p_i$, where p_i is the degree of the polynomial and the origin becomes the fixed point of interest. It is shown in § § 2 and 3 that convergence property depends solely on the coefficients a_j^i. We assume here the presence of chaotic oscillations. Now, in the Newton iteration of zeroes of a single function $f(\mathbf{x})$ [7] the sequence of points is

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \{f'(\mathbf{x}_n)\}^{-1} \cdot f(\mathbf{x}_n),$$

where $f'(\mathbf{x}_n)$ is the Jacobian matrix, which in this case is the diagonal matrix $\operatorname{diag}\{\nabla f(\mathbf{x}_n)\}$. This is of the form (1). Thus we can interpret that the dynamics (1) seeks zeroes of $f(\mathbf{x})$ according to the above rule which can be written as

$$(\mathbf{x}_{n+1} - \mathbf{x}_n)\cdot\nabla f(\mathbf{x}_n) = -f(\mathbf{x}_n). \tag{3}$$

Hence, the sequence of points lie on the solutions of

$$[\mathbf{F}(\mathbf{x}) - \mathbf{x}]\cdot\nabla f = -f, \tag{4}$$

which is a Lagrange linear first order P.D.E [3]. The characteristics of the equation completely determining the general solution of (4) are given by

$$\frac{\mathrm{d}x^i}{F^i(\mathbf{x}) - x^i} = \frac{\mathrm{d}f}{-f}, \quad i = 1,2,\ldots,m. \tag{5}$$

Interestingly, the zeroes of f are the fixed points of (1), if ∇f is finite. Because of this, one may say, the chaotic sequence (1), seeking a zero of f, has a strange attraction caused by the fixed point.

In § § 2 and 3, it is shown that in the general solution of (4) with (2), fractional power functions of the arguments are present in the neighbourhood of the fixed point, namely the origin. The solution in the general m-dimensional case in § 3, is achieved after a linear transformation of the arguments. Suggestive from the forms, new arguments which are power functions of the old ones are then introduced which in effect clear the fractional powers. By subjecting them to the law (5) and the Newton's iterative rule (3) in the new arguments, we obtain a new sequence monotonically convergent in nature. Thus the new sequence even though convergent, is a map of the original obtained by these nonlinear rules. As examples we consider the treatment of the well known logistic map and in two dimensions, the delayed-logistic and the Hénon maps.

In the next section, we consider quadratic systems and look into the effect of stretching by sending the nonzero fixed point to infinity by exponential transformation. As before new sequences are constructed, which in one dimension are oscillatory, confirmed by the logistic map. In two dimensions, the delayed-logistic map is oscillatory but the Hénon map is slowly divergent. Chaotic oscillations in the oscillatory cases is very much reduced as is expected from the nature of transformation. Here, some interesting plots have been obtained.

It is noticed in §§ 2 and 3 that the convergence of the sequence (1), (2) depends on the linear part only. Hence, by introducing an under relaxation factor ω in damped Newton method we can construct fast monotonically convergent sequences. Rearranging the right hand side of (2) of the original sequence on the same basis, the defining equation (1) can be split into a fast monotonically convergent part and a principal highly oscillatory part which is really responsible for the chaotic oscillations. This is discussed in § 5. A consequence is that it should be possible to construct new chaotic sequences by combining the principal oscillatory parts with other monotonically convergent sequences.

2. One-dimension

In one dimension the difference equation (1) is of the form

$$x_{n+1} = (1-a)x_n - \sum g_i x_n^i, \quad 2 \leqslant i \leqslant p. \tag{6}$$

The fixed point $x^* = 0$ is under consideration. By iteration the solution of (6) is

$$x_n = (1-a)^n x_0 + (1-a)^{n-1} O(x_0^2), \tag{7}$$

which converges to the fixed point (or in other words the fixed point is attracting or stable) if $|1-a| < 1$, i.e. $0 < a < 2$. The convergence is monotonic for $0 < a \leqslant 1$ and oscillatory for $1 < a < 2$.

For increasing $a \geqslant 2$, the oscillatory behaviour may persist with bifurcations, chaos and divergence. For such an eventuality, we seek alternative sequence, map of the original one viz. (6) masking these features and converging to the fixed point. If equation (6) defines Newton's iteration for the zeroes of a function $f(x)$, then (3) is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8}$$

and by comparison, f satisfies the differential equation

$$\frac{\mathrm{d}x}{-ax - \sum g_i x^i} = -\frac{\mathrm{d}f}{f}, \tag{9}$$

which is a one parameter (c) family of curves. In the neighbourhood of the fixed point $x = x^* = 0$, dropping the $O(x^2)$ terms,

$$f = c|x|^{1/a}, \tag{10}$$

which is an ath degree parabola through the fixed point. The form of f explains oscillations in the sequence for $a > 1$ seeking the zero by the tangent method of Newton. It is suggestive of transformation of f defined by (9) according to

$$x = |X|^k \operatorname{sgn}(X), \quad k \geqslant a. \tag{11}$$

Equation (9) then becomes

$$\frac{k|X|^{k-1}\mathrm{d}X}{-aX|X|^{k-1} - \sum g_i X^i |X|^{i(k-1)}} = -\frac{\mathrm{d}f}{f}. \tag{12}$$

The corresponding Newton iteration is

$$X_{n+1} = X_n + \frac{1}{k}\left[-aX_n - \sum g_i X_n^i |X_n|^{(i-1)(k-1)}\right]. \tag{13}$$

As in the case of (6), this sequence (13) converges monotonically in as much as $0 \leqslant 1 - a/k < 1$. The sequence will be closely packed near $X^* = 0$ for large enough n. If we want to obtain the original sequence (6) from the monotonically convergent sequence (13), then we retrace the steps by invoking the transformation rule (12), (11), (9) and (8).

2.1 *Example*

The logistic map: This 'map' is the iteratively defined sequence

$$\xi_{n+1} = a\xi_n(1 - \xi_n), \quad a \geqslant 1. \tag{14}$$

The fixed points of the iteration are $\xi^* = 0, (a-1)/a$. The former is unstable and not of much interest. For the latter, the transformation $\xi = x + (a-1)/a$ converts (14) to

$$\begin{aligned} x_{n+1} &= -(a-2)x_n - ax_n^2 \\ &= \{1-(a-1)\}x_n - ax_n^2. \end{aligned} \tag{15}$$

Thus, as in the general discussion, the sequence is monotonic convergent to the fixed point $x^* = 0$ for $1 \leqslant a < 2$ and oscillatory for $2 \leqslant a < 3$. It is for $a \geqslant 3$, much has been written in the earlier literature on chaos. It is known [6] that for $3 \leqslant a \leqslant 1 + \sqrt{6} \approx 3{\cdot}54$, the sequence exhibits bifurcation with a cycle of period 2. For $1 + \sqrt{6} < a \leqslant 3{\cdot}57$, cascade of bifurcations of cycles of periods $2, 4, 8, \ldots, 2^k$ take place and for $3{\cdot}57 < a < 4$ the oscillations become unstable, forming what is called the chaotic régime. For $a = 4$, $x_n = \sin^2(2^n c)$, beyond which there is oscillatory divergence.

From the general theory, the alternative convergent sequence is generated by (cf. equations (9), (11), (12))

$$-\frac{df}{f} = \frac{dx}{-(a-1)x - ax^2} = \frac{k|X|^{k-1}dX}{-(a-1)X|X|^{k-1} - aX^2|X|^{2(k-1)}}, \quad k \geqslant a - 1. \tag{16}$$

Thus, the new Newton scheme yields

$$X_{n+1} = X_n - \frac{X_n}{k}[a - 1 + a|X_n|^k sgn(X_n)], \tag{17}$$

which as in the general theory converges monotonically to the fixed point $X^* = 0$. The sequence can be generated on a computer and the points lie on a smooth trajectory.

3. Multi-dimensions

In m-dimensions with coordinates $(x^1, x^2, \ldots, x^m)$, equation (1) is

$$\begin{aligned} &x^i_{n+1} = x^i_n - \sum_{j=1}^{m} a^i_j x^j_n - \sum g^i_{i_1, i_2, \ldots, i_m} (x^1_n)^{i_1} (x^2_n)^{i_2} \cdots (x^m_n)^{i_m}, \\ &2 \leqslant i_1 + i_2 + i_m \leqslant p_i, \end{aligned} \tag{18}$$

with the fixed point $x^{1*} = x^{2*} = \cdots = x^{m*} = 0$ under consideration. Writing the equation in vector form

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + O(|\mathbf{x}_n|^2)I, \quad A = [\delta^i_j - a^i_j], \tag{19}$$

iteration yields the solution

$$\mathbf{x}_n = A^n \mathbf{x}_0 + O(|\mathbf{x}_0|^2) A^{n-1} I. \tag{20}$$

Thus the vector sequence converges to $\mathbf{x}^* = 0$ if the spectral radius of A is less than unity, that is, the eigenvalues λ_k satisfy $|\lambda_k| < 1$, $k = 1, 2, \ldots, m$. The fixed point is then attracting or stable. The proof follows from representing a vector $\xi = \Sigma_{k=1}^{m} c_k \xi^k$ in terms of eigenvectors ξ^k. Thus

$$A^n \xi = \sum_{k=1}^{m} c_k A^n \xi^k = \sum_{k=1}^{m} c_k \lambda_k^n \xi^k.$$

It also follows that if for any k, $-1 < \lambda_k < 0$, the sequence will oscillate and converge. For decreasing $\lambda_k \leqslant -1$, the oscillatory behaviour may persist leading to chaotic motion and divergence.

As in one dimension, if (18) defines Newton's iteration for the zeroes of $f(x^1, x^2, \ldots, x^m)$, then (3) is

$$\sum_{i=1}^{m} (x_{n+1}^i - x_n^i) \frac{\partial f}{\partial x_n^i} = -f, \tag{21}$$

so that from (18), f satisfies the linear differential equation

$$\sum_{i=1}^{m} \left[-\sum_{j=1}^{m} a_j^i x_n^j - \sum g_{i_1, i_2, \ldots i_m}^i (x_n^1)^{i_1} (x_n^2)^{i_2} \cdots (x_n^m)^{i_m} \right] \frac{\partial f}{\partial x_n^i} = -f. \tag{22}$$

In the neighbourhood of the trivial fixed point, we can drop the nonlinear terms and to facilitate the solution of the resulting equation, perform the linear transformation

$$x^i = \sum_{j=1}^{m} b_j^i y^j; \quad y^j = \frac{1}{\Delta} \sum_{i=1}^{m} B_i^j x^i, \tag{23}$$

where $\Delta = \det(b_j^i)$ and $B_i^j =$ cofactor of b_j^i in Δ. The transformed differential equation for f is

$$\sum_{k=1}^{m} \left[\sum_{l=1}^{m} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} B_i^k a_j^i b_l^j \right\} y^l \right] \frac{\partial f}{\partial yk} = \Delta f. \tag{24}$$

We now assume that the unknown coefficients b_j^i satisfy the $m(m-1)$ nonlinear algebric equations

$$\sum_{i=1}^{m} \sum_{j=1}^{m} B_i^k a_j^i b_l^j = 0, \quad l \neq k, \tag{25}$$

leaving m of the unknowns arbitrary. The roots of these equations in general may be complex. So may be the nature of y^j. For $l = k$ we define the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{m} B_i^k a_j^i b_k^j \stackrel{\text{def}}{=} c_k, \tag{26}$$

and the differential equation (24) reduces to

$$\sum_{k=1}^{m} c_k y^k \frac{\partial f}{\partial yk} = \Delta f. \tag{27}$$

The characteristics of this linear differential equation are given by

$$\frac{\mathrm{d}y^k}{c_k y^k} = \frac{\mathrm{d}f}{\Delta f}, \quad k = 1, 2, \ldots, m. \tag{28}$$

The general solution of (27) is thus given by

$$f^{1/\Delta} = (|y^k|^{1/c_k} \phi\{|y^1|^{1/c_1}/|y^k|^{1/c_k}, |y^2|^{1/c_2}/|y^k|^{1/c_k}, \ldots, |y^m|^{1/c_m}/|y^k|^{1/c_k}\}, \tag{29}$$

where k is fixed and ϕ is an arbitrary function.

The presence of powers of coordinates as in one dimension in (29) is suggestive of power law transformation of coordinates

$$y^k = (Y^k)^{d_k}, \tag{30}$$

in the differential equation for f, or equation of its characteristics in the y^k system

$$\frac{1}{\Delta}\sum_{k=1}^{m}\left[-c_k y^k - \sum_{i=1}^{m} B_i^k \Sigma g^i_{i_1 i_2 \ldots i_m} \prod_{p=1}^{m}\left(\sum_{q=1}^{m} b_q^p y^q\right)^{i_p}\right]\frac{\partial f}{\partial yk} = -f, \tag{31}$$

$$\frac{\mathrm{d}_k (Y^k)^{d_k - 1}\mathrm{d}Y^k}{\frac{1}{\Delta}[-c_k (Y^k)^{d_k} - \Sigma_{i=1}^{m} B_i^k \Sigma g^i_{i_1 i_2 \ldots i_m} \prod_{p=1}^{m}\{\Sigma_{q=1}^{m} b_q^p (Y^q)^{d_q}\}^{i_q}]} = -\frac{\mathrm{d}f}{f}. \tag{32}$$

Thus the alternative sequence following Newton's iteration of the type (21) in these new coordinates is given by

$$Y_{n+1}^k = Y_n^k - \frac{1}{\Delta \mathrm{d}_k}\left[c_k Y_n^k + (Y_n^k)^{1-d_k}\sum_{i=1}^{m} B_i^k \Sigma g^i_{i_1 i_2 \ldots i_m} \prod_{p=1}^{m}\left\{\sum_{q=1}^{m} b_q^p (Y_n^q)^{d_q}\right\}^{i_q}\right]. \tag{33}$$

This sequence which in general is complex will converge (to zero, the fixed point) as before if $|1 - c_k/\Delta d_k| < 1$. Choosing the powers d_k in this manner, we get alternative convergent sequences which are the maps of the original chaotic or even divergent sequences. If we want to obtain the original sequence (18), we have to retrace the steps beginning from (33) through (32), (30), (31), (23), (22) and finally (21).

In the important two dimensional case $m = 2$, the two quadratic equations of (25) yield

$$b_1^2 = \beta_1 b_1^1, \quad b_2^2 = \beta_2 b_2^1, \tag{34}$$

where

$$\beta_1, \beta_2 = \frac{1}{2a_2^1}[a_2^2 - a_1^1 \pm \{(a_2^2 - a_1^1)^2 + 4a_2^1 a_1^2\}^{1/2}]. \tag{35}$$

In this case,

$$c_1 = b_1^1 b_2^1[\beta_2 a_1^1 + \beta_1 \beta_2 a_2^1 - a_1^2 - \beta_1 a_2^2], \tag{36a}$$

$$c_2 = b_1^1 b_2^1[-\beta_1 a_1^1 - \beta_1 \beta_2 a_2^1 + a_1^2 + \beta_2 a_2^2], \tag{36b}$$

$$\Delta = b_1^1 b_2^1 (\beta_2 - \beta_1). \tag{36c}$$

Evidently b_1^1 and b_2^1 can be chosen arbitrarily say 1. By selecting them small, the alterative sequence (33) can be made convergent comparatively rapidly.

3.1 *Examples*

(i) *The delayed-logistic map*: This 'map' is a two dimensional extension of (14):

$$\xi_{n+1} = a\xi_n(1-\eta_n), \quad a \geqslant 1,$$
$$\eta_{n+1} = \xi_n. \tag{37}$$

The fixed points of the iteration are $\xi^* = \eta^* = 0, (a-1)/a$. The map was first proposed by Maynard Smith and subsequently investigated by several authors (Pounder and Rogers [8], Aronson *et al* [1], Rogers and Clarke [9]). It has been shown that the sequence of points (ξ_n, η_n) are attracted to the nontrivial fixed point for $1 \leqslant a < 2$. For $2 \leqslant a < a^*$, $a^* \approx 2{\cdot}27$, the sequence is oscillatory as in limit cycles of convex polygons which degenerate at a^* and the points, in the limit, describe an invariant curve of complicated shape. For $a > a^*$ the sequence diverges.

For our analysis, the transformation of (37) according to $\xi = x + (a-1)/a$, $\eta = y + (a-1)/a$, yields

$$x_{n+1} = x_n - (a-1)y_n - ax_n y_n, \quad a \geqslant 1,$$
$$y_{n+1} = x_n. \tag{38}$$

Here, $a_1^1 = 0$, $a_2^1 = a-1$, $a_1^2 = -1$, $a_2^2 = 1$. For the alternative convergent sequence

$$\beta_1, \beta_2 = \frac{1 \pm i\sqrt{4a-5}}{2(a-1)}, \tag{39}$$

with

$$c_1 = -\bar{c}_2 = \frac{\sqrt{4a-5}}{2(a-1)}[\sqrt{4a-5} - i], \quad \Delta = -\frac{\sqrt{4a-5}}{a-1}i. \tag{40}$$

The alternative sequence showing convergence, namely equation (33) is in this case

$$X_{n+1} = X_n - \frac{1}{\Delta d_1}[c_1 X_n + a\beta_2 X_n^{1-d_1}(X_n + Y_n)(\beta_1 X_n + \beta_2 Y_n)], \tag{41a}$$

$$Y_{n+1} = Y_n - \frac{1}{\Delta d_2}[c_2 Y_n - a\beta_1 Y_n^{1-d_2}(X_n + Y_n)(\beta_1 X_n + \beta_2 Y_n)], \tag{41b}$$

where for convergence it is required that $d_1, d_2 > a-1$. This is also observed in computer simulations. The sequence is complex even though the original is real.

(ii) *The Hénon map*: This well known 'map' can be represented in the standard form

$$x_{n+1} = ax_n + by_n - x_n^2, \tag{42a}$$
$$y_{n+1} = x_n. \tag{42b}$$

For $a = 2{\cdot}1678$, $b = 0{\cdot}3$, it reveals a 'strange attractor', that is, in large number of iterations, the map has a self-similarity and Cantor set-like cross-section. Both the fixed points $(0,0)$ and $(1-a-b, 1-a-b)$ are unstable.

In seeking mapped alternative convergent sequence, we have $a_1^1 = -(a-1)$, $a_2^1 = -b$, $a_1^2 = -1$, $a_2^2 = 1$ and from equations (35) and (36)

$$\beta_1, \beta_2 = -\frac{1}{2b}(a \pm \sqrt{a^2+4b}), \tag{43a}$$

$$c_1 = 2 + \frac{1}{2b}[a^2 - (a-2)\sqrt{a^2+4b}], \tag{43b}$$

$$c_2 = -2 - \frac{1}{2b}[a^2 + (a-2)\sqrt{a^2+4b}], \tag{43c}$$

$$\Delta = \sqrt{a^2+4b}/b. \tag{43d}$$

The alternative sequence (33) becomes in this case

$$X_{n+1} = X_n - \frac{1}{\Delta d_1}[c_1 X_n + \beta_2 X_n^{(1-d_1)}(X_n + Y_n)^2] \tag{44a}$$

$$Y_{n+1} = Y_n - \frac{1}{\Delta d_2}[c_2 Y_n - \beta_1 Y_n^{(1-d_2)}(X_n + Y_n)^2] \tag{44b}$$

and is a real sequence. Since $c_1 > 0$ and $c_2 < 0$, we have to select for convergence $d_1 > c_1/2\Delta$ and $d_2 < c_2/2\Delta$. This may be verified in computer simulations to generate a smooth trajectory of the points.

4. Second degree systems and the exponential transformation

In one dimension, we represent the second degree system as

$$x_{n+1} = (1-a)x_n - gx_n^2. \tag{45}$$

Both the fixed points $x^* = 0$ and $-a/g$ are unstable if $a > 2$. We also assume $g > 0$ for the parabola to be convex upwards in the (x_n, x_{n+1})-plane. The Newton scheme (9) in this case is

$$\frac{\mathrm{d}x}{-x(a+gx)} = -\frac{\mathrm{d}f}{f}. \tag{46}$$

If we perform the transformation $x + a/g = \mathrm{e}^{-X}$, it sends the 'other' fixed point to infinity and the only finite fixed point is $X^* = -\ln(a/g)$. Inserting in (46) and comparing with the form (8), we generate the new sequence

$$X_{n+1} = X_n - (a - g\mathrm{e}^{-X_n}), \tag{47}$$

of which the only fixed point is unstable. Starting near the fixed point the sequence oscillates and cannot diverge. For if $a - g\mathrm{e}^{-X_n} < 0$, $X_{n+1} > X_n$ and in the next iterate, the exponential term diminishes making $X_{n+2} < X_{n+1}$.

In two dimenesions, we suppose that the system is

$$x_{n+1}^1 = x_n^1 - \sum_{j=1}^{2} a_j x_n^j - \sum_{j=1}^{2}\sum_{k=1}^{2} g_{jk} x_n^j x_n^k, \quad g_{jk} = g_{kj}, \tag{48a}$$

$$x_{n+1}^2 = x_n^1, \tag{48b}$$

to which many of the well known forms belong. The fixed points of the system are $x^{1*} = x^{2*} = 0$ and $x^{1*} = x^{2*} = -\Sigma_{j=1}^2 a_j / \Sigma_{j=1}^2 \Sigma_{k=1}^2 g_{jk}$. We assume that both the points are unstable with suitable range of values of the parameters with the transformation $x^j + \Sigma_{j=1}^2 a^j / \Sigma_{j=1}^2 \Sigma_{k=1}^2 g_{jk} = \pm \mathrm{e}^{-X^j}$ (positive or negative sign according to that of the Σ terms) the nontrivial fixed point is removed to infinity and the Newton scheme as

n the derivation of (33) yields

$$X^j_{n+1} = X^j_n - \left[\frac{a^1(g_{11}-g_{22})+2a^2(g_{11}+g_{12})}{\Sigma^2_{j=1}\Sigma^2_{k=1} g_{jk}} + \frac{2a^1(g_{12}+g_{22})-a^2(g_{11}-g_{22})}{\Sigma^2_{j=1}\Sigma^2_{k=1} g_{jk}} e^{X^1_n - X^2_n} - g_{11}e^{-X^1_n} - 2g_{12}e^{-X^2_n} - g_{22}e^{X^1_n - 2X^2_n} \right], \tag{49a}$$

$$X^2_{n+1} = X^2_n - (e^{X^2_n - X^1_n} - 1). \tag{49b}$$

Evidently, the method may be generalized to higher dimensions of similar form. As examples we consider those of §§ 2.1 and 3.1. For the logistic map we get

$$X_{n+1} = X_n - (a - 1 - ae^{-X_n}). \tag{50}$$

Computer simulations show that the fixed point $X^* = -\ln(1-1/a)$ is an attractor for $a < 3$ and for $a \geqslant 3$ the sequence oscillates about X^*. In figure 1, we present the trajectory in the $X = X_n$, $Y = X_{n+1}$ plane of the sequential points for the case $a = 3{\cdot}9$–a value well within the chaotic regime of the original map. We note marked reduction of 'chaos'.

For the delayed-logistic map we get

$$X_{n+1} = X_n - (a - 1 - ae^{-Y_n}), \tag{51a}$$

$$Y_{n+1} = Y_n - (e^{-(X_n - Y_n)} - 1). \tag{51b}$$

In computer simulations, the sequence converges to the attractor $X^* = Y^* = -\ln(1-1/a)$ and oscillates about it for $a \geqslant 2$. In figure 2 its trajectory for $a = 2{\cdot}27$ which shows 'no chaos' is presented.

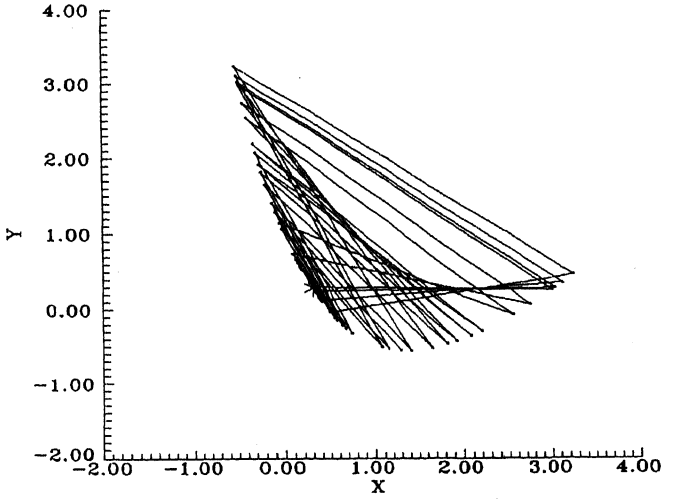

Figure 1. Transformed logistic map (50) for $a = 3{\cdot}9$ and 50 iterations. * is the fixed point $X^* = Y^* = -\ln(1-1/a)$.

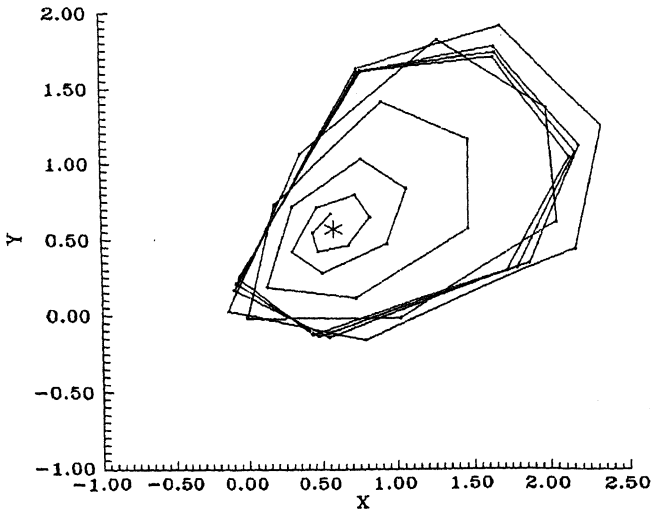

Figure 2. Transformed delayed logistic map (51) for $a = 2{\cdot}27$ and 50 iterations. * is the fixed point $X^* = Y^* = -\ln(1 - 1/a)$.

Transformation of the Hénon map leads to

$$X_{n+1} = X_n + (a + 2b - 1 - be^{X_n - Y_n} - e^{-X_n}), \tag{52a}$$

$$Y_{n+1} = Y_n - (e^{-(X_n - Y_n)} - 1). \tag{52b}$$

For $a = 2{\cdot}1678$, $b = 0{\cdot}3$ the sequence slowly diverges monotonically (in simulations).

5. Damped Newton and principal oscillatory part

If we consider chaotic oscillations in the general form (6) due to failure of convergence of Newton iterations (8), then we can think of a damped sequence

$$x_{n+1} = x_n - \omega(ax_n + \Sigma g_i x_n^i), \tag{53}$$

which fast converges to the fixed point. The fractional power of f, equation (10), implies that $|\omega| < 1$ (under relaxation). Fastest monotonic convergence takes place when the coefficient of the linear term is absent, that is when $\omega = a^{-1}$. Isolating this part in (6) we get

$$x_{n+1} = \left\{-(a-1)x_n\left(1 + \frac{1}{a}\Sigma g_i x_n^i\right)\right\} + \left\{-\frac{1}{a}\Sigma g_i x_n^i\right\}. \tag{54}$$

Thus the generating map of the sequence is decomposed into two; 1) the second expression within braces which is fast monotonically converging and 2) the first expression within braces. The latter must be highly oscillatory if the whole is chaotic. This part may be defined as the principal oscillatory part of the chaotic sequence.

The definition may be extended to two dimensional system (48) with the decomposition

$$x_{n+1}^1 = \left\{-(1-\omega)\left[\sum_{j=1}^{2} a_j x_n^j + \sum_{j=1}^{2}\sum_{k=1}^{2} g_{jk} x_n^j x_n^k\right]\right\} + \left\{x_n^1 - \omega\left[\sum_{j=1}^{2} a_j x_n^j + \sum_{j=1}^{2}\sum_{k=1}^{2} g_{jk} x_n^j x_n^k\right]\right\}, \tag{55a}$$

$$x_{n+1}^2 = x_n^1 \tag{55b}$$

where the under relaxation factor ω, $|\omega| < 1$, is so chosen that the larger of the two eigenvalues

$$\lambda = \frac{1}{2}[1 - \omega a_1 \pm \sqrt{(1-\omega a_1)^2 - 4\omega a_2}], \tag{56}$$

in magnitude is smallest. The right hand side of (55a) thus splits into the fast convergent second part and the principal oscillatory part represented by the first expression within braces, in case (48) is chaotic

The principal oscillatory parts of the three examples of §§ 2.1 and 3.1 are drawn in figures 3, 4, and 5. The under relaxation ω in these cases is respectively

(i) $\omega_L = (a-1)^{-1}$,

(ii) $\omega_D = \frac{1}{4}(a-1)^{-1}$ + a very small fraction,

iii) $\omega_H = -\dfrac{a-1+2b-\sqrt{(a-1+2b)^2-(a-1)^2}}{(a-1)^2}$ − a very small fraction

The necessity of a very small fraction arises from strict inequality required for ω and the three maps are respectively

(i) $$x_{n+1} = -(a-2)x_n\left(1+\frac{a}{a-1}x_n\right), \tag{57}$$

(ii) $$x_{n+1} = -(1-\omega_D)y_n(a-1+ax_n), \tag{58a}$$

$$y_{n+1} = x_n, \tag{58b}$$

iii) $$x_{n+1} = (1-\omega_H)[(a-1)x_n + by_n - x_n^2], \tag{59a}$$

$$y_{n+1} = x_n. \tag{59b}$$

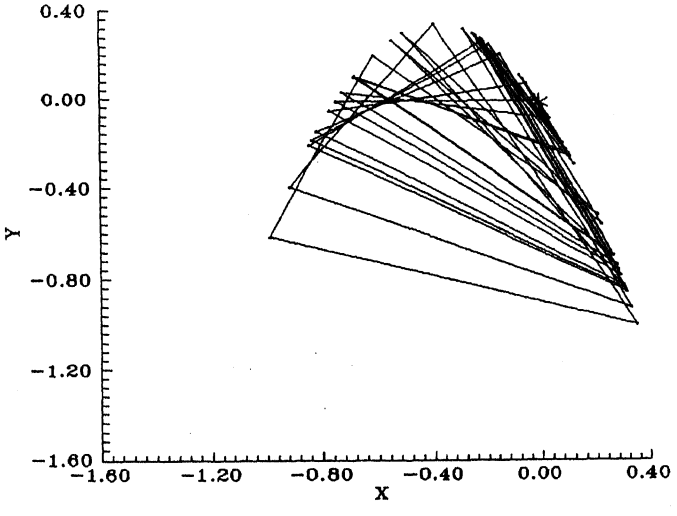

Figure 3. Principal oscillatory part of the logistic map (57) for $a = 3{\cdot}9$ and 50 iterations. * is the fixed point $X^* = Y^* = 0$.

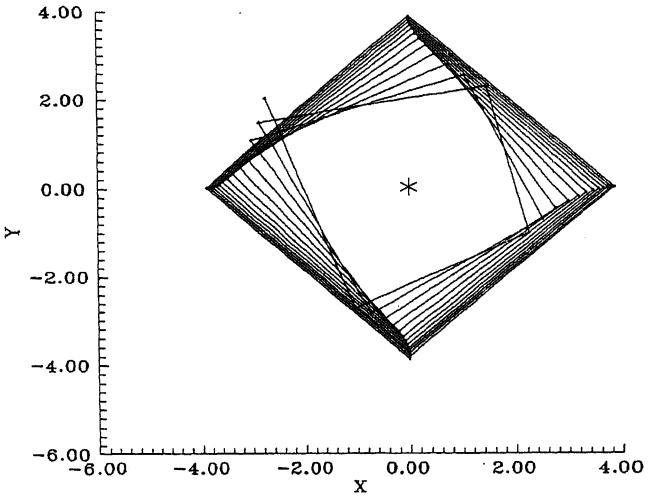

Figure 4. Principal oscillatory part of the delayed logistic map (58) (ten times magnified) for $a = 2{\cdot}27$ and 50 iterations. * is the fixed point $X^* = Y^* = 0$.

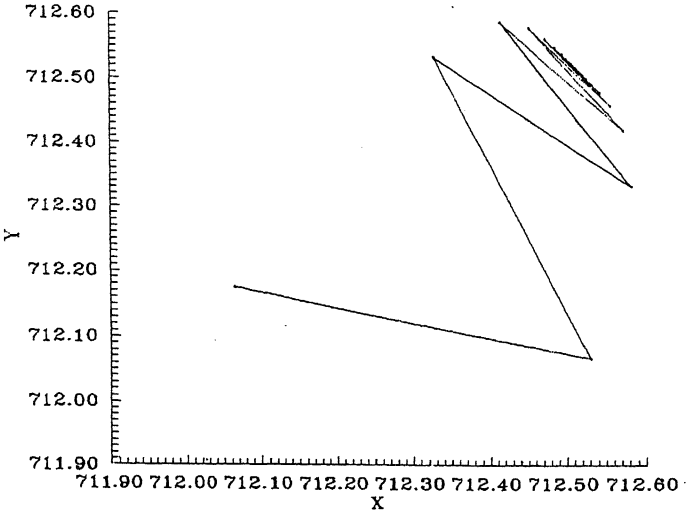

Figure 5. Principal oscillatory part of the Hénon map (59) (one thousand times magnified) for 12–50 iterations.

The extraction of the principal oscillatory part in the general m-dimensional case may be proceeded in a similar way. Finally, we note that since the principal oscillatory part is solely responsible for the chaotic oscillations, it should be possible to generate new chaotic sequences by combining these with other monotonically convergent sequences.

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The "Notes on the preparation of papers" are printed in the last issue of every volume.

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Notes on the preparation of papers

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sup, proj lim	$\overline{\lim}, \underline{\lim}$		
$A \to B$	$A \overset{f}{\to} B$		
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